**Linear Independence and Bases**

August 12th-AM

**Definition**: A finite set of vectors $\{v_1, v_2, v_3, \ldots, v_k\}$ in a real vector space $V$ is said to be *linearly dependent* if there exists $c_1, c_2, c_3, \ldots c_k \in \mathbb{R}$, not all of which are 0, such that $c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0$.

**Examples**:

1. The vectors
$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad x_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$
are linearly dependent since $2x_1 + x_2 - x_3 = 0$.

2. Any set containing the zero vector is linearly dependent.

**Definition**: The set $S = \{v_1, v_2, v_3, \ldots, v_k\}$ is *linearly independent* if the only choice of $c_1, c_2 \ldots c_k \in \mathbb{R}$ such that $c_1x_1 + c_2x_2 + \cdots + c_kx_k = 0$ are $c_1 = c_2 = \cdots = c_k = 0$.

**Examples**:

1. A set consisting of two different vectors is linearly dependent $\iff$ one vector is a nonzero multiple of the other.

2. Find two linearly independent vectors belonging to the null space of
$$A = \begin{pmatrix} 3 & 2 & -1 & 4 \\ 1 & 0 & 2 & 3 \\ -2 & -2 & 3 & -1 \end{pmatrix}$$

3. Are the columns of $A$ linearly independent in $\mathbb{R}^3$?

**Definition**: A set of vectors $S = \{v_1, v_2, v_3, \ldots, v_k\}$ with $v_i \in \mathbb{R}^n$ and $span(S) = V$ is a basis of $V$ if $S$ forms a linearly independent set.

**Definition**: Let $V \neq \{0\}$ be a subspace of $\mathbb{R}^n$. The dimension of $V$ is the number of elements in any basis of $V$.

**Example**: Find the dimension of the null space of
$$A = \begin{pmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$