Homework 3 - Solutions

Sequence Practice

Make sure to justify your solution for each problem. Determine wheter the sequence converges or diverges. If it converges, fint its limit.

1. $a_n = \frac{7+15n^4}{136-22n^3+47n^4}$

This sequence converges to $\frac{15}{47}$. This is because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{7}{n^4}+15}{\frac{136}{n^4}-\frac{22}{n}+47} = \frac{15}{47}$$

2. $a_n = \frac{3^{n+4}}{5^n}$

This sequence converges to zero since it is a gemetric sequence with $r < 1$.

3. $a_n = \sqrt{\frac{n-14}{7n+1}}$

Since $f(x) = \sqrt{x}$ is continuous on its domain, we have

$$\lim_{n\to\infty} \sqrt{\frac{n-14}{7n+1}} = \sqrt{\lim_{n\to\infty} \frac{n-14}{7n+1}} = \sqrt{\frac{1}{7}}$$

4. $a_n = ne^{-n}$

This sequence converges to zero by L'Hopital's rule.

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

5. $\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, ...\}$

Notice that when n is odd, $a_n = \frac{2}{n+1}$ and when n is even, $a_n = \frac{2}{n+4}$. Since both of these sequences converge to zero, the entire sequence converges to zero by the squeeze theorem.

6. $a_n = \frac{1+2+3+...+(n-1)}{n!}$

Recall that $1 + 2 + 3 + ... + (n-1) = \frac{n(n-1)}{2}$. So we can rewrite the series as $a_n = \frac{n(n-1)}{2n!} = \frac{1}{2(n-2)!}$. This sequence converges to zero.

7. $a_n = \frac{\cos(n)}{n^2}$

This converges to zero by the squeeze theorem.

8. $a_n = \frac{2+3^n}{2+3^{n+1}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{2}{3^n}+1}{\frac{2}{3^n}+3} = \frac{1}{3}$$

Double Integral Practice

9. $\int_0^3 \int_0^1 (16 - x^2 - 3y^2)\, dydx$

$$\int_0^3 \int_0^1 (16 - x^2 - 3y^2)\, dydx = \int_0^3 \left[\int_0^1 (16 - x^2 - 3y^2)\, dy \right] dx = \int_0^3 \left[\left(16y - x^2y - y^3 \Big|_0^1 \right) \right] dx$$

$$= \int_0^3 \left(15 - x^2\right)\, dx = \left(15x - \frac{x^3}{3}\right)\Bigg|_0^3 = 15(3) - 9 = 36$$

Change of Variables

10 . Prove that the change of variables formula from rectangular to polar coordinates is $dx\, dy = r\, dr\, d\theta$.

The change of coordinates is determined by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The Jacobian is

$$\begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(x) + r\sin^2(x) = r$$