Section 1.1

1. Given both of the equations $y' = 4 - 2y$ and $y' = 3y - 3$, draw a direction field for the differential equation. Based on the direction field, determine the behavior of $y$ as $t \to \infty$. Does this behavior depend on the initial value of at $t=0$? If so, how?

Solution:

In this case $y$ approaches 2 as $t$ approaches infinity. This is independent of the initial value of $y(0)$.

Here there are three cases. If $y(0) = 1$ then $y$ is constant and $y$ remains constant as $t$ approaches infinity. If $y(0) < 1$ then $y \to -\infty$ as $t \to \infty$. If $y(0) > 1$ then $y \to \infty$ as $t \to \infty$.
2. (a) Give two different differential equations of the form \( \frac{dy}{dt} = ay + b \) such that all solutions approach \( y = 5 \).

Solution: The general solution to \( \frac{dy}{dt} = ay + b \) is \( y = ce^{at} - \frac{b}{a} \). It follows that if \( a < 0 \) and \( \frac{b}{a} = -5 \) then the solution will behave as specified in the question. Two examples are \( \frac{dy}{dt} = -y + 5 \) and \( \frac{dy}{dt} = -2y + 10 \).

(b) Give two differential equation of the form \( \frac{dy}{dt} = ay + b \) such that all solutions diverge from \( y = \frac{1}{2} \).

Solution: By similar logic to the above if \( a > 0 \) and \( \frac{b}{a} = -\frac{1}{2} \) the solution will satisfy the above conditions. Two examples are \( \frac{dy}{dt} = 2y - 1 \) and \( \frac{dy}{dt} = 4y - 2 \).

3. Problem 15-20 from section 1.1 of Boyce and Diprima.

Solution: Found in the back of B&D.

4. Optional Hard Problem (Problem 22 from B&D) A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution: The equation for the volume of a sphere is \( V = \frac{4}{3}\pi r^3 \) and the equation for the surface area is \( S = 4\pi r^2 \). Solving the volume equation for \( r \) we have \( r = \left(\frac{3V}{4\pi}\right)^{\frac{2}{3}} \). Plugging into the equation for Surface area we have \( S = \pi^{\frac{1}{2}}6^{\frac{2}{3}}V^{\frac{2}{3}} \). We are given that \( \frac{dV}{dt} = -kS \) where the negative sign is used (really as a notational device) because the volume is decreasing and has a negative derivative with respect to time. Putting it all together yields \( \frac{dV}{dt} = -\tilde{k}V^{\frac{2}{3}} \). Here \( \tilde{k} = \pi^{\frac{1}{2}}6^{\frac{2}{3}}k \). It is common to continually absorb all constants into one undetermined constant term, and often we don’t even bother to rename the constants as we do this. So the final solution is \( \frac{dV}{dt} = -kV^{\frac{2}{3}} \).
5. (Problem 3 from B&D) Consider the differential equation \( \frac{dy}{dt} = -ay + b \) where both \( a \) and \( b \) are positive numbers.

(a) Solve the differential equation.

**Solution:** Some algebraic manipulation gives \( \frac{1}{y - \frac{b}{a}} \frac{dy}{dt} = -a \). We can then integrate both sides to get \( \ln(|y - \frac{b}{a}|) = -at + c \) and \( y - \frac{b}{a} = \pm e^{-at+c} \). The final solution is then \( y = ce^{-at} + \frac{b}{a} \) where \( c \) can be any real number.

Notice that the relative value of the constant \( c \) changed throughout the solution process, and the reason we don’t take care to keep track of the exact value is the following. Given an initial condition the final answer will come out to be the same regardless of how \( c \) is represented. It is worth mentioning, though, why \( c \) can be any real number. Notice first that \( e^x \) ranges over all positive real numbers so \( \pm e^x \) ranges over all nonzero real numbers. Where do we get 0 as a possibility in our final solution? When \( c = 0 \) we have the equilibrium solution, which (think about why this happened) we lost during the solving process.

(b) Sketch the solution for several different initial conditions.

**Solution:** Here are two variations plotted.
(c) Describe how the solutions change under each of the following conditions:

i. $a$ increases

**Solution:** If $a$ increases the value that each solution converges to decreases and the rate at which the solutions converge increases.

ii. $b$ increases

**Solution:** If $b$ increases the value that each solution converges to increases.

iii. Both $a$ and $b$ increase, but the ration $\frac{b}{a}$ remains the same.

**Solution:** If both $a$ and $b$ increase, but the ration $\frac{b}{a}$ remains the same then the rate at which the solutions converge increases.

6. (Problem 7 from B&D) Suppose a field mouse population satisfies the differential equation $\frac{dp}{dt} = 0.5p - 450$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.

**Solution:** As mentioned above, the general solution to $\frac{dy}{dt} = ay + b$ is $y = ce^{at} - \frac{b}{a}$. Therefore, the solution in this case is $p(t) = ce^{-5t} + 900$. Plugging the initial condition into the equation yields $850 = c + 900$ and $c = -50$. So we want to solve $p(s) = 0$ for $s$. That is $0 = -50e^{-5t} + 900$ which happens when $t = 2 \ln(18) \approx 5.780743516$ months.

(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

**Solution:** Plugging the initial condition into $p(t) = ce^{-5t} + 900$ yields $p_0 = c + 900$ and $c = p_0 - 900$. In order to find the time of extinction, we need to solve $P(T) = 0$ for $T$. That is $0 = (p_0 - 900)e^{-5T} + 900$ and $T = 2 \ln \left( \frac{-900}{p_0 - 900} \right)$. Note that the initial restriction on $p_0$ is necessary here so that the term inside the logarithm is positive.

(c) Find the initial population $p_0$ if the population is to become extinct in 1 year.
Solution: For this we need to solve \( P(12) = 0 \) for \( p_0 \). Here we are assuming that \( t \) is in months, but if you assumed that \( t \) was in years or days or scores that’s fine because we didn’t specify. We then have \( 0 = (p_0 - 900)e^6 + 900 \) and \( p_0 = 900 - \frac{900}{e^6} \approx 897.7691230 \)

7. (Problem 8 from B&D) Consider a population \( p \) of field mice that rows at a rate proportional to the current population, so that \( \frac{dp}{dt} = rp \)

(a) Find the rate constant \( r \) if the population doubles in 30 days.

Solution: Solving the differential equation gives \( p(t) = ce^{rt} \). Assuming \( t \) is in days we are trying to find \( r \) so that \( p(30) = 2p(0) \). That is \( ce^{30r} = 2ce^{0r} = c2 \) and \( r = \frac{\ln(2)}{30} \approx 0.03465735903 \)

(b) Find \( r \) if the population doubles in \( N \) days.

Solution: Following the same steps as above you should find the solution to be \( \frac{\ln(2)}{N} \).

Section 1.3

8. For each of the following, determine the order of the differential equation and whether the equation is linear or nonlinear.

(a) \( t^3 \frac{d^2y}{dt^2} + (t + 1) \frac{dy}{dt} + 5y = \sin(t) \)

Solution: Linear, order 2

(b) \( y \frac{d^3y}{dt^3} + \frac{dy}{dt} = t^2 \)

Solution: Nonlinear, order 3

(c) \( \frac{dy}{dt} + \sin(y) = 0 \)

Solution: Nonlinear, order 1
9. (Problem 9-10 from B&D) For each of the following, verify that the given function is a solution of the differential equation.

(a) \( t y' - y = t^2 \); \( y = 3t + t^2 \)

**Solution:** Noting that \( y' = 3 + 2t \) and plugging in, we can verify that \( t y' - y = t(3 + 2t) - (3t + t^2) = t^2 \)

(b) \( y'''' + 4y''' + 3y = t \); \( y_1(t) = \frac{t}{3}, \; y_2(t) = e^{-t} + \frac{t}{3} \)

**Solution:** Notice the following.
\[
y_1(t) = \frac{t}{3} \\
y_1'(t) = \frac{1}{3} \\
y_1''(t) = y_1'''(t) = y_1''''(t) = 0
\]
From the above it is clear that \( y_1 \) is a solution. Now let \( y = e^{-t} \) and notice that \( y'''' + 4y''' + 3y = 0 \). By linearity of differentiation we have that \( e^{-t} + \frac{t}{3} \) is also a solution to the above differential equation.

10. (Problem 17 from B&D) Determine the values of \( r \) for which \( y'' + y' - 6y \) has solutions of the form \( y = e^{rt} \).

**Solution:** Plugging \( e^{rt} \) into the equation, we need \( r \) so that \( r^2e^{rt} + re^{rt} - 6e^{rt} = 0 \). Because \( e^{rt} \) is never zero we can divide through by it and are left with \( r^2 + r - 6 = 0 \). It follows that \( r = -3 \) and \( r = 2 \) are the only solutions that work.

11. For each part below, give an example of an ODE satisfying the stated conditions:

(a) 2nd order linear equation
(b) 5th order linear equation involving a trig function
(c) 3rd order nonlinear equation involving a trig function.
(d) 4th order nonlinear equation not involving a trig function
Solution: Many correct solutions exist but here are a few:

(a) \( y'' + y' + 5y = 6 \)
(b) \( \frac{d^2y}{dt^2} + \sin(t) \cdot \frac{dy}{dt} + y = t^2 \)
(c) \( y'' + ty' = \sin(y) \)
(d) \( \frac{d^4y}{dt^4} + y \cdot \frac{d^2y}{dt^2} + y' = 0 \)

Section 2.1
Each answer must be supported by a sufficient amount of work. Because many of the answers to these questions are in the back of your textbook, answers without work will not receive any credit.

12. Consider the differential equation

\[ \mu'(t) = \mu(t)p(t) \]

where \( p(t) \) is any integrable function and \( \mu(t) > 0 \) for all \( t \in \mathbb{R} \).

(a) Solve the differential equation by first dividing both sides by \( \mu(t) \) (it is given that it is never 0), then integrating both sides, and finally exponentiating both sides.

Solution: Following the steps outlined above we have

\[ \frac{\mu'(t)}{\mu(t)} = p(t). \]

Integrating yields

\[ \ln(\mu(t)) = \int p(t) \, dt. \]

We do not need the absolute value within the logarithm because \( \mu(t) \) is positive everywhere. Exponentiating gives

\[ \mu(t) = e^{\int p(t) \, dt} \]

and completes the solution process.
(b) Determine \( \int \mu(t)y'(t) + \mu(t)p(t)y(t) \, dt \). (Hint – what is \( \mu(t)p(t) \) equal to?)

**Solution:** Noticing that \( \mu(t)p(t) = \mu'(t) \) we have

\[
\int \mu(t)y'(t) + \mu'(t)y(t) \, dt.
\]

Considering the product rule for differentiating, we have that

\[
\int \mu(t)y'(t) + \mu'(t)y(t) \, dt = \mu(t)y(t).
\]

(c) Explain how the above two problems relate to solving first order linear differential equations.

**Solution:** The above two problems outline the derivation of the integrating factor and illustrate how multiplication by the integrating factor makes the left hand side \( (y' + p(x)y) \) of linear first order differential equations (with a leading coefficient of 1) integrable. Furthermore, part b gives the integral explicitly.

13. (problem 5 from B&D) Consider the following differential equation (assume \( y \) is a function of \( t \)).

\[
y' - 2y = 3e^t
\]  

(1)

(a) Draw a direction field for equation (1). If you choose, you may have a computer do this (using Maple, Matlab, Mathematica, Wolfram Alpha, etc.).

**Solution:**
(b) Based on an inspection of the direction field, describe how solutions behave for large $t$.

**Solution:** From the direction field, it appears that $\lim_{t \to \infty} y(t) = \pm \infty$ depending on the initial condition.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t$ approaches infinity.

**Solution:** The integrating factor $\mu(t)$ is

$$e^{\int -2 \, dt} = e^{-2t}$$
(remember that we can let the constant of integration be zero when finding the integrating factor). Multiplying both sides of the differential equation by $\mu(t)$ we have

$$e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$$

Integrating both sides yields

$$e^{-2t}y = -3e^{-t} + c.$$ 

Remember that the integral of the left hand side is always $\mu(t)y$. The final answer is then

$$y(t) = -3e^t + ce^{2t}.$$ 

The limit as $t$ tends to infinity is $\pm\infty$ depending on the sign of $c$.

14. (from problem 6 in B&D) Find the general solution of the following differential equation:

$$ty' + 2y = \sin(t), \quad t > 0.$$ 

**Solution:** Dividing through by $t$ we have

$$y' + \frac{2y}{t} = \frac{\sin(t)}{t}.$$ 

We can divide through by $t$ as it is nonzero on the region $(0, \infty)$. Our integrating factor, $\mu(t)$, is then

$$e^{\int \frac{2}{t} \, dt} = e^{\ln(t^2)} = t^2.$$ 

We need not worry about the absolute value as $t > 0$. Integrating both sides, we have

$$t^2y = \int t \sin(t) \, dt = -t \cos(t) - \int -\cos(t) \, dt = \sin(t) - t \cos(t) + c.$$ 

Dividing through by $\mu(t)$ gives a final answer of

$$\frac{\sin(t)}{t^2} - \frac{\cos(t)}{t} + \frac{c}{t^2}; \quad t > 0$$
15. Find the general solution to the following differential equation:
\[ y'\sqrt{1 + t^2} + y = 1 \]

Here are some useful hints and identities for this one.

(a) You will probably need to use trig substitution (with \( t = \tan(u) \)).

(b) \( \int \sec(t) \, dt = \ln(|\sec(t) + \tan(t)|) + C \)

(c) Your final answer should not include the absolute value of anything (be sure to justify why).

**Solution:** Dividing through by \( \sqrt{1 + t^2} \) gives

\[ y' + \frac{y}{\sqrt{1 + t^2}} = \frac{1}{\sqrt{1 + t^2}} \]

Our integrating factor, \( \mu(t) \), is then

\[ \exp\left( \int \frac{1}{\sqrt{1 + t^2}} \, dt \right) \]

where \( \exp(x) = e^x \). We then need to find \( \int \frac{1}{\sqrt{1 + t^2}} \, dt \). Start by letting \( u = \arctan(x) \). Then \( \tan(u) = x \) and \( \sec^2(u) \, du = dx \). It follows that

\[ \int \frac{1}{\sqrt{1 + t^2}} \, dt = \int \frac{\sec^2(u)}{\sqrt{1 + \tan^2(u)}} \, du = \int \sec(u) \, du. \]

Integrating \( \sec(u) \) is such a fun problem that it has its own Wikipedia page. For more fun readers are referred to the Wikipedia page for the integral of \( \sec^3(x) \).

\[ \int \sec(u) \, du = \int \frac{1}{\cos(u)} \, du = \int \frac{\cos(u)}{\cos^2(u)} \, du = \int \frac{\cos(u)}{1 - \sin^2(u)} \, du \]

Making the substitution \( v = \sin(u) \) we have \( dv = \cos(u) \, du \) and

\[ \int \frac{\cos(u)}{1 - \sin^2(u)} \, du = \int \frac{1}{1 - v^2} \, dv = \frac{1}{2} \int \frac{1}{1 - v} + \frac{1}{1 + v} \, dv = \frac{1}{2}(-\ln|1-v| + \ln|1+v|) \]

\[ = \frac{1}{2} \ln \left| \frac{1 + v}{1 - v} \right| = \frac{1}{2} \ln \left| \frac{1 + \sin(u)}{1 - \sin(u)} \right| = \frac{1}{2} \ln \left| \frac{(1 + \sin(u))^2}{1 - \sin^2(u)} \right| = \ln \left| \frac{1 + \sin(u)}{\cos(u)} \right| \]

\[ = \ln |\sec(u) + \tan(u)| = \ln \left| \sqrt{1 + \tan^2(u)} + \tan(u) \right| = \ln \left( \sqrt{1 + t^2} + t \right). \]
We can ignore the absolute value here as $\sqrt{1+t^2} + t$ is positive everywhere. It follows that our integrating factor is

$$\mu(t) = \exp(\ln(\sqrt{1+t^2} + t)) = \sqrt{1+t^2} + t$$

Integrating both sides of $\mu(t)y'(t) + \frac{\mu(t)y(t)}{\sqrt{1+t^2}} = \frac{\mu(t)}{\sqrt{1+t^2}}$ we have

$$(\sqrt{1+t^2} + t)y = \int \frac{\sqrt{1+t^2} + t}{\sqrt{1+t^2}} \, dt = \int 1 \, dt + \int \frac{t}{\sqrt{1+t^2}} \, dt = t + \sqrt{1+t^2} + c.$$ To integrate $\int \frac{t}{\sqrt{1+t^2}} \, dt$ we used the substitution $u = 1+t^2$. Dividing through by $\mu(t)$ gives a final answer of

$$y(t) = 1 + \frac{c}{t + \sqrt{1+t^2}}.$$

16. (B&D #17) Find the solution to this initial value problem.

$$y' - 2y = e^{2t}, \quad y(0) = 2$$

**Solution:** Like problem 2, the integrating factor $\mu(t)$ is $e^{-2t}$. Dividing through by $\mu(t)$ and integrating we have

$$y e^{-2t} = \int 1 \, dt = t + c.$$ The general solution is then $y(t) = te^{2t} + ce^{2t}$. With the initial condition we have

$$y(0) = 2 = 0e^{2\cdot0} + ce^{2\cdot0} = c.$$ The final answer is then

$$y(t) = (t + 2)e^{2t}$$
17. (a) (B&D #20) Find the solution to this initial value problem.

\[ ty' + (t + 1)y = t, \quad y(\ln(2)) = 1, \quad t > 0 \]

\textbf{Solution:} Dividing through by \( t \) leaves us with

\[ y' + \frac{t + 1}{t} y = 1. \]

The integrating factor \( \mu(t) \) is then

\[ \mu(t) = \exp \left( \int \frac{t + 1}{t} \, dt \right) = \exp \left( \int 1 + \frac{1}{t} \, dt \right) = te^t. \]

Dividing through by our integrating factor and integrating both sides yields

\[ te^t y(t) = \int te^t \, dt = te^t - \int e^t \, dt = (t - 1)e^t + c. \]

The general solution is then

\[ y(t) = \frac{t - 1}{t} + \frac{c}{te^t}. \]

Plugging in for the initial condition gives

\[ y(\ln(2)) = 1 = \frac{\ln(2) - 1}{\ln(2)} + \frac{c}{\ln(2)e^{\ln(2)}} = \frac{2\ln(2) - 2 + c}{2\ln(2)}, \]

implying that \( 2\ln(2) = 2\ln(2) - 2 + c \) and \( c = 2 \). The final solution then becomes

\[ \frac{t - 1}{t} + \frac{2}{te^t}. \]

(b) Briefly explain why the condition \( t > 0 \) was given.

\textbf{Solution:} This condition was given as \((0, \infty)\) is the largest open interval containing \( \ln(2) \) (where the initial value was specified) on which \( t \) (the leading coefficient) is nonzero and \( \frac{t + 1}{t} \) is continuous. We also use the fact that \( t \) was positive in the above solution when we neglected to use absolute values when integrating \( \frac{1}{t} \).
18. **Optional Hard Problem** (B&G #30) Find the value of $y_0$ for which the solution of the initial value problem

$$y' - y = 1 + 3 \sin(t), \quad y(0) = y_0$$

remains finite as $t \to \infty$.

**Solution:** We will start by solving the differential equation. The integrating factor is $\mu(t) = e^{-t}$. Multiplying through by $\mu(t)$ and integrating both sides yields

$$ye^{-t} = \int e^{-t} + 3e^{-t} \sin(t) \, dt = -e^{-t} + 3 \int e^{-t} \sin(t) \, dt$$

Notice the following.

$$\int e^{-t} \sin(t) \, dt = -e^{-t} \sin(t) - \int -e^{-t} \cos(t) \, dt = -e^{-t} \sin(t) - e^{-t} \cos(t) - \int e^{-t} \sin(t) \, dt$$

implying that

$$2 \int e^{-t} \sin(t) = -e^{-t} (\sin(t) + \cos(t)) + c$$

and

$$\int e^{-t} \sin(t) = -\frac{1}{2} e^{-t} (\sin(t) + \cos(t)) + c.$$ 

It follows that

$$ye^{-t} = -e^{-t} - \frac{3}{2} e^{-t} (\sin(t) + \cos(t)) + c$$

and

$$y(t) = -1 - \frac{3}{2} (\sin(t) + \cos(t)) + ce^t.$$ 

Plugging in the initial condition $y(0) = y_0$ we have

$$y_0 = -1 - \frac{3}{2} + c$$

implying that

$$c = y_0 + \frac{5}{2}.$$ 

In order for $-1 - \frac{3}{2} (\sin(t) + \cos(t)) + ce^t$ to stay finite as $t \to \infty$, $c$ must be zero. It follows that the value of $y_0$ that we are looking for is $-\frac{5}{2}$. 

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19. (Problem 8c and 11c from B&D) For each of the following, solve the differential equations determine its behavior as $t \to \infty$.

(a) $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
(b) $y' + y = 5 \sin(2t)$

Solution:

(a) We begin by dividing everything by $1 + t^2$ which gives us the equation

$$y' + \frac{4t}{1 + t^2}y = (1 + t^2)^{-3}$$

(2)

Note that we don’t have to worry about the domain of $t$ since $1 + t^2$ is never zero. Next, we need to find an integrating factor $\mu(t) = e^{\int \frac{4t}{1 + t^2}dt}$. To solve $\int \frac{4t}{1 + t^2}dt$, use $u$-substitution with $u = 1 + t^2$ and $du = 2tdt$ to write the integral as $\int \frac{2u}{u}du = 2 \ln|u| + C$. Thus $\mu(t) = e^{2 \ln|u|+C} = C_1(u^{\ln|u|})^2 = C_1|u|^2$ which becomes $(1 + t^2)^2$ by choosing $C_1 = 0$ and noting that $|u^2| = u^2$ for any $u$. Then equation (1) becomes

$$(1 + t^2)^2y' + 4t(1 + t^2)y = \frac{1}{1 + t^2}$$

(3)

Integrating both sides gives

$$(1 + t^2)^2y = \int \frac{1}{1 + t^2} = \arctan(t) + C$$

(4)

so we have

$$y = \frac{\arctan(t) + C}{(1 + t^2)^2}$$

(5)

Finally, we need to consider the behavior of $y$ as $t \to \infty$. However, we know that $-\frac{\pi}{2} < \arctan(t) < \frac{\pi}{2}$ so by the Squeeze Theorem, we see

$$\lim_{t \to \infty} \frac{-\frac{\pi}{2} + C}{(1 + t^2)^2} \leq \lim_{t \to \infty} \frac{\arctan(t) + C}{(1 + t^2)^2} \leq \lim_{t \to \infty} \frac{\frac{\pi}{2} + C}{(1 + t^2)^2}$$

$$\implies 0 \leq \lim_{t \to \infty} \frac{\arctan(t) + C}{(1 + t^2)^2} \leq 0$$

$$\implies \lim_{t \to \infty} \frac{\arctan(t) + C}{(1 + t^2)^2} = 0$$

Thus, regardless of initial conditions, $y \to 0$ as $t \to \infty$. 

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(b) For \( y' + y = 5 \sin(2t) \), our integrating factor is just \( e^t \) so we need to find the integral \( \int 5 \sin(2t) e^t \, dt \). We use integration by parts to see
\[
\int 5 \sin(2t) e^t \, dt = 5 \sin(2t) e^t - \int 10 \cos(2t) e^t \, dt
\]
Using integration by parts again on \( \int 10 \cos(2t) e^t \, dt \) we get
\[
\int 10 \cos(2t) e^t \, dt = 10 \cos(2t) e^t - \int -20 \sin(2t) e^t \, dt
\]
Thus we finally have
\[
\int 5 \sin(2t) e^t \, dt = 5 \sin(2t) e^t - 10 \cos(2t) e^t + C
\]
\[
\Rightarrow \int 25 \sin(2t) e^t \, dt = 5 \sin(2t) e^t - 10 \cos(2t) e^t + C
\]
\[
\Rightarrow \int 5 \sin(2t) e^t \, dt = \sin(2t) e^t - 2 \cos(2t) e^t + C
\]
Therefore we have
\[
e^t y = \int 5 \sin(2t) e^t \, dt = \sin(2t) e^t - 2 \cos(2t) e^t + C
\]
\[
\Rightarrow y = \sin(2t) - 2 \cos(2t) + C e^{-t}
\]
As \( t \) gets large, \( y(t) \) approaches \( \sin(2t) - 2 \cos(2t) \) since \( \lim_{t \to \infty} C e^{-t} = 0 \).

Section 2.2

20. (B&D #1) Solve the following differential equation.
\[
y' = \frac{x^2}{2}
\]

**Solution:** The above contains a typo and is (as a result) a Calc I problem. It should read
\[
y' = \frac{x^2}{y},
\]
in which case we have

\[ y' = x^2. \]

Integrating both sides yields

\[ \frac{1}{2} y^2 = \frac{1}{3} x^3 + c. \]

Cleaning up a bit and remembering that \( y \) cannot be zero, we have a final solution of

\[ 3y^2 - 2x^3 = c; \quad y \neq 0. \]

21. (B&D #5) Solve the following differential equation.

\[ y' = (\cos^2(x)) (\cos^2(2y)) \]

**Solution:** The natural first step in the solution process is to divide through by \( \cos^2(2y) \), but first we have to consider the case where \( \cos^2(2y) = 0 \). This happens when \( y = \frac{\pi}{4} + k\frac{\pi}{2} \). In this case we have a constant solution.

Now assume that \( \cos(2y) \neq 0 \). Rearranging the separable equation yields

\[ \sec^2(2y)y' = \cos^2(x). \]

The integral of the left hand side is \( \frac{1}{2} \tan(2y) \). The integral of the right hand side requires just a bit of work.

\[
\int \cos^2(x) \, dx = \int \cos^2(x) + \frac{1}{2} \, dx = \int \cos^2(x) + \frac{1}{2} - \frac{1}{2} \sin^2(x) + \cos^2(x) \, dx \\
= \int \frac{1}{2} \cos^2(x) - \sin^2(x) + \frac{1}{2} \, dx = \frac{1}{2} \int \cos(2x) + 1 \, dx = \frac{1}{4} \sin(2x) + \frac{1}{2} x + c.
\]

Equating the two sides and making the solution look all nice and tidy we have

\[ 2 \tan(2x) - \sin(2x) - 2x = c; \quad y \neq \frac{\pi}{4} + k\frac{\pi}{2} \]

\[ y = \frac{\pi}{4} + k\frac{\pi}{2}; \quad k \in \mathbb{Z}. \]

22. (B&D #9)

(a) Find the solution of the following initial value problem in explicit form.

\[ y' = (1 - 2x)y^2, \quad y(0) = \frac{1}{6} \]
Solution: Note first that we have an equilibrium solution \( y(x) = 0 \). Assuming then that \( y \neq 0 \) we can rearrange to yield
\[
\frac{y'}{y^2} = 1 - 2x.
\]
Integrating both sides we have
\[
\frac{-1}{y} = x - x^2 + c
\]
and
\[
y = \frac{1}{x^2 - x + c}.
\]
Plugging in the initial value we have that \( c = 6 \), so the final answer is
\[
y = \frac{1}{x^2 - x + 6}.
\]

(b) Determine the interval in which the solution is defined.

Solution: Note that this problem differs slightly from the one in the book (in the book the initial value given is \( y(0) = -\frac{1}{6} \)). In this case, the solution is defined for all \( x \in \mathbb{R} \) as \( x^2 - x + 6 \) has no real roots. It is worthwhile to consider the problem as it is given in the book.

23. (B&D #20)

(a) Find the solution of the following initial value problem in explicit form.

\[
y^2(1 - x^2)^{\frac{1}{2}} \frac{dy}{dx} = \arcsin(x), \quad y(0) = 1
\]

Solution: Assuming \( 1 - x^2 > 0 \), we have
\[
y^2 y' = \frac{\arcsin(x)}{\sqrt{1 - x^2}}.
\]
Integrating both sides (using substitution with \( u = \arcsin(x) \) on the left hand side) yields

\[
\frac{1}{3}y^3 = \frac{1}{2} \arcsin^2(x) + c.
\]

Plugging in our initial value gives that \( c = \frac{1}{3} \) and our final solution becomes

\[
y = \sqrt[3]{3} \arcsin(x) + 1.
\]

(b) Determine the interval in which the solution is defined.

**Solution:** The domain of the differential equation was initially restricted by the \( \arcsin(x) \) on the right hand side and the radical on the right. We are further restricted because \( \frac{dy}{dx} = 0 \) when \( x = \pm 1 \) (consider why this is). Hence the interval in which the solution is defined is \((-1, 1)\).

24. (B&D #32)

(a) Show that the following differential equation is homogeneous.

\[
\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}
\]

**Solution:** Notice that

\[
\frac{x^2 + 3y^2}{2xy} = \frac{1}{2} \left( \frac{x}{y} + 3 \frac{y}{x} \right) = \frac{1}{2} \left( \left( \frac{y}{x} \right)^{-1} + 3 \frac{y}{x} \right).
\]

(b) Solve the above differential equation.

**Solution:** Making the substitution \( v = \frac{y}{x} \) we have \( y = vx \) and \( \frac{dy}{dt} = x \frac{dv}{dt} + v \). The differential equation then becomes

\[
x \frac{dv}{dt} = \frac{1}{2} \left( v^{-1} + 3v \right) - v = \frac{1}{2} (v^{-1} + v) = \frac{1 + v^2}{2v}.
\]

Some rearrangement yields

\[
\frac{2v}{1 + v^2}v' = \frac{1}{x}
\]
and we can integrate both sides.

\[ \ln |1 + v^2| = \ln |x| + c. \]

Exponentiating both sides and noticing that \( 1 + v^2 > 0 \) for all \( v \in \mathbb{R} \) we have

\[ 1 + v^2 = c|x|; \quad c > 0 \]

and

\[ x^2 + y^2 = cx^2|x|. \]

What can we do about those ugly absolute values? Well the original differential equation had issues when \( x = 0 \) or \( y = 0 \). In particular, we will always need to specify an open interval not containing zero on which our solution is defined. On these intervals \( x \) will always have the same sign, so we can absorb it in \( c \). So the final solution becomes

\[ x^2 + y^2 = cx^3; \quad c > 0, y \neq 0. \]

25. (Problem 2 from B&D) Solve the separable differential equation \( y' = \frac{x^2}{y(1+x^3)} \)

**Solution:** We can separate this as

\[ ydy = \frac{x^2}{1 + x^3}dx \]

and immediately integrate each side. For the right side, just use \( u \)-substitution with \( u = 1 + x^3, du = 3x^2 \) to get

\[ \int \frac{x^2}{1 + x^3}dx = \int \frac{1}{3} \frac{du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |1 + x^3| + C \]

Then we have

\[ \frac{1}{2}y^2 = \frac{1}{3} \ln |1 + x^3| + C \]

\[ \implies \frac{1}{2}y^2 - \frac{1}{3} \ln |1 + x^3| = C \]

26. (Problem 34 from B&D) Show that the equation \( \frac{dy}{dx} = -\frac{4x+3y}{2x+y} \) is homogeneous and use this property to solve it.
Solution: We want to make the change of variables \( v = \frac{y}{x} \). Dividing the top and bottom of the right side by \( x \) and using the identity \( \frac{dy}{dx} = v + x \frac{dv}{dx} \) gives us

\[
v + x \frac{dv}{dx} = -\frac{4 + 3v}{2 + v}
\]

\[\Rightarrow \frac{dv}{dx} = -\frac{4 + 3v}{2 + v} - v
\]

\[\Rightarrow \frac{dv}{dx} = -\frac{(4 + 3v) + (2v + v^2)}{2 + v} = -\frac{4 + 5v + v^2}{2 + v}
\]

\[\Rightarrow -\frac{2 + v}{4 + 5v + v^2} \, dv = \frac{1}{x} \, dx
\]

However, we can factor \( 4 + 5v + v^2 = (v + 4)(v + 1) \) and use partial fractions decomposition. Then

\[
\frac{A}{v + 4} + \frac{B}{v + 1} = \frac{2 + v}{4 + 5v + v^2}
\]

so we get \( Av + A + Bv + 4B = 2 + v \) so \( Av + Bv = v \) and \( A + 4B = 2 \). Then solving the second equation for \( A \) we get \( A = 2 - 4B \) so \( (2 - 4B)v + Bv = v \) implying \( 2 - 3B = 1 \) which in turn implies \( 1 = 3B \). Thus \( B = \frac{1}{3} \) so \( A = 2 - 4(\frac{1}{3}) = \frac{2}{3} \). Therefore

\[
\frac{2}{3(v + 4)} + \frac{1}{3(v + 1)} = \frac{2 + v}{4 + 5v + v^2}
\]

giving us the differential equation

\[
\frac{-2}{3(v + 4)} + \frac{-1}{3(v + 1)} \, dv = \frac{1}{x} \, dx
\]

Integrating both sides gives

\[
-\frac{2}{3} \ln |v + 4| + -\frac{1}{3} \ln |v + 1| = \ln |x| + C
\]

Putting both sides as the exponent of \( e \), we can reduce to

\[
|(v + 4)^{\frac{2}{3}}||(v + 1)^{\frac{1}{3}}| = C_1 |x|
\]

We can then cube both sides to get

\[
|v + 4|^{-2} \cdot |(v + 1)|^{-1} = C_2 |x|^3
\]

\[\Rightarrow C_3 = |v + 4|^2 \cdot |(v + 1)||x|^3
\]

\[\Rightarrow C_3 = \left( |x|^2 |v + 4|^2 \right) \cdot \left( |x||v + 1| \right) = \left( |x(v + 4)|^2 \right) \cdot \left( |x(v + 1)| \right)
\]
Substituting $\frac{y}{x}$ back in for $v$ gives

$$C_3 = \left( |x(\frac{y}{x} + 4)|^2 \right) \cdot \left( |x(\frac{y}{x} + 1)| \right)$$

$$\implies C_3 = |y + 4x|^2 \cdot |y + x|$$

Section 2.3

27. (B&D #1) Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

Solution: Let $Q(t)$ denote the quantity of dye (in grams) in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$ 

There is no dye being added to the tank, so the rate in is zero. The rate out is $\frac{Q(t)}{100 \text{ min}}$. So we have

$$\frac{dQ}{dt} = -\frac{Q}{100}$$

which is both linear and separable. Solving the equation we have

$$Q(t) = ce^{-\frac{t}{100}}.$$ 

From the problem statement we have $Q(0) = 200$ so $c = 200$. Determining when the amount of dye in the tank is at 1% of its initial value amounts to solving

$$200e^{-\frac{t}{100}} = 2$$

for $t$. So the final answer is $t = 200 \ln(10) \text{ min} \approx 460.5170186 \text{ min}.$

28. (B&D #9) A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate $k$, determine the payment rate $k$ that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.
Solution: The differential equation modeling this situation is
\[
\frac{dD}{dt} - 1D = -k,
\]
where \( t \) is in years and \( D(t) \) is the debt at time \( t \) in dollars. We will solve the linear equation using integrating factor \( \mu(t) = e^{-1t} \). The general solution is then
\[
D(t) = ce^{\frac{t}{10}} + 10k.
\]
Plugging in the initial condition \( D(0) = 8000 \) we have \( c = 8000 - 10k \) and a solution of
\[
D(t) = (8000 - 10k)e^{\frac{t}{10}} + 10k.
\]
In order to determine the annual payment required to pay off the loan in 3 years, we solve for \( k \) with the condition \( D(3) = 0 \). We have
\[
0 = (8000 - 10k)e^{\frac{3}{10}} + 10k
\]
and
\[
8000e^{3} = 10k(e^{3} - 1).
\]
Our final solution is then
\[
k = \frac{800e^{3}}{e^{3} - 1} \approx \$3086.636728.
\]
The interest paid is \( 3k - 8000 \approx \$1259.910184 \).

29. (B&D #12) Radiocarbon dating is a method of dating organic material by determining the ratio of carbon-14 to other carbon isotopes present in the material. The half-life of carbon-14 is approximately 5730 years (i.e it takes 5730 years for half of the carbon-14 to decay). Let \( Q(t) \) be the amount of carbon-14 remaining at time \( t \).

(a) Assuming that \( Q \) satisfies the differential equation \( Q' = -rQ \), determine the decay constant \( r \) for carbon-14.

Solution: Solving the equation we have
\[
Q(t) = ce^{-rt}.
\]
From the problem statement we know \( Q(0) = 2Q(5730) \). Solving this we have
\[
2e^{-r5730} = 1
\]
and
\[ r = \frac{\ln(2)}{5730} \text{ yr}^{-1}. \]

(b) Find an expression for \( Q(t) \) at any time \( t \), if \( Q(0) = Q_0 \).

**Solution:** Plugging the initial condition in we have
\[ Q_0e^{\frac{\ln(2)}{5730}t}. \]

30. **Optional Hard Problem** Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surroundings. Use this law to answer the following question.

Two people sitting at a café order coffee. The first person orders creme with the coffee and promptly adds some when the coffee arrives. The second person decides, after a moment of letting the coffee cool, to order creme and adds some when the server brings it. Assuming that the coffees were 200° when they were first brought, the creme is always served at 40°, the temperature of the room is 70°, and both people add the same quantity of creme to the same quantity of coffee; whose coffee is hotter after person two adds creme? Note that the temperature of the coffee and creme mixture is
\[ T = \frac{T_oC_o + T_rC_r}{C_o + C_r} \]
where \( T_o \) is the temperature of the coffee, \( T_r \) is the temperature of the creme, \( C_o \) is the amount of coffee, and \( C_r \) is the amount of creme.

**Solution:** We begin by modeling the temperature of a cup of coffee with initial temperature \( T_0 \). From the problem statement we have that the temperature of the surroundings is 70°, so by Newton’s law of cooling,
\[ \frac{dT}{dt} = -k(T - 70) \]
where \( T(t) \) is the temperature of the coffee at time \( t \). Solving the equation we have
\[ T(t) = ce^{-kt} + 70; \quad c > 0. \]

Plugging in our initial condition yields
\[ T(t) = (T_0 - 70)e^{-kt} + 70. \]
We can then model the temperature of person 1’s coffee (after the creme is added) by the equation

\[ T_1(t) = \left( \frac{200C_o + 40C_r}{C_o + C_r} - 70 \right) e^{-kt} + 70. \]

With \( C_o \) and \( C_r \) as indicated in the problem statement. Pulling out \( \frac{1}{C_o + C_r} \) yields.

\[ T_1(t) = \frac{1}{C_o + C_r} \left[ (200C_o + 40C_r - 70(C_o + C_r)) e^{-kt} + 70(C_o + C_r) \right] \]

Collecting terms we have

\[ T_1(t) = \frac{1}{C_o + C_r} \left[ C_o \left( (200 - 70)e^{-kt} + 70 \right) + C_r \left( (40 - 70)e^{-kt} + 70 \right) \right]. \]

This indicates that mixing the creme and coffee and letting the mixture cool is equivalent (under our assumptions) to letting the creme heat up and the coffee cool and then mixing them together at time \( t \) (the above equation is the result of plugging the equations for temperature of coffee and creme at a given time with initial temperatures 200\(^\circ\)/40\(^\circ\) into the equation for the temperature of two fluids after mixing). It follows that person 1’s coffee will be warmer because his situation is equivalent to letting the creme heat up and then adding it, while person 2’s creme is kept cold.

31. (Problem 8 from B&D) A college student with no initial capital invests \( k \) dollars per year at an annual rate of return \( r \). Assume the investments are made continuously and that the return is compounded continuously. Then

(a) Determine the sum \( S(t) \) accumulated at any time \( t \)

(b) If \( r = 7.5\% \), determine \( k \) so that $1 million will be available for retirement in 40 years.

(c) If \( k = $2000 \) per year, determine the return rate \( r \) that must be obtained to have $1 million available in 40 years.

Solution:

(a) We see that \( \frac{dS}{dt} = k + rS = r(S + \frac{k}{r}) \) so we separate the equation and get

\[ \frac{dS}{S + \frac{k}{r}} = rdt \]
Integrating both sides gives

$$\ln |S + \frac{k}{r}| = rt + C \implies S + \frac{k}{r} = C_1 e^{rt}$$

$$\implies S = C_1 e^{rt} - \frac{k}{r}$$

Note that $S_0 = S(0) = 0$ so

$$0 = C_1 - \frac{k}{r} \implies C_1 = \frac{k}{r}$$

Thus

$$S(t) = \frac{k}{r} \left( e^{rt} - 1 \right)$$

(b) If $r = 7.5\% = 0.075$ and $S(40) = 1,000,000$ we have

$$S(40) = 1,000,000 = \frac{k}{0.075} \left( e^{0.075(40)} - 1 \right)$$

$$\implies (0.075)1,000,000 \frac{1}{e^{0.075(40)} - 1} = k$$

$$\implies k \approx \$3930$$

(c) If we now assume $k = \$2000$ and $S(40) = \$1,000,000$, then

$$1,000,000 = \frac{2,000}{r} \left( e^{40r} - 1 \right)$$

$$\implies 500r = e^{40r} - 1$$

Unfortunately, a computer is needed to solve this equation. We get $r = 0.0977 = 9.77\%$.

32. (Problem 5a,c from B&D) A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of \( \frac{1}{4}(1 + \frac{1}{2}\sin(t)) \) oz gal flows into the tank at a rate of \( \frac{2}{\text{min}} \) gal, and the mixture in the tank flows out at the same rate.

(a) Find the amount of salt $Q(t)$ in the tank at any time.

(b) Note that the long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?
Solution:

(a) As usual with tank problems, \( \frac{dQ}{dt} = \text{rate in} - \text{rate out} \). Here

"rate in" = 2 \cdot \frac{1}{4} (1 + \frac{1}{2} \sin(t)) = \left( \frac{1}{2} + \frac{1}{4} \sin(t) \right) \frac{oz}{min}

and

"rate out" = 2 \cdot \frac{Q(t)}{100} = \frac{Q(t)}{50} \frac{oz}{min}

Therefore we have that

\[
\frac{dQ}{dt} = \left( \frac{1}{2} + \frac{1}{4} \sin(t) \right) - \frac{Q(t)}{50}
\]

\[\Rightarrow \frac{dQ}{dt} + \frac{Q(t)}{50} = \left( \frac{1}{2} + \frac{1}{4} \sin(t) \right)\]

This is a first order linear equation so we can solve it with an integrating factor. Here \( \mu(t) = e^{\int \frac{1}{50} dt} = e^{\frac{t}{50}} \). Thus the differential equation becomes

\[
e^{\frac{t}{50}} \frac{dQ}{dt} + \frac{1}{50} e^{\frac{t}{50}} Q(t) = \frac{1}{2} e^{\frac{t}{50}} + \frac{1}{4} e^{\frac{t}{50}} \sin(t)
\]

Integrating both sides gives

\[
e^{\frac{t}{50}} Q(t) = \frac{1}{2} \int e^{\frac{t}{50}} dt + \frac{1}{4} \int e^{\frac{t}{50}} \sin(t) dt
\]

We note that \( \frac{1}{2} \int e^{\frac{t}{50}} dt = 25 e^{\frac{t}{50}} + C \), and proceed to use integration by parts to solve \( \frac{1}{4} \int e^{\frac{t}{50}} \sin(t) dt \).

\[
\int e^{\frac{t}{50}} \sin(t) dt = 50 e^{\frac{t}{50}} \sin(t) - 50 \int e^{\frac{t}{50}} \cos(t) dt
\]

Integration by parts again gives

\[
\int e^{\frac{t}{50}} \cos(t) dt = 50 e^{\frac{t}{50}} \cos(t) + 50 \int e^{\frac{t}{50}} \sin(t) dt
\]

so we finally have

\[
\int e^{\frac{t}{50}} \sin(t) dt = 50 e^{\frac{t}{50}} \sin(t) - (50)^2 e^{\frac{t}{50}} \cos(t) - (50)^2 \int e^{\frac{t}{50}} \sin(t) dt
\]

\[\Rightarrow \left( (50)^2 + 1 \right) \int e^{\frac{t}{50}} \sin(t) dt = 50 e^{\frac{t}{50}} \sin(t) dt - (50)^2 e^{\frac{t}{50}} \cos(t) + C\]
\[ \Rightarrow \int e^{\sqrt[50]{t}} \sin(t) dt = \frac{50}{(50)^2 + 1} e^{\sqrt[50]{t}} \sin(t) - \frac{50^2}{(50)^2 + 1} e^{\sqrt[50]{t}} \cos(t) + C \]

At last we can write

\[ e^{\sqrt[50]{t}} Q(t) = \frac{1}{2} \int e^{\sqrt[50]{t}} dt + \frac{1}{4} \int e^{\sqrt[50]{t}} \sin(t) dt \]

\[ e^{\sqrt[50]{t}} Q(t) = 25 e^{\sqrt[50]{t}} + \frac{50}{4(50)^3 + 1} e^{\sqrt[50]{t}} \sin(t) - \frac{50^2}{4(50)^3 + 1} e^{\sqrt[50]{t}} \cos(t) + C \]

\[ \Rightarrow Q(t) = 25 + \frac{50}{4(50)^2 + 1} \sin(t) - \frac{50^2}{4(50)^2 + 1} \cos(t) + Ce^{-\sqrt[50]{t}} \]

which we could reduce to

\[ Q(t) = 25 + \frac{25}{5002} \sin(t) - \frac{625}{2501} \cos(t) + Ce^{-\sqrt[50]{t}} \]

However, our initial conditions were \( Q(0) = 50 \) so

\[ 50 = 25 - \frac{625}{2501} + C \]

\[ \Rightarrow C = 25 - \frac{625}{2501} = \frac{63150}{2501} \]

so all told we have

\[ Q(t) = 25 + \frac{25}{5002} \sin(t) - \frac{625}{2501} \cos(t) + \frac{63150}{2501} e^{-\sqrt[50]{t}} \]

(b) As \( t \to \infty, \frac{63150}{2501} e^{-\sqrt[50]{t}} \to 0 \). Thus we oscillate around \( Q(t) = 25 \). To find the amplitude, we must find the max/min of

\[ f(t) = \frac{25}{5002} \sin(t) - \frac{625}{2501} \cos(t) \]

To do this, we take the derivative and set it equal to 0.

\[ f'(t) = \frac{25}{5002} \cos(t) + \frac{625}{2501} \sin(t) \]

\[ 0 = \frac{25}{5002} \cos(t) + \frac{625}{2501} \sin(t) \]
\[ -50 \sin(t) = \cos(t) \]
\[ \Rightarrow \tan(t) = -\frac{1}{50} \]

Thus \( f(t) \) has a min/max at \( \arctan\left(-\frac{1}{50}\right) \). We then have
\[
f(\arctan\left(-\frac{1}{50}\right)) = \frac{25}{5002} \sin(\arctan\left(-\frac{1}{50}\right)) - \frac{625}{2501} \cos(\arctan\left(-\frac{1}{50}\right))
\]
To find \( \sin(\arctan\left(-\frac{1}{50}\right)) \), suppose \( u = \arctan\left(-\frac{1}{50}\right) \). Draw a right triangle with \( u \) as one of the angles. Since \( u = \arctan\left(-\frac{1}{50}\right) \), it follows that \( \tan(u) = -\frac{1}{50} \) so let the leg of the triangle opposite \( u \) have length \(-1\) and the other leg have \( 50 \). Then the hypotenuse, must have length \( \sqrt{1 + 50^2} \). Thus \( \sin(u) = \frac{-1}{\sqrt{1+50^2}} \). Similarly, we can see \( \cos(u) = \frac{50}{\sqrt{1+50^2}} \).

\[
f(\arctan\left(-\frac{1}{50}\right)) = \frac{25}{5002} \frac{-1}{\sqrt{1+50^2}} - \frac{625}{2501} \frac{50}{\sqrt{1+50^2}}
\]
\[
= \frac{62525\sqrt{2501}}{(5002)(2501)} = \frac{25\sqrt{2501}}{5002}
\]

so we see the amplitude of the oscillation is
\[
\frac{25\sqrt{2501}}{5002} \approx 0.24995
\]

**Section 2.4**

33. (Problems 1 and 3 from B&D) For each of the following initial value problems, find an interval of \( t \) in which Theorem 2.4.1 from the book (Theorem 1 from lecture notes) guarantees the existence of the a (unique) solution.

(a) \( (t - 3)y' + \ln(t)y = 2t \); \( y(1) = 2 \)
(b) \( y' + \tan(t)y = \sin(t) \); \( y(\pi) = 0 \)

**Solution:**

(a) To apply Theorem 2.4.1, we need to get the equation into the form \( \frac{dy}{dt} = f(t, y) \). Thus we consider

\[
y' = \frac{2t}{t-3} - \frac{\ln(t)}{t-3}
\]
Thus we cannot have \( t = 3 \). Furthermore, the presence of \( \ln(t) \) means we must require \( t > 0 \). Therefore we can either choose \((0, 3)\) or \((3, \infty)\). However, we need the interval to include our initial condition \( y(1) = 2 \). Thus we choose \((0, 3)\) since it contains \( t = 1 \).

(b) We proceed similarly and get
\[
y' = \sin(t) - \tan(t) y
\]
This has a discontinuity at \( t = \frac{(2k+1)\pi}{2} \) for \( k \in \mathbb{Z} \) since this is exactly where \( \cos(t) = 0 \) and \( \tan(t) = \frac{\sin(t)}{\cos(t)} \). Because our initial condition is \( y(\pi) = 0 \), we must choose the interval \( \frac{\pi}{2} < t < \frac{3\pi}{2} \).

34. (Problems 7 and 8 from B&D) For each of the following initial value problems, state where in the \( ty \)-plane the hypothesis of Theorem 2.4.2 from the book (Theorem 2 from lecture notes) are satisfied.

(a) \( y' = \frac{t-y}{2t+5y} \)

(b) \( y' = (1-t^2-y^2)^{\frac{1}{2}} \)

Solution:

(a) According to Theorem 2.4.2, we must find a rectangle \( \alpha < t < \beta, \gamma < y < \delta \) in which both \( f \) and \( \frac{\partial f}{\partial y} \) are continuous. Here
\[
f = \frac{t-y}{2t+5y}; \quad \frac{\partial f}{\partial y} = \frac{-7t}{(2t+5y)^2}
\]
Thus both are discontinuous exactly when \( 2t + 5y = 0 \) or equivalently on the line \( y = -\frac{2}{5}t \). Thus our rectangle should either be completely in the region \( 2t + 5y > 0 \) or the region \( 2t + 5y < 0 \).

(b) We proceed similarly, noting
\[
f = y' = (1-t^2-y^2)^{\frac{1}{2}}; \quad \frac{\partial f}{\partial y} = \frac{2y}{2(1-t^2-y^2)^{\frac{3}{2}}}
\]
Thus we must have that \( 1 - t^2 - y^2 > 0 \) or equivalently, \( t^2 + y^2 < 1 \). Note that this is just the interior of the unit circle in the \( ty \)-plane.
35. (Problems 13 and 16 from B&D) For each of the following, solve the initial value problem and determine how the interval in which the solution exists depend on the initial value $y_0$.

(a) $y' = -\frac{4t}{y}$; \hspace{1cm} y(0) = y_0

(b) $y' = \frac{t^2}{y(1+t^3)}$; \hspace{1cm} y(0) = y_0

Solution:

(a) This equation is separable, so we write it as

$$ydy = -4tdt$$

Integrating both sides gives

$$\frac{1}{2}y^2 = -2t^2 + C$$

Applying our initial conditions show $\frac{1}{2}y_0^2 = C$, giving us the equation

$$\frac{1}{2}y^2 = -2t^2 + \frac{1}{2}y_0^2$$

$$\implies y = \pm \sqrt{y_0^2 - 4t^2}$$

Thus we must have $|t| < \frac{1}{2}|y_0|$. Furthermore, since $f(t, y) = -\frac{4t}{y}$, we must have $y \neq 0$.

(b) This equation is also separable, becoming

$$ydy = \frac{t^2}{1+t^3} dt$$

$$\implies \frac{1}{2}y^2 = \frac{1}{3} \ln(1+t^3) + C$$

Thus applying our initial condition gives $C = \frac{1}{2}y_0^2$ (remember $\ln(1) = 0$). Therefore we have

$$y = \pm \sqrt{\frac{2}{3} \ln(1+t^3) + y_0^2}$$

Thus we must have $\frac{2}{3} \ln(1+t^3) + y_0^2 > 0$. (Note that $y = 0$ is not allowed since our original equation included division by $y$.) Therefore we have

$$\ln(1+t^3) > -\frac{3y_0^2}{2} \implies 1+t^3 > e^{-\frac{3y_0^2}{2}} \implies t^3 > -1 + e^{-\frac{3y_0^2}{2}}$$
Note that in particular this requires $t > -1$ so $1 + t^3 > 0$ as we needed to ensure we were not dividing by 0 or having the argument of $\ln(1 + t^3)$ be non-positive.

36. **Optional Hard Problem** Suppose we have a tank of water which starts with an initial concentration of $\frac{1}{10}$ lb of salt per gallon, and initially contains 100 gallons of water. Suppose water with a salt concentration of $\frac{1}{5}$ lb gal pours into the tank at a rate of $(200 - 100 \sin(t) \cos(t)) \text{ gal/hr}$. Assume water in the tank is mixed completely and instantaneously. Suppose water drains from the tank at a rate of 200 gal/hr. Let $S(t)$ be the amount of salt in pounds contained in the tank.

(a) Construct a differential equation to model this situation. (Hint: The amount of water in the tank is not constant.)

(b) Note that the equation is a first order linear equation and find an integrating factor for it. You should explicitly solve the integral involved.

(c) Find an expression for $S(t)$. This expression may still contain an integral.

**Solution:**

(a) As usual with tank problems, $\frac{dS}{dt} = \text{rate in} - \text{rate out}$. Here we have

$$\text{rate in} = \frac{1}{5} \left(200 - 100 \sin(t) \cos(t)\right) \text{lbs/hr}$$

The tricky part is finding the rate out since the amount of water in the tank is not constant. However, we note that the total change in the amount of water in the tank is

$$\int_0^t \left(200 - 100 \sin(x) \cos(x)\right) - 200 dx = \int_0^t -100 \sin(x) \cos(x) dx = -50 \sin^2(t)$$

Thus the amount of salt leaving the tank

$$\text{rate out} = 200 \cdot \frac{S(t)}{100 - 50 \sin^2(t)} = \frac{200 \cdot S(t)}{100 - 50 \sin^2(t)} \text{ lbs/hr}$$
Therefore
\[ \frac{dS}{dt} = \frac{1}{5} \left( 200 - 100 \sin(t) \cos(t) \right) - \frac{200 \cdot S(t)}{100 - 50 \sin^2(t)} \]
\[ \Rightarrow \frac{dS}{dt} + \frac{200}{100 - 50 \sin^2(t)} \cdot S(t) = \frac{1}{5} \left( 200 - 100 \sin(t) \cos(t) \right) \]

(b) We showed in part (a) that the differential equation was linear and therefore can be solved using an integrating factor. Thus we need to find
\[ \mu(t) = e^{\int \frac{200}{100 - 50 \sin^2(t)} \, dt} \]

To solve the integral, multiply top and bottom by \( \sec^2(t) \) to get
\[ \int \frac{200}{100 - 50 \sin^2(t)} \, dt = 100 \int \frac{2}{1 - \frac{1}{2} \sin^2(t)} \, dt = 100 \int \frac{2 \sec^2(t)}{\sec^2(t) - \frac{1}{2} \tan^2(t)} \, dt \]

Then remember the identity \( \sec^2(t) = 1 + \tan^2(t) \) to rewrite the denominator as
\[ 1 + \tan^2(t) - \frac{1}{2} \tan^2(t) = 1 + \frac{1}{2} \tan^2(t) \]

so we get
\[ 100 \int \frac{2 \sec^2(t)}{1 + \frac{1}{2} \tan^2(t)} \, dt \]

From here, it is just a \( u \)-substitution problem. It may be useful to remember (or if memory fails, look up) the derivative of \( \arctan(t) \). Let \( u = \frac{1}{\sqrt{2}} \tan(t) \) meaning \( du = \frac{1}{\sqrt{2}} \sec^2(t) \). Thus we get
\[ 100 \int \frac{2 \sqrt{2}}{1 + u^2} \, du = 200 \sqrt{2} \int \frac{du}{1 + u^2} = 200 \sqrt{2} \arctan(u) \]

\[ = 200 \sqrt{2} \arctan \left( \frac{1}{\sqrt{2}} \tan(t) \right) \]

Thus the integrating factor is
\[ \mu(t) = e^{200 \sqrt{2} \arctan \left( \frac{1}{\sqrt{2}} \tan(t) \right)} \]

(c) Now our differential equation becomes
\[ \mu(t) \frac{dS}{dt} + \mu(t) \frac{200}{100 - 50 \sin^2(t)} \cdot S(t) = \mu(t) \frac{1}{5} \left( 200 - 100 \sin(t) \cos(t) \right) \]
Integrating both sides gives

\[ e^{200\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\tan(t)\right)} S(t) = \int e^{200\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\tan(t)\right)} \frac{1}{5} \left(200 - 100 \sin(t) \cos(t)\right) dt \]

Thus

\[ S(t) = e^{-200\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\tan(t)\right)} \int e^{200\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\tan(t)\right)} \frac{1}{5} \left(200 - 100 \sin(t) \cos(t)\right) dt \]