Calculus IV - HW 5

Due 8/3

Section 6.3

1. For each of the following, sketch the graph of the given function on the interval $t \geq 0$.

   (a) $g(t) = u_2(t) + 2u_5(t) - 5u_7(t)$
   (b) $g(t) = u_\pi(t) \sin(t)$
   (c) $g(t) = u_\pi(t) \sin(t - \pi)$

2. Define $f(t)$ to be

   $$f(t) = \begin{cases} 
   1, & 0 \leq t < 3 \\
   -5, & 3 \leq t < 4 \\
   2, & t \geq 4 
   \end{cases}$$

   (a) Sketch the graph of the $f(t)$.
   (b) Express $f(t)$ in terms of the unit step functions (also called Heaviside functions) $u_c(t)$.

3. (Problems 16 and 18 from B&D) For each of the following, find the Laplace transform of the given function.

   (a) $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
   (b) $f(t) = t - u_1(t)(t - 1), \quad t \geq 0$

4. (Problems 20, 22, and 24 from B&D) For each of the following, find the inverse Laplace transform of the given function.

   (a) $F(s) = \frac{e^{-2s}}{s^2 + s - 2}$
   (b) $F(s) = \frac{2e^{-2s}}{s^4 - 4}$
   (c) $F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$
5. **Optional** Find the Laplace transform of
\[ f(t) = 1 - u_1(t) + u_2(t) - \ldots - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t). \]

**Section 6.4**

6. (Problem 1a, 4a from B&D) For each of the following, find the solution of the given initial value problem.

   (a) \( y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1 \) where \( f(t) = \begin{cases} 1, & 0 \leq t < 3 \\ 0, & 3\pi \leq t < \infty \end{cases} \).

   (b) \( y'' + 4y = \sin(t) + u_\pi(t) \sin(t - \pi); \quad y(0) = 0, \quad y'(0) = 0. \)

7. **Optional** Suppose you are given the initial value problem
\[ y'' + 16y = g(t); \quad y(0) = 0, \quad y'(0) = 0, \]
where \( g(t) = \begin{cases} 0, & 0 \leq t < 3 \\ \frac{t-3}{6}, & 3 \leq t < 9 \\ 1, & t \geq 9 \end{cases} \).

   (a) Show that \( g(t) \) can be written as
\[ g(t) = \frac{u_3(t) \cdot (t-3) - u_9(t) \cdot (t-9)}{6}. \]

   (b) Use the Laplace transform to solve the initial value problem. **(Hint:** It may be helpful to use Example 2 from the book as a reference.)

**Section 6.5**

8. (Problem 1a from B&D) Find the solution of the initial value problem
\[ y'' + 2y' + 2y = \delta(t - \pi); \quad y(0) = 1, \quad y'(0) = 0. \]

9. Consider the initial value problem
\[ y'' + y = \delta(t - 2\pi) \cos(t); \quad y(0) = 0, \quad y'(0) = 1. \]

   (a) Explain why
\[ \mathcal{L}\{\delta(t - 2\pi) \cos(t)\} = \int_0^\infty \delta(t - 2\pi) \cos(t)e^{-st}dt = \int_{-\infty}^\infty \delta(t - 2\pi) \cos(t)e^{-st}dt. \]
(b) Recall that we proved in class that
\[ \int_{-\infty}^{\infty} \delta(t - 2\pi)f(t)dt = f(t_0). \]

Use this fact to calculate \( \mathcal{L}\{\delta(t - 2\pi)\cos(t)\} \).

(c) Use your answer from part (b) to solve the initial value problem using the Laplace transform.

Section 6.6

10. (Problem 4-7 from B&D) For each of the following, find the Laplace transform of the given function using Theorem 6.6.1.

(a) \( f(t) = \int_{0}^{t} (t - \tau)^2 \cos(2\tau)d\tau \)

(b) \( f(t) = \int_{0}^{t} e^{-(t-\tau)} \sin(\tau)d\tau \)

(c) \( f(t) = \int_{0}^{t} (t - \tau)e^{\tau}d\tau \)

(d) \( f(t) = \int_{0}^{t} \sin(t - \tau) \cos(\tau)d\tau \)

11. (Problem 8-11 from B&D) For each of the following, find the inverse Laplace transform of the given function using Theorem 6.6.1. You may leave your answers as unevaluated integrals.

(a) \( F(s) = \frac{1}{s^4(s^2+1)} \)

(b) \( F(s) = \frac{s}{(s+1)(s^2+4)} \)

(c) \( F(s) = \frac{1}{(s+1)^2(s^2+4)} \)

(d) \( F(s) = \frac{G(s)}{s^2+1} \) where \( G(s) = \mathcal{L}\{g(t)\} \).

Section 5.1

12. (B&D 1, 3, 8) Determine the radius of convergence of the following power series.

(a) \( \sum_{n=0}^{\infty} (x - 3)^n \)

(b) \( \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \)
\[ \sum_{n=1}^{\infty} \frac{n!x^n}{n^n} \] \[ \text{Hint: } \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \]

13. (B&D 21, 24, 27) Rewrite each of the following expressions as a sum whose generic term involves \( x^n \).

(a) \[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

(b) \[ (1 - x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \]

(c) \[ x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \]

**Section 5.2**

14. (B&D 1, 3, 9, 12) In each of the following problems

(a) Seek power series solutions to the given differential equation about the given point \( x_0 \); find the recurrence relation.

(b) Find the first four terms in each of two solutions \( y_1 \) and \( y_2 \) (your solution from part (a) should be in the form \( a_0 y_1 + a_1 y_2 \)).

(c) By evaluating the Wronskian \( W(y_1, y_2)(x_0) \), show that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

(d) If possible, find the general term in each solution.

i. \( y'' - y = 0, \quad x_0 = 0 \)

ii. \( y'' - xy' - y = 0, \quad x_0 = 1 \)

iii. \( (1 + x^2)y'' - 4xy' + 6y = 0, \quad x_0 = 0 \)

iv. \( (1 - x)y'' + xy' - y = 0, \quad x_0 = 0 \)

15. **Optional** Consider the following equation.

\[ (1 + x)y'' = \sin(x), \quad x_0 = 0 \]

(a) Find a power series solution to the differential equation by giving a recurrence relation on the coefficients and give the first six terms of the series.

(b) What is the general solution to the differential equation?

16. (B&D 4, 10) In each of the following problems
(a) Seek power series solutions to the given differential equation about the given point \( x_0 \); find the recurrence relation.

(b) Find the first four terms in each of two solutions \( y_1 \) and \( y_2 \) (your solution from part (a) should be in the form \( a_0 y_1 + a_1 y_2 \)).

(c) By evaluating the Wronskian \( W(y_1, y_2)(x_0) \), show that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

(d) Find the general term in each solution.

   i. \( y'' + k^2 x^2 y = 0, \quad x_0 = 0, \quad k \) a constant
   
   ii. \( (4 - x^2)y'' + 2y = 0, \quad x_0 = 0 \)

17. (B&D 22) Consider the initial value problem \( y' = \sqrt{1 - y^2}, \ y(0) = 0 \).

   (a) Show that \( y = \sin(x) \) is a solution to this initial value problem.

   (b) Look for a solution of this initial value problem in the form of a power series about \( x = 0 \). Find the coefficients up to the \( x^3 \) term.

   **Hints**
   
   i. Square both sides
   
   ii. In order to find the first four coefficients (i.e. \( a_0, a_1, a_2, a_3 \)) you only need to consider the partial sums of each side up to the \( x^2 \) term
   
   iii. Use the Cauchy product
   
   iv. If you pay close attention you’ll see that you need to assume that \( a_1 > 0 \)

**Section 5.3**

18. (B&D 4) Determine \( \phi''(x_0), \ \phi'''(x_0), \ \phi^{(4)}(x_0) \) for the given point \( x_0 \) if \( y = \phi(x) \) is a solution of the given initial value problem.

\[ y'' + \sin(x)y' + \cos(x)y = 0; \quad y(0) = 0, \quad y'(0) = 1 \]

19. (B&D 5-7) In each of the following problems determine a lower bound for the radius of convergence of series solutions about each given point \( x_0 \) for the given differential equation.

   (a) \( y'' + 4y' + 6xy; \quad x_0 = 0, \quad x_0 = 4 \)
   
   (b) \( (x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4 \quad x_0 = 0 \)
   
   (c) \( (1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2 \)

20. (B&D 11) Find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

\[ y'' + \sin(x)y = 0 \]