Homework 6 - Solutions

1. Determine if the columns of A form a linearly independent set.

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

Solution: A is row-equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since every column has a pivot, the columns of A form a linearly independent set.

2. Prove that if $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$, then there exists $\mathbf{v_k}$ in S such that $\text{Span}(S \backslash \{\mathbf{v_k}\}) = \text{Span}(S)$.

Proof: S is linearly dependent; therefore, there exists some solution to

$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

with some k such that $x_k \neq 0$. Then,

$$-x_k\mathbf{v_k} = x_1\mathbf{v_1} + \cdots + x_{k-1}\mathbf{v_{k-1}} + x_{k+1}\mathbf{v_{k+1}} + \cdots + x_p\mathbf{v_p}$$

and hence

$$\mathbf{v_k} = -\frac{x_1}{x_k}\mathbf{v_1} - \cdots - \frac{x_{k-1}}{x_k}\mathbf{v_{k-1}} - \frac{x_{k+1}}{x_k}\mathbf{v_{k+1}} - \cdots - \frac{x_p}{x_k}\mathbf{v_p}$$

For any $\mathbf{w} \in \text{Span}(S)$, we have

$$\mathbf{w} = w_1\mathbf{v_1} + \cdots + w_p\mathbf{v_p}$$

$$= w_1\mathbf{v_1} + \cdots + w_k\left(-\frac{x_1}{x_k}\mathbf{v_1} - \cdots - \frac{x_{k-1}}{x_k}\mathbf{v_{k-1}} - \frac{x_{k+1}}{x_k}\mathbf{v_{k+1}} - \cdots - \frac{x_p}{x_k}\mathbf{v_p}\right) + \cdots + w_p\mathbf{v_p}$$

Hence, it is also true that $\mathbf{w} \in \text{Span}(S \backslash \{\mathbf{v_k}\})$. Thus, $\text{Span}(S \backslash \{\mathbf{v_k}\}) = \text{Span}(S)$.

3. Find a basis for each of the following subspaces of $\mathbb{R}^n$.

   (a) All vectors whose components are equal in $\mathbb{R}^4$.
   Solution: This is the set

$$\left\{\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}\right\} = \left\{a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

. The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent and therefore a basis.

(b) All vectors whose components add up to zero in $\mathbb{R}^4$.

Solution: This is the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\} = \left\{ \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\} = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

The set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent and therefore a basis for the space.

4. Consider the matrix $A = \begin{bmatrix} 2 & 5 & -8 & 7 \\ -1 & 5 & 4 & 7 \\ 0 & 5 & 0 & 7 \end{bmatrix}$.

(a) Find two different bases for $ColA$.

Solution: Row-reduce A to locate the pivot columns, which will form one basis for $ColA$. Reorder the columns of A to get a matrix A', and then row-reduce A' to locate the pivot columns, which will form a different basis for $ColA$.

(b) Find two different bases for $NulA$.

Solution: Row-reducing A produces the matrix

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is straightforward to show that

$$NulA = \text{Span}\left(\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.4 \\ 0 \\ 1 \end{bmatrix} \right)$$

and that $\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1.4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set and hence a basis for $NulA$. Replace at least one of the basis vectors with a scalar multiple of itself to obtain a different basis for $NulA$.

5. Suppose S is a 5-dimensional subspace of $\mathbb{R}^6$. Prove that every basis for S can be extended to a basis for $\mathbb{R}^6$ by adding one more vector.

   Proof: Suppose $\{\mathbf{v_1}, \ldots, \mathbf{v_5}\}$ is a basis for S, and let $\mathbf{w}$ be a vector in $\mathbb{R}^6$ that is not in Span(S). Then, $\mathbf{w}$ cannot be written as a linear combination of the vectors in S. Since S is already a linearly independent set, $S \cup \{\mathbf{w}\}$ is also linearly independent. The set $S \cup \{\mathbf{w}\}$ is a linearly independent set of 6 vectors in a 6-dimensional vector space, thus it is a basis for $\mathbb{R}^6$.

6. Find the eigenvalues of

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

   Solution: It is straightforward to show that the characteristic polynomial of B is $\lambda^3(4 - \lambda)$, so the eigenvalues of B are 0 (with algebraic multiplicity 3) and 4 (with algebraic multiplicity 1).

7. Prove that the eigenvalues of A are the same as the eigenvalues of A^T for any square matrix A.

   Proof: Note that $(A - \lambda I)^T = A^T - \lambda I$. Expanding $\det(\ (A - \lambda I)^T\ )$ along the first row and $\det(\ A^T - \lambda I\ )$ along the first column will lead to the same polynomial. Thus, both A and A^T have the same characteristic polynomial and, hence, the same eigenvalues.