

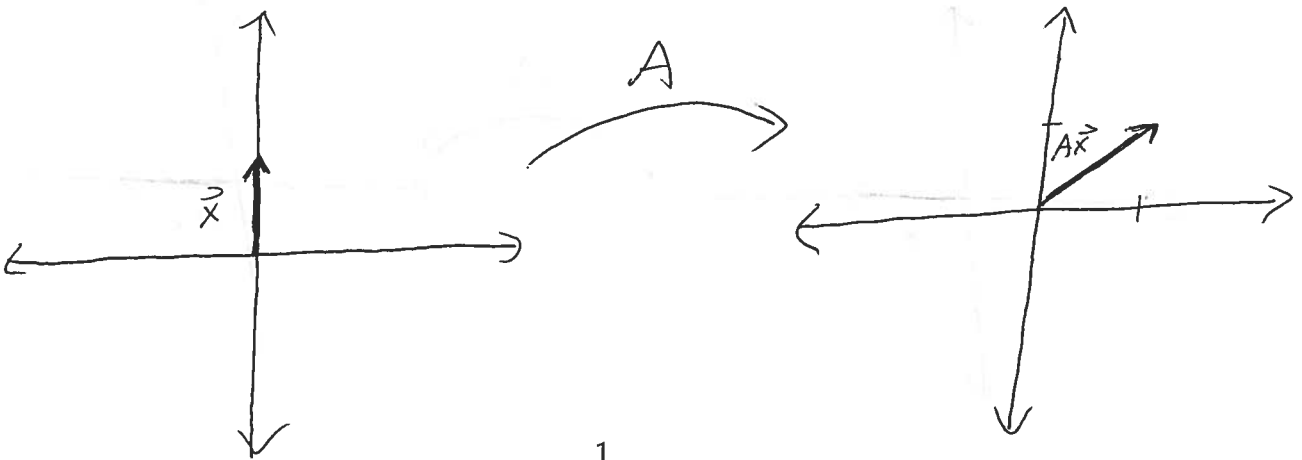
Eigenvalues and Eigenvectors

When a square matrix A acts upon a vector x , it generally outputs a new vector Ax . Usually this new vector will be stretched and rotated. For example, if we take the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

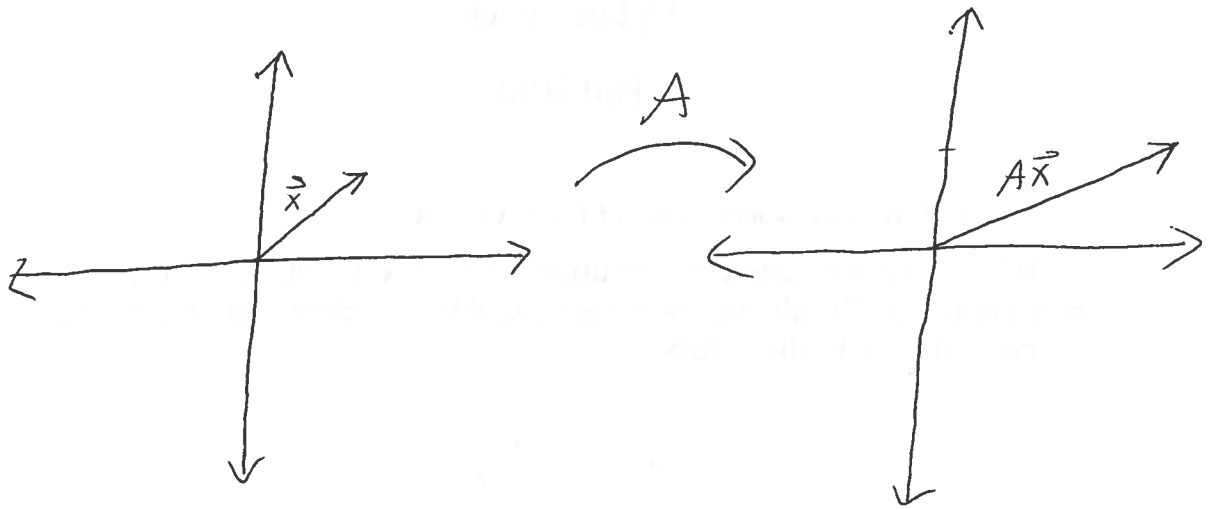
and apply it to the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we get:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



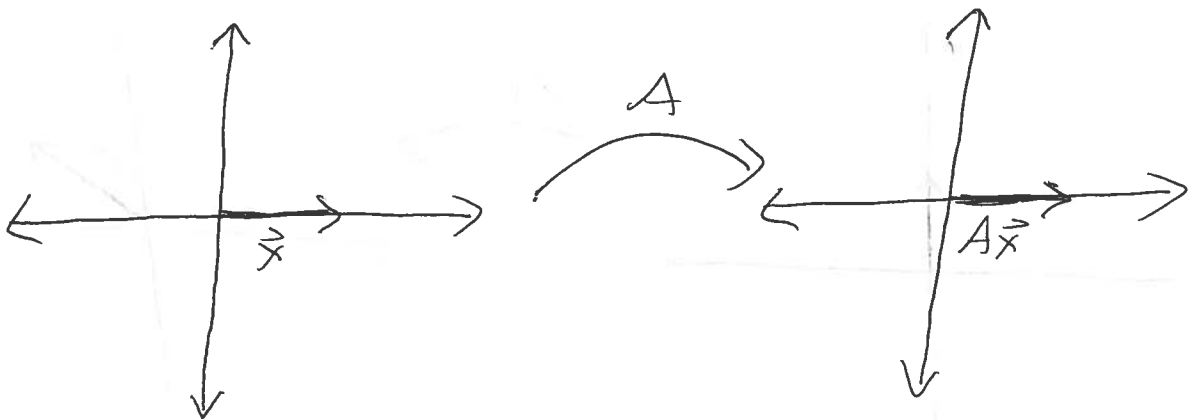
If we apply A to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



However, if we apply A to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



The matrix A leaves the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ unchanged! In particular, it does not rotate the vector. When a matrix A acts upon a vector and does not rotate it, we have $A\mathbf{x} = \lambda\mathbf{x}$, where λ is a scaling factor. In our example $\lambda = 1$. We call such a vector an *eigenvector* for the matrix A , and the associated scaling factor λ an *eigenvalue*. This section, and in fact this chapter, explore eigenvectors and eigenvalues.

Note we'll assume throughout this lecture, and in fact throughout all lectures about chapter 6, that all matrices are square.

The assigned problems for this section are:

Section 6.1 - 2, 3, 5, 16, 17

1 Eigenvectors and Eigenvalues

The basic equation for eigenvectors and eigenvalues is

$$A\mathbf{x} = \lambda\mathbf{x}$$

where A is a matrix and λ is a number. One property we see right away is

$$A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

and in general

$$A^n\mathbf{x} = \lambda^n\mathbf{x}.$$

So, the eigenvectors of A^n are the same as the eigenvectors of A , while the eigenvalues for A^n are the eigenvalues for A raised to the n th power.

OK, now let's see how we actually calculate what these eigenvalues and eigenvectors are. The first thing I want to stress is that we calculate the eigenvectors from the eigenvalues, so the first thing we want to do is calculate the eigenvalues. How do we do this? Well, we note that if

$$A\mathbf{x} = \lambda\mathbf{x}$$

then

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

So, the matrix $(A - \lambda I)$ has a nontrivial nullspace, and therefore must be singular. So,

$$\det(A - \lambda I) = 0.$$

So, if λ is an eigenvalue of A then $\det(A - \lambda I) = 0$. It turns out that this equation can be used to calculate *every* eigenvalue. The equation $\det(A - \lambda I) = 0$ will be a polynomial equation in λ , and its roots will give us all the eigenvalues. We call this polynomial equation the *characteristic equation* of A .

For example, the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is

$$\begin{vmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{vmatrix} = (1 - \lambda)(4 - \lambda) - 4 = \lambda^2 - 5\lambda.$$

The roots of the polynomial $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$ are $\lambda = 0$ and $\lambda = 5$, and these will be the eigenvalues of A . From these we can compute the eigenvectors

$$\lambda = 0. (A - 0I)\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

yields the solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, which is an eigenvector for $\lambda = 0$.

$$\lambda = 5. (A - 5I)\mathbf{x} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

yields the solution $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which is an eigenvector for $\lambda = 5$.

Note that these eigenvectors are *not* unique. In fact, any non-zero multiple $c\mathbf{x}$ ($c \neq 0$) of an eigenvector is another eigenvector.

Oh, and we should probably mention right now that if $\mathbf{x} = 0$ then $A\mathbf{x} = \lambda\mathbf{x}$ for *any* λ . So, we have to put in a qualifier that *the zero vector is never an eigenvector*. Please keep in mind that the number 0 can certainly be an eigenvalue.

Example - Find the eigenvalues and associated eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(2-\lambda) - 2(2-\lambda) = (2-\lambda)(6-5\lambda+\lambda^2)+2\lambda-4$$

$$= 12 - 10\lambda + 2\lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 + 2\lambda - 4$$

$$= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda-1)(\lambda^2-6\lambda+8)$$

$$= -(\lambda-1)(\lambda-4)(\lambda-2) \quad \text{Eigenvalues } \lambda = 1, 4, 2$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \text{Eigenvector for } \lambda = 1$$

$$\begin{pmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{Eigenvector for } \lambda = 2$$

$$\begin{pmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -7 \\ 4 \\ -2 \end{pmatrix} \rightarrow \text{Eigenvector for } \lambda = 4$$

2 Some Facts About Eigenvectors

First, some bad news. We cannot use elimination to calculate eigenvalues. Sorry. If we use elimination to convert a matrix A into an upper triangular matrix U , the eigenvalues of U could be different than the eigenvalues of A . However, the two will not be completely unrelated.

What relates them is the amazing fact that the determinant of a matrix is equal to the product of its eigenvalues. That is to say, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of a matrix A , then

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

So, because A and U have the same determinant, the product of their eigenvalues will be the same.

Example - Calculate the determinant for the matrix A from the previous example, and verify that the determinant is, in fact, equal to the product of the eigenvalues.

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 2 \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 2(6-2) = 8 \\ &= 1 \times 2 \times 4 \end{aligned}$$

Finally, we note that the trace of a matrix is defined as being the sum of the diagonal elements.

$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

The trace of a matrix will be equal to the sum of the eigenvalues of the matrix

$$\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Example - Verify this for the matrix

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$1 + 2 + 4 = 2 + 3 + 2. \quad \checkmark$$