Sequences and Sequential Limits
August 7 - PM

Sequences A real *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$.

Example: Find the first five terms of the sequence $a_n = n!$.

The first five terms are determined by substituting $n = 0, 1, 2, 3, 4$ for n. So we have the first five terms are $1, 1, 2, 6, 24$.

There are also **recursive** formulas for sequences.

Example: Let $a_0 = 1$ and $a_n = 2a_{n-1} + 1$. Find a_1 and a_2.

$$a_1 = 2a_0 + 1 = 2 + 1 = 3 \qquad a_2 = 2a_1 + 1 = 6 + 1 = 7$$

Example: Fibonacci's sequence is recursive. It is defined by

$$F_0 = 1 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2}$$

The first few terms are $1, 1, 2, 3, 5, 8, 13, 21, ...$

Sequential Limits:

The most common question that arises in the setting of sequences is whether a given sequence converges.

Definition: A sequence a_n converges to a limit L if $\forall \epsilon > 0 \, \exists N$ such that $n > N \implies |a_n - L| < \epsilon$.

Example: The sequence $a_n = \frac{1}{n}$ converges to zero.

Let $\epsilon > 0$ be given. Let $N = \frac{1}{\epsilon}$. When $n > N$ we have

$$|a_n - 0| = |\frac{1}{n}| < \frac{1}{N} = \epsilon$$

The most common way to find sequential limits is to use the following theorem and then apply all the limit tools we have already developed.

Theorem: If $f(n) = a_n$ and $\lim_{s\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Example: Find the limit of the sequence $a_n = \frac{n+\ln(n)}{n^2}$.

$$\lim_{n\to\infty} \frac{n + \ln(n)}{n^2} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2n} = 0$$

We used L'Hopital's rule for the first equality.

Geometric Sequence: Suppose $a_n = cr^n$ for some real numbers c, r. Then

$$\lim_{n\to\infty} a_n = \begin{cases} c & r = 1 \\ 0 & r < 1 \\ \infty & r > 1 \end{cases}$$

Refer to the limit laws handout for more rules.

Example: Show $a_n = \frac{\sin(n)}{n}$ converges to zero.

This follows directly from the squeeze theorem by using the fact that $|\sin(n)| \leq 1$ for all natural numbers.