# TAYLOR AND MACLAURIN SERIES

## 1. Basics and examples

Consider a function $f$ defined by a power series of the form

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \tag{1}$$

with radius of convergence $R>0$. If we write out the expansion of $f(x)$ as

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^2+c_4(x-a)^4\ldots,$$

we observe that $f(a)=c_0$. Moreover

$$\begin{aligned} f^{'}(x)&=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\ldots, \\ f^{(2)}(x)&=2c_2+2\cdot 3\cdot c_3(x-a)+3\cdot 4\cdot c_4(x-a)^2+\ldots \\ f^{(3)}(x)&=2\cdot 3\cdot c_3+2\cdot 3\cdot 4\cdot c_4(x-a)^2+\ldots \end{aligned}$$

After computing the above derivatives we observe that

$$\begin{aligned} f(a)&=c_0, \\ f^{'}(a)&=c_1, \\ f^{(2)}(a)&=2\Longrightarrow c_2=\frac{f^{(2)}(a)}{2!}, \\ f^{(3)}(a)&=2\cdot 3\cdot c_3\Longrightarrow c_3=\frac{f^{(3)}(a)}{3!}. \end{aligned}$$

In general we have

$$f^{(n)}(a)=n!c_n\Longrightarrow c_n=\frac{f^{(n)}(a)}{n!},$$

We have shown the following

**Theorem 1** (Taylor-Maclaurin series)**.** *Suppose that $f(x)$ has a power series expansion at $x=a$ with radius of convergence $R>0$, then the series expansion of $f(x)$ takes the form*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f^{'}(a)(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\ldots, \tag{2}$$

*that is, the coefficient* $c_n$ *in the expansion of* $f(x)$ *centered at* $x = a$ *is precisely* $c_n = \dfrac{f^{(n)}(a)}{n!}$. *The expansion* (2) *is called* **Taylor series**. *If* $a = 0$, *the expansion*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f^{'}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \dots,$$

*is called* **Maclaurin Series**.

Let us now consider several classical Taylor series expansions. For the following examples we will assume that all of the functions involved can be expanded into power series.

**Example 1.** The function $f(x) = e^x$ satisfies $f^{(n)}(x) = e^x$ for any integer $n \geq 1$ and in particular $f^{(n)}(0) = 1$ for all $n$ and then the Maclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

observe that the radius of convergence of $f(x)$ is computed by noting that $c_n x^n = \dfrac{x^n}{n!}$ so that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0,$$

and the radius of convergence is $R = \infty$ since the above computation shows that the series converges absolutely for any $x$. Note that for any other center, say $x = a$ we have $f^{(n)}(a) = e^a$, so that the Taylor expansion of $f(x)$ is

$$e^x = \sum_{n=0}^{\infty} \frac{e^a (x-a)^n}{n!}.$$

and this series also has radius of convergence $R = \infty$.

**Example 2.** Compute the Maclaurin series of the function $f(x) = \cos(x)$. Note that $f(x)$ satisfies

$$\begin{cases} f^{'}(x) &= -\sin(x) \\ f^{(2)}(x) &= -\cos(x) \\ f^{(3)}(x) &= \sin(x) \\ f^{(4)}(x) &= \cos(x) \end{cases}$$

and the above pattern is periodic, in fact, we will have

$$f^{(2n)}(x) = (-1)^n \cos(x) \Longrightarrow f^{(2n)}(0) = (-1)^n$$
$$f^{(2n+1)}(x) = (-1)^n \sin(x) \Longrightarrow f^{(2n+1)}(0) = 0,$$

and therefore

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Note that $\cos(x)$ is an even function in the sense that $\cos(-x) = \cos(x)$ and this is reflected in its power series expansion that involves only even powers of $x$. The radius of convergence in this case is also $R = \infty$.

**Example 3.** Compute the Maclaurin series of $f(x) = \sin(x)$. For this case we note that

$$f^{(2n)}(x) = (-1)^n \sin(x) \Longrightarrow f^{(2n)}(0) = 0$$
$$f^{(2n+1)}(x) = (-1)^n \cos(x) \Longrightarrow f^{(2n+1)}(0) = (-1)^n,$$

and therefore

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

The radius of convergence is again $R = \infty$.

**Example 4.** Compute the Maclaurin series of the following functions

(1) $\dfrac{\sin(x)}{x}$
(2) $\dfrac{\sin(x^2)}{x^2}$
(3) $\displaystyle\int_0^x \frac{\sin(s^2)}{s^2} ds$

For (1) we use the the expansion $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ so that

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}.$$

For (2) we replace $x$ by $x^2$ and obtain for $x > 0$ the series

$$\frac{\sin(x^2)}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{4n}}{(2n+1)!}.$$

Finally, for (3) we integrate the Maclaurin series of $\dfrac{\sin(x^2)}{x^2}$

$$\int_0^x \frac{\sin(s^2)}{s^2} ds = \sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{(s)^{4n}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{4n+1}}{(4n+1) \cdot (2n+1)!}.$$

**Remark:** For a function that has an even expansion like $f(x) = \dfrac{\sin(x)}{x}$, we can also expand $f(\sqrt{x})$ as a power series. As an **exercise**, compute the Maclaurin expansion of $\displaystyle\int_0^x \frac{\sin(\sqrt{s})}{\sqrt{s}} ds$.

1.1. **Taylor polynomials and Maclaurin polynomials.** The partial sums of Taylor (Maclaurin) series are called Taylor (Maclaurin) polynomials. More precisely, the Taylor polynomial of degree $k$ of $f(x)$ at $x = a$ is the polynomial

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and the Maclaurin polynomial of degree $k$ of $f(x)$ (at $x = 0$) is the polynomial

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

An important question about Taylor polynomials is how well they approximate the functions that generate them. In fact we have the following error estimate

**Theorem 2.** *Consider the interval $(x_0, x_1)$ with $x_0 < a < x_1$ and suppose that $f(x)$ is differentiable to any order on $(x_0, x_1)$ and continuous on $[x_0, x_1]$. Fix $k \geq 1$ and let $M > 0$ be a constant such that $\max_{[x_0,x_1]} |f^{(k+1)}(x)| \leq M$. Then for any $x$ in $(x_0, x_1)$ we have*

$$|f(x) - p_k(x)| \leq \frac{M|x-a|^{k+1}}{(k+1)!}.$$

On the other hand, when it comes to the practical computation of Taylor or Maclaurin polynomials it may not be necessary to compute all of the derivatives of $f(x)$.

**Example 5.** Compute the Maclaurin polynomial of degree 4 for the function $f(x) = \cos(x)\ln(1-x)$ for $-1 < x < 1$.

**Idea:** In order to compute the Maclaurin polynomial of degree 4 of $f(x)$ we will multiply out the series expansions of the functions $\cos(x)$ and $\ln(1-x)$ thus obtaining a new power series, however we will only keep those terms in the expansion of the new series that have degree **at most 4**. In other words, if after multiplying the power series expansions of $\cos(x)$ and $\ln(1-x)$ we manage to write out the power series expansion of $\cos(x)\ln(1-x)$ in the form

$$f(x) = \cos(x)\ln(1-x) = \underbrace{c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4} + c_5 x^5 + \dots$$

then the Maclaurin polynomial $p_4$ of degree 4 of $f(x)$ is

$$p_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4.$$

Note that for $-1 < x < 1$ we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\ln(1-x) = -\int_0^x \frac{ds}{(1-s)} = -\sum_{n=0}^{\infty} \int_0^x s^n ds$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

on the other hand

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots, \tag{3}$$

let us use $(*)$ to denote the expansion in (3), meaning that $(*) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$, so that after multiplying both series we have

$$\begin{aligned}
\cos(x)\ln(1-x) &= \underbrace{(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots)}_{(*)}(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots) \\
&= -x(*) - \frac{x^2}{2}(*) - \frac{x^3}{3}(*) - \frac{x^4}{4}(*) - \dots \\
&\underbrace{\left(-x + \frac{x^3}{2} - \frac{x^5}{4!} + \dots\right)}_{-x(*)} + \underbrace{\left(-\frac{x^2}{2} + \frac{x^4}{2 \cdot 2!} - \frac{x^6}{2 \cdot 4!} + \dots\right)}_{-\frac{x^2}{2}(*)} \\
&+ \underbrace{\left(-\frac{x^3}{3} + \frac{x^5}{3 \cdot 5!} - \dots\right)}_{-\frac{x^3}{3}(*)} + \underbrace{\left(-\frac{x^4}{4} + \frac{x^6}{4 \cdot 2!} - \dots\right)}_{-\frac{x^4}{4}(*)} \\
&= \left(-x + \frac{x^3}{2}\right) + \left(-\frac{x^2}{2} + \frac{x^4}{2 \cdot 2!}\right) + \left(-\frac{x^3}{3}\right) + \left(-\frac{x^4}{4}\right) + \dots \\
&= \underbrace{-x - \frac{x^2}{2} + \frac{x^3}{6}}_{p_4(x)} + \dots
\end{aligned}$$

We have used the color blue to highlight those terms of degree **at most** 4 in the multiplication of the two series. It follows that the Maclaurin polynomial of order 4 of $f(x) = \cos(x)\ln(1-x)$ is

$$\boxed{p_4(x) = -x - \frac{x^2}{2} + \frac{1}{6}x^3}$$

**Remark:** The radius of convergence of $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ and this is also the case for $-\ln(1-x) = \sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n+1}$, however the interval of convergence of this last series is $[-1, 1)$ (closed on the left and open on the right) because for $x = -1$ the series is $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ which converges conditionally but for $x = 1$ the series is $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

**Exercise:** Compute the first four terms in the power series expansion of $f(x) = \dfrac{\ln(1+x)}{1+x}$.

**Example 6.** Compute the limit

$$\lim_{x\to 0} \frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}}.$$

Note that in this case using a L'Hospital rule is extremely tedious. An alternative approach is to expand $\cos(x^4) - 1 + \frac{1}{2}x^8$ as a power series

$$\cos(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n}}{(2n)!} = 1 - \frac{1}{2}x^8 + \frac{x^{16}}{4!} - \dots,$$

so that

$$\lim_{x\to 0} \left( \frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}} \right) = \frac{1}{4!}.$$