Homework **5** - Solutions

1. Let V be the set of all linear combinations of the vectors $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Is V a vector space? Prove or disprove.

   Proof: By definition, $V = \text{Span}\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix} \right\}$. Every span is subspace of its ambient space and, hence, must be a vector space.

2. Let W be the set of points on the line $y = 2x + 1$. That is, let

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x + 1 \right\}.$$

   Is W a vector space? Prove or disprove.

   Disproof: The zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in W, hence W cannot be a vector space. Alternatively, it is easy to show that W is not closed under vector addition nor under scalar multiplication.

3. Prove that the set of all differentiable functions on $\mathbb{R}$ is a vector space over $\mathbb{R}$.

   Proof: Let $\mathbb{D}$ denote the set of all differentiable functions on $\mathbb{R}$. It is clear that $\mathbb{D}$ inherits most of the required properties from the vector space of all functions on $\mathbb{R}$, so we only need to verify that $\mathbb{D}$ is a subspace. Let $f(x)$ and $g(x)$ be in $\mathbb{D}$, and let c be an arbitrary real number. We know that

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

   and

$$\frac{d}{dx}(cf) = c \cdot \frac{df}{dx},$$

   which means that $\mathbb{D}$ is closed under vector addition and scalar multiplication. Since all constant functions are differentiable, we have that the zero function is also in $\mathbb{D}$. By the Subspace Criteria, $\mathbb{D}$ is a subspace of the vector space of all functions on $\mathbb{R}$ and, thus, is a vector space in its own right.

4. Prove that the set of all rational numbers is NOT a vector space over $\mathbb{R}$.

   Proof: The number 1 is a rational number, and $\sqrt{2}$ is a irrational number. Since $\sqrt{2} \cdot 1 = \sqrt{2}$ is irrational, the set of all rational numbers is not closed under scalar multiplication with scalars drawn from $\mathbb{R}$. Hence, the set of all rational numbers is not a vector space over $\mathbb{R}$.

5. Prove that the map $T : \mathbb{R}^2 \to \mathbb{R}^3$ that maps a vector $\mathbf{v}$ to the vector $\mathbf{0}$ is a linear transformation.

Proof: Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors in $\mathbb{R}^2$, and let c be an arbitrary scalar from $\mathbb{R}$. Notice that

$$T(\mathbf{u}+\mathbf{v}) = 0 = 0 + 0 = T(\mathbf{u}) + T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = 0 = c \cdot 0 = c \cdot T(\mathbf{u}).$$

Since these two statements are true for all appropriate vectors and scalars, T must be a linear transformation by definition.

6. Suppose that T is the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^5$ that maps a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \text{ to } \begin{bmatrix} -v_1 - 7v_3 \\ v_2 \\ v_1 \\ 0 \\ v_2 - v_3 \end{bmatrix}.$$

Find a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$.

Solution: The required matrix is

$$\begin{bmatrix} -1 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

7. Suppose that T is the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$ that takes the coefficient vector $\mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ of a polynomial $f(x) = v_0 + v_1 x + v_2 x^2 + v_3 x^3 + v_4 x^4$ and maps it to the coefficient vector of $\displaystyle\int_0^x f(x)\,dx$. Find a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$.

Solution: The required matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

8. Is the vector $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ in $\text{Span}\left\{ \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$?

Solution: No. If we row-reduce the matrix

$$\begin{bmatrix} -4 & 1 & | \; 2 \\ 7 & 2 & | \; 3 \\ 8 & 3 & | \; 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | \; 0 \\ 0 & 1 & | \; 0 \\ 0 & 0 & | \; 1 \end{bmatrix},$$

we see that the three vectors are linearly independent. Therefore, $\mathbf{w}$ cannot be in $\text{Span}\left\{ \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.