Homework 4 - Solutions

1. Prove or disprove: If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} (a_n + b_n)$ is divergent.

   Proof: Given two sequences $\sum_{n=1}^{\infty} a_n$, which is convergent to L, and $\sum_{n=1}^{\infty} b_n$, which is divergent, suppose that $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent to L'. Then, we have

   $$\sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + b_n - a_n) = \sum_{n=1}^{\infty} b_n = L - L'$$

   is convergent. This contradicts the assumption that $\sum_{n=1}^{\infty} b_n$ is divergent. Hence, if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} (a_n + b_n)$ is divergent.

2. Find a value of c such that $\sum_{n=1}^{\infty} (1 + c)^{-n} = 2$.

   If the geometric series converges, then

   $$\begin{aligned}
   \sum_{n=1}^{\infty} (1+c)^{-n} &= \sum_{n=1}^{\infty} \left(\frac{1}{1+c}\right)^n \\
   &= \frac{1}{1 - \frac{1}{1+c}} - 1 \\
   2 &= \frac{1}{1 - \frac{1}{1+c}} - 1 \\
   3 &= \frac{1}{1 - \frac{1}{1+c}} \\
   1 - \frac{1}{1+c} &= \frac{1}{3} \\
   \frac{2}{3} &= \frac{1}{1+c} \\
   1 + c &= \frac{3}{2} \\
   c &= \frac{1}{2}
   \end{aligned}$$

To check, if $c = \frac{1}{2}$, then $\frac{1}{1+c} = \frac{2}{3}$, the geometric series does converge.

3. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{(n+1)^2}$

   Notice that

   $$\lim_{n\to\infty} \frac{n^2+2}{(n+1)^2} = 1 \neq 0.$$

   Thus, the series diverges by the Divergence Test.

   (b) $\displaystyle\frac{1}{1} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$

   First, observe that this series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so the series converges.

   (c) $\displaystyle\sum_{n=6}^{\infty} \frac{\sqrt{n}}{n-5}$

   Observe that

   $$\sum_{n=6}^{\infty} \frac{\sqrt{n}}{n-5} > \sum_{n=6}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=6}^{\infty} \frac{1}{n^{1/2}}.$$

   Since the latter series is a divergent p-series ($p = 1/2 < 1$), the first series diverges by the Comparison Test.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

   Consider the function $f(x) = \dfrac{1}{x(\ln(x))^2}$. Observe that $f(n) = a_n$ and that $f(x)$ is positive, decreasing, and continuous on $[2, \infty)$. We have

   $$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\, dx = \frac{1}{\ln(2)}$$

   via u-substitution with $u = \ln(x)$. Since the improper integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges by the Integral Test.

   (e) $\displaystyle\sum_{n=1}^{\infty} \left(\int_n^{n+1} \frac{dx}{x^{5/3}} \right)$

Consider the partial sums of this series:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\int_k^{k+1} \frac{dx}{x^{5/3}} \right) \\ &= \int_1^{n+1} \frac{dx}{x^{5/3}} \\ &= \frac{3}{8} \left(1 - \frac{1}{(n+1)^{8/3}} \right) \\ \lim_{n\to\infty} S_n &= \frac{3}{8} \end{aligned}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \left(\int_n^{n+1} \frac{dx}{x^{5/3}} \right)$ converges to $\frac{3}{8}$.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

First, compare $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3}$ using the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{n^2}{n^3}} = \lim_{n\to\infty} \frac{n^3+n}{n^3+1} = 1$$

We also have that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Then, by the Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ also diverges.

4. Can you find a sequence $\{a_n\}$ converging to 0 such that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges?

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

5. Find the Maclaurin series representation for each of the following series. (Hint: It is unnecessary to take any derivatives.)

(a) $f(x) = \dfrac{1}{x+10}$

$$\begin{aligned}\frac{1}{x+10} &= \frac{1}{10(1-(-\frac{x}{10}))} \\ &= \frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{-x}{10}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{10^{n+1}}\end{aligned}$$

(b) $f(x) = \dfrac{x}{2x^2+1}$

$$\begin{aligned}\frac{x}{2x^2+1} &= x\frac{1}{1-(-2x^2)} \\ &= x\sum_{n=0}^{\infty}(-2x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}\end{aligned}$$

6. Evaluate the indefinite integral as a power series.

$$\int \frac{\ln 1 - t}{t}\, dt$$

$$\begin{aligned}\int \frac{\ln(1-t)}{t}dt &= \int \frac{1}{t}\sum_{n=1}^{\infty} -\frac{t^n}{n}dt \\ &= \int \sum_{n=1}^{\infty} -\frac{t^{n-1}}{n}dt \\ &= \sum_{n=1}^{\infty} - \int \frac{t^{n-1}}{n}dt \\ &= \sum_{n=1}^{\infty}\left(-\frac{x^n}{n^2} + C_n\right) \\ &= \sum_{n=1}^{\infty} -\frac{x^n}{n^2} + C\end{aligned}$$

7. Find the Taylor series for each of the following functions at the indicated center.

(a) $\cos(x), \quad c = \pi/4$

For $f(x) = \cos(x)$, we have

$$f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n \equiv 0 \mod 4 \\ -\sin(x) & \text{if } n \equiv 1 \mod 4 \\ -\cos(x) & \text{if } n \equiv 2 \mod 4 \\ \sin(x) & \text{if } n \equiv 3 \mod 4 \end{cases}$$

and hence

$$f^{(n)}\left(\frac{\pi}{4}\right) = \begin{cases} \frac{\sqrt{2}}{2} & \text{if } n \equiv 0,3 \mod 4 \\ -\frac{\sqrt{2}}{2} & \text{if } n \equiv 1,2 \mod 4 \end{cases}$$

Therefore, the Taylor Series expansion for $\cos(x)$ centered at $\frac{\pi}{4}$ is

$$\begin{aligned} \cos(x) &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2!} + \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{3!} + - - + \cdots \\ &= \frac{\sqrt{2}}{2}\left(1 - (x - \frac{\pi}{4}) - \frac{1}{2!}(x - \frac{\pi}{4})^2 + \frac{1}{3!}(x - \frac{\pi}{4})^3 + \frac{1}{4!}(x - \frac{\pi}{4})^4 - - + + \cdots\right) \\ &= \frac{\sqrt{2}}{2}\left(\sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{4})^{2n}}{(2n)!} - \sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{4})^{2n+1}}{(2n+1)!}\right) \\ &= \frac{\sqrt{2}}{2}\left(\sum_{n=0}^{\infty}(-1)^{\lceil\frac{n}{2}\rceil}\frac{(x-\frac{\pi}{4})^n}{n!}\right) \end{aligned}$$

(b) $e^x + e^{-x}, \quad c = 0$

We have that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and hence $e^x = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$. Therefore,

$$\begin{aligned} e^x + e^{-x} &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n + (-x)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n(1 + (-1)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{2x^{2n}}{(2n)!} \\ &= 2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \end{aligned}$$