Double Integrals, Change of Variables,
August 7 - AM

Double Integrals
Double integrals geometrically measure the volume of a shape in space. Double integrals are similar to partial derivatives in the fact that you hold one variable constant while integrating with respect to the other variable.

Example: Evaluate $\int_2^4 \int_1^9 ye^x \, dy \, dx$

$$\int_2^4 \int_1^9 ye^x \, dy \, dx = \int_2^4 \left[\int_1^9 ye^x \, dy \right] \, dx = \int_2^4 \left[\frac{y^2}{2} e^x \Big|_1^9 \right] \, dx$$

$$= \int_2^4 \left[\frac{9^2}{2} e^x - \frac{1^2}{2} e^x \right] \, dx = \int_2^4 40e^x \, dx = 40e^x \Big|_2^4 = 40(e^4 - e^2)$$

Example: Evaluate $\int_1^9 \int_2^4 ye^x \, dx \, dy$

$$\int_1^9 \int_2^4 ye^x \, dx \, dy = \int_1^9 \left[\int_2^4 ye^x \, dx \right] \, dy = \int_1^9 \left[ye^x \Big|_2^4 \right] \, dy$$

$$= \int_1^9 \left[y(e^4 - e^2) \right] \, dy = \frac{y^2}{2}(e^4 - e^2) \Big|_1^9 = 40(e^4 - e^2)$$

Fubini's Theorem If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$ then

$$\int \int_R f(x, y) dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Example: Evaluate $\int_1^3 \int_1^{x^2} xy \, dy \, dx$.

$$\int_1^3 \left[\int_1^{x^2} xy \, dy \right] \, dx = \int_1^3 \left[\frac{xy^2}{2} \Big|_1^{x^2} \right] \, dx = \int_1^3 \frac{x^5}{2} - \frac{x}{2} \, dx$$

$$= \frac{x^6}{12} - \frac{x^2}{4} \Big|_1^3 = \frac{243}{4} - \frac{9}{4} = \frac{176}{3}$$

Note: Triple integrals (etc.) work the same way.

Change of Variables: Double integrals in x, y coordinates which are taken over circular regions, or have integrands involving the combination $x^2 + y^2$, are often better done in polar coordinates. The change of rectangular to polar coordinates is done using the following.

$$\int \int_R f(x, y) \, dx \, dy = \int \int_R g(r, \theta) r \, dr \, d\theta$$

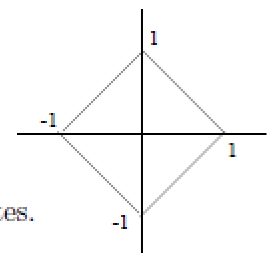
$$r^2 = x^2 + y^2 \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad x = r\cos(\theta) \quad y = r\sin(\theta)$$

In the same way, double integrals involving other types of regions or integrands can sometimes be simplified by changing the coordinate system from x, y to one better adapted to the region or integrand. Call the new coordinates u, v. Then there will be equations introducing the new coordinates as functions of the old coordinates and visa versa.

$$u = u(x, y) \quad v = v(x, y) \quad x = x(u, v) \quad y = y(u, v)$$

We need to supply the area element as with the case of polar coordinates. In this general case, define the Jacobian matrix by $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$. Then,

$$\int\int_R f(x, y)dx\, dy = \int\int_R g(u, v) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\, dv$$



Example Evaluate $\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\, dy$ over the region R pictured.

Solution. This would be a painful integral to work out in rectangular coordinates. But the region is bounded by the lines

$$x + y = \pm 1, \qquad x - y = \pm 1$$

and the integrand also contains the combinations $x - y$ and $x + y$. These powerfully suggest that the integral will be simplified by the change of variable (we give it also in the inverse direction, by solving the first pair of equations for x and y):

$$u = x + y, \qquad v = x - y; \qquad\qquad x = \frac{u+v}{2}, \qquad y = \frac{u-v}{2}.$$

We will also need the new area element;

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2};$$

Thus the new area element is

$$dA = \frac{1}{2}\, du\, dv.$$

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\, dy = \iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv.$$

In uv-coordinates, the boundaries of the region are simply $u = \pm 1$, $v = \pm 1$, so the integral becomes

$$\iint_R \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv = \int_{-1}^{1}\int_{-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2}\, du\, dv$$

We have

$$\text{inner integral} = -\frac{v^2}{2(u+2)}\Bigg]_{u=-1}^{u=1} = \frac{v^2}{3}; \qquad \text{outer integral} = \frac{v^3}{9}\Bigg]_{-1}^{1} = \frac{2}{9}.$$