Research Description

Margaret A. Readdy

What follows is a sampling of my research activities since promotion to Associate Professor.

Poset transforms and topological combinatorics

Björner and Welker recently initiated a study to generalize concepts from commutative algebra to the area of poset topology. One such poset product they discovered, called the Rees product, models the Rees algebra occurring in commutative algebra. One of the fundamental topological results they show is that the poset theoretic Rees product preserves the Cohen-Macaulay property. This property is important in that it immediately implies well-behaved homology groups, that is, the order complex $\Delta(P)$ of a rank n Cohen-Macaulay poset P has vanishing homology groups everywhere except for the top homology. Furthermore, the dimension of the top homology group is given by $\dim \widetilde{H}_{n-2}(\Delta(P)) = (-1)^n \cdot \mu_P(\hat{0}, \hat{1})$, where μ_P denotes the poset-theoretic Möbius function.

Very little is known about the Rees product of two arbitrary posets. The examples studied to-date by Jonsson, Muldoon Brown-Readdy, and Shareshian-Wachs have yielded rich enumerative and q-enumerative results and have implications for studying homological and representation-theoretic questions. Jonsson showed the Möbius function of the Rees product of the rank n Boolean algebra with the chain C_n (equivalently, the Euler characteristic of its order complex) equals $(-1)^n$ times the nth derangement number, settling a conjecture of Björner and Welker. In [23*] *Rees product of posets*, my graduate student Muldoon Brown and I established the signed version of Jonsson's results, that is, the Rees product of the the face lattice of the n-dimensional cube $\mathscr{C}_n$ with the n-chain has the Euler characteristic of its order complex equal to n times a signed derangement number. We also discovered a short and elegant bijective proof of Jonsson's theorem and studied topological aspects of the order complex of the Rees product of $\mathscr{C}_n$ with the chain.

In order to understand the Rees product topologically, one must first understand how this poset product changes the flag f-vector of the original posets. We show the Möbius function of *any graded poset* with the chain, and more generally, with the t-ary tree coincides with the Möbius function of its dual with the tree. This is quite unexpected as the two resulting posets are in general not isomorphic. My next step is to find a homotopy equivalence between the order complexes of these two Rees products. More interestingly would be to consider the case when the poset P under consideration is not Cohen-Macaulay.

I have recently discovered a new poset product, the k-interpolated Rees product, which interpolates between the Segre and Rees products. It remains for me to determine what topological properties it preserves, such as the homotopic Cohen-Macaulay and Cohen-Macaulay properties, and if there is a commutative algebraic notion corresponding to the k-interpolated Rees product.

Given the importance of Eulerian posets, one would like to understand poset transforms which preserve the Eulerian property. Two such poset transforms, discovered by Hetyei, are studied in-depth study in the paper [22*] *The Tchebyshev transforms of the first and second kind*. For the Tchebyshev transform of the first kind, we show it is a linear transformation of the flag vector. When restricted to Eulerian posets it corresponds to the classical Billera-Ehrenborg-Readdy omega map linking the flag f-vector of a hyperplane arrangement with its oriented matroid. The Tchebyshev transform of the first kind is also shown to preserve nonnegativity of the **cd**-index and EL-shellability. These are important properties when one is studying flag f-vector inequalities. The Tchebyshev transform U of the second kind is shown to be a Hopf algebra endomorphism on the space of QSym of quasi-symmetric functions. When restricted to Eulerian posets, it coincides with Stembridge's peak enumerator. The complete spectrum of U is determined, generalizing work of Billera, Hsiao and van Willigenburg. The type B quasisymmetric function of a poset is introduced and, like Ehrenborg's classical quasisymmetric function of a poset, it is a comodule morphism with respect to the quasisymmetric functions QSym.

Mathematical physics

Beginning with the paper [13], I became interested in algebraic and topological structures motivated by physics. In [16*] *The pre-WDVV ring of physics and its topology,* I focus on a simplicial complex associated with the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations of string theory. This tree space, first studied by Boardman and better known as the Whitehouse complex, has reappeared in many guises, including representation theory (by Robinson-Whitehouse and generalized by Hanlon), geometric group theory (especially work of Vogtmann), and phylogenetic trees (see work of Ardila-Klivans, as well as that of Billera-Holmes-Vogtmann). Using discrete Morse theory, I give elementary proofs of the topological structure of the Whitehouse complex, including that it is homotopy equivalent to a wedge of spheres and satisfies the Cohen-Macaulay property. Face enumeration of the complex and the Hilbert series of its Stanley-Reisner ring are also determined.

In [28*] *Enumerative and asymptotic analysis of a moduli space*, I am studying the Hilbert series of the cohomology ring of the moduli space of stable n-pointed curves of genus zero. This moduli space is connected with the WDVV equations. By considering the exponential generating function formed by taking diagonal entries in the triangle of Hilbert series coefficients, I prove these series satisfy an integral operator identity. This enables me to glean enumerative and asymptotic behavior of this family of series. I then study what the physicists call the "total dimension", that is, the sum of the coefficients of the Hilbert series and show its asymptotic behavior is controlled by the Lambert W function.

Structure theorems for posets

Understanding the structure of posets is central to combinatorics. One example is due to Doubilet, Rota and Stanley, who introduced the notion of binomial posets in part to explain why certain families of generating functions "naturally" occur in combinatorics. These posets require every interval of length n to have the same number $B(n)$ of maximal chains. The function $B(n)$ is called the factorial function. This notion was generalized simultaneously in [2] and by Reiner to Sheffer posets, where one makes a distinction between whether or not the n-intervals start from the minimal element of the poset. The face lattice of an n-dimensional cube is the classical example of a Sheffer paper.

In [17*] *Classification of the factorial functions of Eulerian binomial and Sheffer posets*, we are able to completely classify the factorial functions of binomial and Sheffer posets which are *Eulerian*, that is, which satisfy the Euler-Poincaré relation in every interval. Additionally, if the binomial factorial function of an Eulerian Sheffer poset is $B(n) = n!$, respectively $B(n) = 2^{n-1}$, we classify the Sheffer factorial function $D(n)$ to one of three, respectively two possibilities. Imposing the further condition that an Eulerian binomial or Sheffer poset is a lattice forces the poset to be the infinite Boolean algebra or the infinite cubical lattice.

Stanley's theory of exponential structures, that is, families of posets modeled upon the partition lattice, was developed to explain certain permutation phenomena. As his work motivated many researchers to continue to study these posets from representation theoretic and topological considerations, it makes sense to broaden this class. Recall the Dowling lattice arises from the intersection lattice of the complex hyperplane arrangement

$$\begin{cases} z_i = \zeta^h \cdot z_j & \text{for } 1 \leq i < j \leq n \text{ and } 0 \leq h \leq s-1, \\ z_i = 0 & \text{for } 1 \leq i \leq n. \end{cases}$$

Here ζ is a primitive sth root of unity and the case $s = 1$ yields the familiar partition lattice. Analogous to Stanley's work on exponential structures, in [19*] *Exponential Dowling structures* we define the notion of an exponential Dowling structure. The results include finding a compositional formula for these structures and applying this to give the generating function for their Möbius values. The notion of the r-divisible partition lattice is also extended, and Wach's EL-labeling of the r-divisible partition lattice is shown to extend to this generalization.

A second paper motivated by Stanley's work on the r-divisible partition lattice is [18*] *The Möbius function of partitions with restricted block sizes.* In this paper the subposet of set partitions whose elements are restricted by type, that is, the multiset of cardinalities of blocks, is examined. Sylvester in the case $r = 2$, and later Stanley for general r, showed the Möbius function of set partitions with blocks having cardinality divisible by r is given by permutations having descent set $\{r, 2r, 3r, \ldots\}$. In our case we show the Möbius function computation is reduced to understanding the descent set statistics and the Möbius function of the (easier) lattice of integer compositions. For the case of knapsack partitions, we utilize a topological argument on subcomplexes of the boundary of the dual of the permutahedron to aid in the Möbius function computation.

Descent sets and permutations

For a permutation $\pi = \pi_1 \cdots \pi_n$ in the symmetric group $\mathfrak{S}_n$ on n elements, the descent set of π is $\{i : \pi_i > \pi_{i+1}\}$. Observe the descent set is a subset $S \subseteq \{1, \ldots, n-1\}$. Given a subset $S \subseteq \{1, \ldots, n-1\}$, the beta invariant $\beta(S)$ enumerates all permutations in the symmetric group $\mathfrak{S}_n$ having descent set equal to S. The descent set statistic is usually encoded using the Eulerian polynomial $E_n(t) = \sum_S \beta(S) \cdot t^{|S|}$.

In the paper [20*] *Cyclotomic factors of the descent set polynomial* we introduce the *nth descent set polynomial* $Q_n(t) = \sum_S t^{\beta(S)}$ as an alternative way to encode the sizes of descent classes of permutations. These polynomials exhibit interesting factorization patterns. The question of when particular cyclotomic factors divide these polynomials is explored. For example, we deduce the proportion of odd entries in the descent set statistics in the symmetric group on n elements only depends on the number on 1's in the binary expansion of n. Similar properties are determined for the signed descent set statistics.

Extensions of classical hyperplane arrangement theory

In the paper [21*] *Affine and toric hyperplane arrangements*, my coauthors and I extend the oriented matroids ideas developed in [5] to affine subspace arrangements in d-dimensional Euclidean space and subspace arrangements on the d-dimensional torus. Studying toric arrangements is an intellectual shift in our field, as until recently most combinatorialists studied sphere-like objects, rather than more general manifolds.

For toric arrangements, we show that the characteristic polynomial is given by the valuation of the complement of the arrangement, analogous to the valuation interpretation of the characteristic polynomial in [9], and that the number of regions is up to a sign given by setting $t = 0$ in the characteristic polynomial. (There is an analogous result for affine arrangements, which I omit here.) This latter result is the toric analogue of Zaslavsky's classical result that the number of regions in a hyperplane arrangement is found by setting $t = -1$ in its characteristic polynomial. Fundamentally this works because the reduced Euler characteristic of an open n-dimensional ball is $(-1)^n$, while that of an n-dimensional toric subspace is zero.

Coalgebraic techniques are used to extend the Billera-Ehrenborg-Readdy [5] omega map of oriented matroids between the flag f-vector and intersection poset for the toric and affine arrangements. For any regular CW-complex whose geometric realization is a compact n-dimensional manifold, we show the **ab**-index can be written as a **cd**-index when n is odd. When n is even, the **ab**-index has the form

$$\Psi(P) = \left(1 - \frac{\chi(M)}{2}\right) \cdot (\mathbf{a} - \mathbf{b})^{n+1} + \frac{\chi(M)}{2} \cdot \mathbf{c}^{n+1} + \Phi,$$

where Φ is a homogeneous **cd**-polynomial of degree $n + 1$ which does not have the term $\mathbf{c}^{n+1}$. This paper ends with a Pandora's box of open problems involving regular subdivisions of manifolds and inequalities for flag vectors of manifolds.

Eulerian posets, Bruhat graphs and Kazhdan-Lusztig theory

A major project I am working on is to understand the notoriously difficult Kazhdan-Lusztig polynomials of Coxeter groups. These polynomials arose out of Kazhdan and Lusztig's study of the Springer representations of the Hecke algebra of a Coxeter group. The Kazhdan-Lusztig polynomials have many applications, including to Verma modules and the algebraic geometry and topology of Schubert varieties.

One of the main obstacles to understanding the Kazhdan-Lusztig polynomials is that in general they are difficult to compute. Recent work of Billera and Brenti have shown one can calculate the Kazhdan-Lusztig polynomials by defining a complete **cd**-index of Bruhat intervals in terms of a quasisymmetric function. This work motivated Ehrenborg and I to pursue a more general study of labeled directed graphs. In the paper [26*] *Balanced and Bruhat graphs* we prove that the Billera-Brenti results can be greatly simplified by our recently-discovered and more general setting of balanced graphs, of which the Bruhat graphs are particular instances.

A tantalizing research problem looms in the background of this work. Currently there is no known combinatorial proof of the non-negativity of the coefficients of Kazhdan-Lusztig polynomials. The first proof for finite Weyl groups arose using local intersection cohomology. We hope that approaching the Kazhdan-Lusztig polynomials combinatorially via the **cd**-index may lead to a combinatorial proof. Such a proof would be widely recognized as a major research result.

A beginning step in this direction is to prove the non-negativity of the **cd**-index of bounded acyclic digraphs having what we call a balanced linear order. We hope to prove this conjecture using a combinatorial argument, such as the notion of shelling. A second project is to develop Kazhdan-Lusztig polynomials for directed graphs. Another avenue of research concerns Brenti, Caselli and Marietti's theory of special matchings of a Hasse diagram of a poset which parallels the notion of perfect matchings in the Bruhat graph. Can this be extended to balanced graphs? There also may be some connection with Stembridge's recent work on W-admissible graphs.

Research monograph

One of the recurring themes of my research is to study the face incidence structure of polytopes and more general objects. This face enumerative data is encoded in the flag f-vector, that is, for a d-dimensional polytope and $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \ldots, d-1\}$, the flag f-vector entry f_S enumerates the number of chains of faces $F_1 \subset \cdots \subset F_k$ with the dimension of the face F_i equal to s_i. There are linear relations which hold among the face data of a polytope. The most well-known is the Euler-Poincaré relation, which for a d-dimensional polytope is the alternating sum $f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d$, where f_i equals the number of i-dimensional faces. For the flag f-vector there are additional linear relations which hold among the entries. These are called the generalized Dehn-Sommerville relations and are due to Bayer and Billera.

One soon realizes this facial data becomes unwieldy since a d-dimensional polytope has 2^d flag vector entries associated to it. Bayer and Klapper showed there is a non-commutative polynomial called the **cd**-index in which one can record this information compactly, that is, which removes all the linear redundancies described by the generalized Dehn-Sommerville relations. Encoding the face incidence data using the **cd**-index turns out to be advantageous. In the seminal paper on coproducts, Ehrenborg and I discovered that the **cd**-index has an inherent coalgebra structure. This has enabled us to introduce coalgebraic techniques into the field of polytopes.

The Ehrenborg-Readdy coalgebraic techniques have been successfully applied to a host of fundamental problems. One of the most striking early applications of these techniques was by Billera and Ehrenborg who settled the Stanley conjecture for Gorenstein* lattices in the case of polytopes. This result gives a lower bound on the individual **cd**-index coefficients, and ultimately, a lower bound on the flag f-vector entries. Ehrenborg has continued to use the coalgebraic techniques to study the general question of determining all the *inequalities* which hold among the flag vectors of polytopes. This question is already open for 4-dimensional polytopes. The coalgebraic techniques also give very elegant proofs of old results, including Billera-Ehrenborg-Readdy's concise proof that flag vectors of zonotopes span the space of all flag vectors of polytopes, vastly simplifying the original proof by Bayer and Billera.

The study of face incidence information in combinatorics using a coalgebraic viewpoint is a very rich and interdisciplinary area involving algebra, combinatorics, geometry and topology. There is much more work to be done, but what is severely lacking is a resource for researchers to learn the results and techniques in order to further advance the field.

Billera, Ehrenborg and I have coordinated our sabbatical leaves for 2010–2011 so that we can write such a research monograph. It will give the core results in the theory of coalgebras applied to polytopes and more general flag enumeration in posets, and include new research directions which embrace quasisymmetric functions, topology, algebraic geometry, recent results in Kazhdan-Lusztig theory and polynomial flag vector inequalities.