Derivatives and Integrals
Aug 6 - AM

Average Rate of Change: The average rate of change of a function $f(x)$ from a to $a+h$ is

$$\frac{f(a+h)-f(a)}{h}$$

Instantaneous Rate of Change: The instantaneous rate of change of a function $f(x)$ at $x=a$ is defined as

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

This is also called the derivative of $f(x)$ at $x=a$ and is denoted by $f'(a)$.

L'Hopital's Rule:
Suppose we want to evaluate

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

If either

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

or

$$\lim_{x\to c} |f(x)| = \lim_{x\to c} |g(x)| = \infty$$

Then,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Provided the limit exists.

Partial Derivatives: Partial derivatives apply to functions more than one variable. They are instantaneous rates of change with respect to a specific variable. Let $f(x,y)$ be a function. Then we can defined the partial derivative of f with respect to x (similarly y) as

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

Example: Compute the partial derivatives of $f(x,y) = x^2y^5$.

$$f_x(x,y) = 2xy^5$$

$$f_y(x,y) = 5x^2y^4$$

See handout for list of important derivative rules.

Integration: The integral of a piece-wise continuous function $f(x)$ over an interval $[a, b]$ is the total signed area bounded between the curve and the x-axis and is denoted

$$\int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus: If $f(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Also, if we define $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

U-Substitution

Example: Evaluate $\int_2^4 x^2 e^{x^3}\,dx$.

We need to substitute part of the function with u. But it isn't random. We want the derivative of u with respect to x to match up with the other part of the function. In this case, we let $u = x^3$. Then $\frac{du}{dx} = 3x^2$ or $\frac{du}{3x^2} = dx$. Also, if $x = 2$ then $u = 8$ and if $x = 4$, then $u = 64$. We can now rewrite the integral as follows.

$$\int_2^4 x^2 e^{x^3}\,dx = \int_8^{64} x^2 e^u \frac{du}{3x^2} = \int_8^{64} e^u \frac{du}{3} = \left.\frac{e^u}{3}\right|_8^{64} = \frac{e^{64}}{3} - \frac{e^8}{3}$$