Vector Spaces, Subspaces, Span, Matrices, Linear Tranformations, Null Spaces, Column Spaces
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Definition

A *vector space* V over a field $\mathbb{F}$ is a nonempty set on which two operations are defined - addition and scalar multiplication. Addition is a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, and scalar multiplication is a rule for associating with each scalar $k \in \mathbb{F}$ and each object $\mathbf{u}$ in V an object $k\mathbf{u}$ such that

1. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u}+\mathbf{v} \in V$.
2. If $\mathbf{u} \in V$ and $k \in \mathbb{F}$, then $k\mathbf{u} \in V$.
3. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$
4. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$
5. There is an object $\mathbf{0}$ in V, called a **zero vector** for V, such that $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V.
6. For each $\mathbf{u}$ in V, there is an object -$\mathbf{u}$ in V, called the **additive inverse** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u}) = -\mathbf{u}+\mathbf{u} = \mathbf{0}$;
7. $k(l\mathbf{u}) = (kl)\mathbf{u}$
8. $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$
9. $(k+l)\mathbf{u} = k\mathbf{u}+l\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Remark The elements of the underlying field $\mathbb{F}$ are called scalars and the elements of the vector space are called vectors. Note also that we often restrict our attention to the case when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Examples of Vector Spaces

A wide variety of vector spaces are possible under the above definition as illustrated by the following examples. In each example we specify a nonempty set of objects V. We must then define two operations - addition and scalar multiplication, and as an exercise we will demonstrate that all the axioms are satisfied, hence entitling V with the specified operations, to be called a vector space.

1. The set of all n-tuples with entries in the field $\mathbb{F}$, denoted $\mathbb{F}^n$ (especially note
2. The set of all $m \times n$ matrices with entries from the field $\mathbb{F}$, denoted $M_{m\times n}(\mathbb{F})$.
3. The set of all real-valued functions defined on the real line $(-\infty, \infty)$.
4. The set of polynomials with coefficients from the field $\mathbb{F}$, denoted $P(\mathbb{F})$.
5. (Counter example) Let $V = \mathbb{R}^2$ and define addition and scalar multiplication oparations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0).$$

Theorem *If $u, v, w \in V$ (a vector space) such that $u + w = v + w$, then $u = v$.*

Corollary *The zero vector and the additive inverse vector (for each vector) are unique.*

Theorem *Let V be a vector space over the field $\mathbb{F}$, $\mathbf{u} \in V$, and $k \in \mathbb{F}$. Then the following statement are true:*

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-k)\mathbf{u} = -(k\mathbf{u}) = k(-\mathbf{u})$

(d) If $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = 0$.

Subspaces

- A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In general, all ten vector space axioms must be verified to show that a set W with addition and scalar multiplication forms a vector space. However, if W is part of a larget set V that is already known to be a vector space, then certain axioms need not be verified for W because they are inherited from V. For example, there is no need to check that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (axiom 3) for W because this holds for all vectors in V and consequently holds for all vectors in W. Likewise, axioms 4, 7, 8, 9 and 10 are inherited by W from V. Thus to show that W is a subspace of a vector space V (and hence that W is a vector space), only axioms 1, 2, 5 and 6 need to be verified. The following theorem reduces this list even further by showing that even axioms 5 and 6 can be dispensed with.

Theorem *If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.*

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Examples of Subspaces

1. A plane through the origin of $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$. This is evident geometrically as follows: Let W be any plane through the origin and let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in W other than the zero vector. Then $\mathbf{u} + \mathbf{v}$ must lie in W because it is the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, and $k\mathbf{u}$ must lie in W for any scalar k because $k\mathbf{u}$ lies on a line through $\mathbf{u}$. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^3$.
2. A line through the origin of $\mathbb{R}^3$ is also a subspace of $\mathbb{R}^3$. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well. Thus, W is closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^3$.

Definitions

- A vector $\mathbf{w}$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_r$ if it can be expressed in the form

$$\mathbf{w} = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Example

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in $\mathbb{R}^3$. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and that $\mathbf{w}' = (4, -1, 8)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Span

If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ are vectors in a vector space V, then generally some vectors in V may be linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ and others may not. The following theorem shows that if a set W is constructed consisting of all those vectors that are expressible as linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, then W forms a subspace of V.

Theorem 1.6. *If* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ *are vectors in a vector space* V*, then:*

(a) The set W *of all linear combinations of* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ *is a subspace of* V*.*

(b) W *is the smallest subspace of* V *that contains* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ *every other subspace of* V *that contains* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ *must contain* W

Definitions

- If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the **space spanned** by $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, and it is said that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ **span** W. To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ the below notation is used.

$$W = span(S) \quad or \quad W = span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$$

Example The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. This can be denoted by writing

$$P_n = span\{1, x, x^2, \ldots, x^n\}$$

Spanning sets are not unique. For example, any two noncolinear vectors that lie in the $x - y$ plane will span the $x - y$ plane. Also, any nonzero vector on a line will span the same line.

Column Space and Nullspace

If A is an $m \times n$ matrix, then the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the *column space*. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$, is called the *nullspace*.

Definition: A system of equations $A\mathbf{x} = \mathbf{b}$ is consistent if there is a solution(s).

Theorem: A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Example: Let $A\mathbf{x} = \mathbf{b}$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A and express $\mathbf{b}$ as a linear combination of the column vectors of A.

Solving the system by Gaussian elimination yields

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

Since the system is consistent, $\mathbf{b}$ is in the column space of A. Moreover,

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Linear Transformations:
If $T : V \to W$ is a function between vector spaces, then T is called a **linear transformation** from V to W if for all vectors $\mathbf{u}, \mathbf{v}$ in V and all scalars c

$$(a)\ T(\mathbf{u} + \mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v}) \quad \text{and} \quad (\mathbf{b})\ \mathbf{T}(\mathbf{cu}) = \mathbf{cT}(\mathbf{u})$$

Example: Define the *orthogonal projection* of $\mathbf{v}$ onto $\mathbf{w}$ by $\text{proj}_{\mathbf{w}}\mathbf{v} = (\mathbf{v} \cdot \mathbf{w})\frac{\mathbf{w}}{||\mathbf{w}||^2}$. Then for a fixed $\mathbf{w}$, $T(\mathbf{v}) = \text{proj}_{\mathbf{w}}\mathbf{v}$ is a linear transformation.

Example: Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ maps

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{to} \begin{bmatrix} x - y \\ 0 \\ 2x + 3y + z \\ y + 4z \end{bmatrix}$$

Find a matrix A such that $T(\mathbf{v}) = \mathbf{Av}$.
Answer: The required matrix is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$