1. a) \( \frac{dy}{dt} = y(y-1)(y-2) \); \( y_0 \geq 0 \)

Critical points are at \( y = 0 \), \( y = 1 \), and \( y = 2 \).
16) \[ \frac{\text{d}y}{\text{d}t} = e^{-y} - 1; \quad -\infty < y_0 < \infty \]

\[ \text{t stable} \]
2a) \[ \frac{dy}{dt} = y^2 (y^2 - 1) \quad \text{for} \quad -\infty < y < \infty \]
2b) \( \frac{dy}{dc} = y^2(1-y^2) \quad -\infty < y_0 < \infty \)
3a)

\[ \frac{dy}{dt} = rY \ln \left( \frac{K}{Y} \right) \]

Diagram 1:
- Plot of \( f(Y) \)
- X-axis: \( t \)
- Y-axis: \( Y \)
- Unstable region
- Stable region
- Point labeled \( K \)

Diagram 2:
- Plot of \( Y \) vs. \( t \)
- Unstable region
- Stable region
- Point labeled \( K \)
Section 2.5

3. (Problems 16 from B&D) Another equation that has been used to model population growth is the Gompertz equation

\[
\frac{dy}{dt} = r y \ln \left( \frac{K}{y} \right)
\]

where \( r \) and \( k \) are positive constants.

(a) Sketch the graph of \( f(y) \) versus \( y \), find the critical points, and determine whether each is asymptotically stable or unstable.

(b) For \( 0 \leq y \leq K \), determine where the graph of \( y \) versus \( t \) is concave up and where it is concave down.

(c) For each \( y \) in \( 0 < y \leq K \), show that \( \frac{dy}{dt} \) as given by the Gompertz equation is never less than \( \frac{dy}{dt} \) as given the logistic equation \( \frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y \).

Solution:

(b) Remember that \( \frac{d^2y}{dt^2} = \frac{d}{dt} \frac{dy}{dt} = \frac{df}{dy} \frac{dy}{dt} = f'(y) \frac{dy}{dt} = f'(y) f(y) \). Note that concavity can only change where \( \frac{d^2y}{dt^2} = 0 \). However this would imply either \( f'(y) = 0 \) or \( f(y) = 0 \). Since the solutions to \( f(y) = 0 \) are just our critical points, we need only find where \( f'(y) = 0 \). Our course we must first determine \( f'(y) \). Applying the product rule, we get

\[
f'(y) = \frac{d}{dy} f(y) = \frac{d}{dy} \left( r y \ln(K/y) \right)
\]

\[
= r \ln(K/y) + r y \left( \frac{y}{K} \cdot \frac{-K}{y^2} \right) = r \ln(K/y) - r
\]
Thus we need to solve the equation

\[ 0 = r \ln(\frac{K}{y}) - r \Rightarrow r = r \ln(\frac{K}{y}) \Rightarrow 1 = \ln(\frac{K}{y}) \]

\[ \Rightarrow e = \frac{K}{y} \Rightarrow y = \frac{K}{e} \]

Take a value, say \( \frac{K}{2e} \), in the interval \( 0 < y < \frac{K}{e} \). Then

\[
\frac{d^2y}{dt^2}\left(\frac{K}{2e}\right) = r\left(\frac{K}{2e}\right) \ln\left(\frac{K}{\frac{K}{2e}}\right) \cdot r \left( \ln\left(\frac{K}{\frac{K}{2e}}\right) - 1 \right) \\
= r^2\left(\frac{K}{2e}\right) \ln(2)\left( \ln(2) - 1 \right) > 0
\]

Thus the graph of \( y \) versus \( t \) is concave up on \( 0 < y < \frac{K}{e} \). Similar computations show that the graph is concave down on \( \frac{K}{e} < y < K \).

(c) This amounts to showing that

\[ ry \ln(\frac{K}{y}) \geq ry\left(1 - \frac{y}{K}\right) \]

which is equivalent to showing

\[ \ln(\frac{K}{y}) \geq (1 - \frac{y}{K}) \]

Here we use a trick which involves considering the difference of the two functions

\[ D(y) = \ln(\frac{K}{y}) - (1 - \frac{y}{K}) = \ln(\frac{K}{y}) + \frac{y}{K} - 1 \]

and taking the derivative getting

\[ D'(y) = -\frac{1}{y} + \frac{1}{K} \]

Setting it equal to 0 we get

\[ 0 = -\frac{1}{y} + \frac{1}{K} \Rightarrow y = K \]

Thus \( y = K \) is either a maximum or minimum for the difference function \( D(y) \). However, we note that \( D'(y) < 0 \) for all \( y < K \). Thus we see \( y = K \) must be a minimum of the difference equation. However,

\[ D(K) = \ln(\frac{K}{K}) + \frac{K}{K} - 1 = 0 + 1 - 1 = 0 \]
Thus we see

\[ D(y) = \ln(K/y) + \frac{y}{K} - 1 \geq 0 \text{ for } 0 < y \leq K \]

\[ \implies \ln(K/y) \geq 1 - \frac{y}{K} \implies ry \ln(K/y) \geq ry(1 - \frac{y}{K}) \]

4. (Problem 17a from B&D) Solve the Gompertz equation

\[ \frac{dy}{dt} = ry \ln(K/y) \]

subject to the initial condition \( y(0) = y_0 \).

Hint: You may want to let \( u = \ln(y/K) \).

**Solution:**

This equation is separable so we write it as

\[ \frac{1}{y \ln K/y} = r dt \]

Thus we need to integrate both sides. For the left side, we let \( u = \ln K/y \) which, applying the chain rule, gives \( du = \frac{1}{y} \cdot \frac{-K}{y^2} dy = -\frac{1}{y} dy \). Thus

\[ \int \frac{1}{y \ln(K/y)} dy = \int \frac{-1}{u} du = -\ln|u| = -\ln|\ln(K/y)| \]

so \( \ln|\ln(K/y)| = -rt + C \) implying \( |\ln(K/y)| = C_1 e^{-rt} \). Note that \( \ln(K/y) \) only switches from negative to positive at the equilibrium solutions \( K \). Since solutions do not cross the equilibrium solutions, we may drop the absolute values, getting \( \ln(K/y) = C_1 e^{-rt} \). Furthermore, applying our initial conditions implies \( C_1 = \pm \ln(K/y_0) \), with we choose the positive if \( y_0 < k \) and negative if \( y_0 > k \). Assuming \( y_0 < K \), this implies \( \frac{K}{y} = e^{\ln(K/y_0)e^{-rt}} = \left( \frac{K}{y_0} \right)^{e^{-rt}} \) meaning

\[ y = K \left( \frac{K}{y_0} \right)^{e^{-rt}} \]

Similarly, if \( y_0 < k \), we have

\[ y = K \left( \frac{K}{y_0} \right)^{tge^{-rt}} \]
5. **Optional Hard Problem** (Problems 18 from B&D) A pond forms as water collects in a conical depression of radius $a$ and depth $h$. Suppose that water flows in at a constant rate $k$ and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time $t$ satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha \pi \left(\frac{3a}{\pi h}\right) V^{\frac{2}{3}}$$

where $\alpha$ is the coefficient of evaporation.

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

(c) Find a condition that must be satisfied if the pond is not to overflow.

**Solution:**

(a) First, let’s clarify notation. The radius $a$ and height $h$ are fixed constants giving the dimensions of the pond, and thus serving to provide us with a fixed ratio between height and radius. We will denote the radius at any given time as $r$ and the height as $g$. We note that $\frac{dV}{dt}$ is just the rate water enters minus the rate it leaves. It is given to us that water enters at a rate $k$. Thus we need to find the rate at which it evaporates. To do this, we need to find the surface area. Remember that the volume of a cone is $V(r) = \frac{1}{3} \pi r^2 g$ and the surface area of the top is just $S(r) = \pi r^2$. Note that there is a linear relationship between $g$ and $r$. In particular, $g = r \cdot \frac{h}{a}$ (consider the similar triangles). Thus the volume equation becomes

$$V = \frac{1}{3} \pi r^2 \cdot r \cdot \frac{h}{a} = \frac{\pi h}{3a} r^3$$

If we now solve the volume equation for $r$, we get $r = \left(\frac{3a}{\pi h}\right)^{\frac{1}{3}} V^{\frac{1}{3}}$. Plugging this into the surface area equation gives

$$S = \pi r^2 = \pi \left(\frac{3a}{\pi h}\right)^{\frac{2}{3}} V^{\frac{2}{3}}$$

Thus multiply by our coefficient of evaporation, we get that

$$\frac{dV}{dt} = k - \alpha \pi \left(\frac{3a}{\pi h}\right) V^{\frac{2}{3}}$$
(b) Note that the equilibrium solution occurs when \( \frac{dy}{dt} = 0 \) or equivalently

\[ k = \alpha \pi \left( \frac{3a}{\pi h} \right)^{\frac{2}{3}} V^{\frac{2}{3}} \]

We now want to express \( V \) in terms of \( g \). Using similar triangles again, we can find that \( r = \frac{ag}{h} \). Plugging this into the volume equation gives

\[ V = \frac{1}{3} \pi \frac{a^2 g^2}{h^2} \cdot g = \frac{1}{3} \pi \frac{a^2 g^3}{h^2} \]

We can now write the equation as

\[ k = \alpha \pi \left( \frac{3a}{\pi h} \right)^{\frac{2}{3}} \left( \frac{1}{3} \pi \frac{a^2 g^3}{h^2} \right)^{\frac{2}{3}} = \alpha \pi \left( \frac{3a}{\pi h} \right)^{\frac{2}{3}} \left( \frac{1}{3} \pi \frac{a^2}{h^2} \right)^{\frac{2}{3}} g^2 \]

Solving for \( g \) we get

\[ g = \left( \frac{k}{\alpha \pi \left( \frac{3a}{\pi h} \right)^{\frac{2}{3}} \left( \frac{1}{3} \pi \frac{a^2}{h^2} \right)^{\frac{1}{3}}} \right)^{\frac{1}{2}} = \frac{\sqrt{k}}{\alpha \pi \left( \frac{3a}{\pi h} \right)^{\frac{2}{3}} \left( \frac{1}{3} \pi \frac{a^2}{h^2} \right)^{\frac{1}{3}}} = \frac{\sqrt{k}}{\sqrt{\alpha \pi \left( \frac{a}{h^3} \right)^{\frac{1}{2}}}} = \frac{\sqrt{k}}{\sqrt{\alpha \pi \frac{a}{h^3}}} = \sqrt{\frac{k}{\alpha \pi a}} \]

Thus we need only determine if this solution is asymptotically stable or unstable. However, it is fairly clear that \( \frac{dV}{dt} \) is negative if \( g \) is greater than the equilibrium solution and positive if it is smaller. Thus the solution is asymptotically stable.

(c) In order for the pond not to overflow, we need the \( g \) value of the equilibrium solution to be less than \( h \). Thus we require

\[ \sqrt{\frac{k}{\alpha \pi a}} \leq h \quad \implies \quad \sqrt{k} \leq \frac{k}{\alpha \pi a} \]

\[ \implies k \leq \alpha \pi a^2 \]
6. Suppose we are given an autonomous differential equation \( \frac{dy}{dt} = f(y) \) with the following graph of \( f(y) \) vs \( y \).

(a) Find all the critical points, draw the phase lines of the graph, and determine whether each critical point cooresponds to a stable, unstable, or semistable equilibrium solution.

Solution:

(b) Between each pair of critical points, estimate the \( y \) value at which solutions whose initial values are in that interval switch concavity.

Solution: Given that \( \frac{dy}{dt} = f(y) \), we have \( \frac{d^2y}{dt^2} = \frac{d}{dt} f(y) = f'(y) \frac{dy}{dt} = f'(y)f(y) \). Now we know that when \( f(y) = 0 \) we have an equilibrium solution, so no solutions change concavity at these points. It follows that the only possible \( y \) values for which concavity is going to change are when \( f'(y) = 0 \). From the graph we can see that this happens at approximately \( y = 17 \) and \( y = 63 \). Furthermore, we can see that these will, in fact, be inflection points.
(c) Use parts (a) and (b) to sketch solutions to the differential equation in the $ty$ plane given different initial conditions $y(0) = y_0$. Include several solutions with initial conditions $y(0) = y_0$ between each pair or equilibrium solutions.

**Solution:**

```
Several Solutions
```

(d) Recall that a threshold for a population is value above which the population grows but below which the population becomes extinct. Use part (c) to determine the threshold and carrying capacity for the population described by this differential equation.

**Solution:** From the above plot we can see that the threshold is 40 and the carrying capacity is 80.

(e) Based off your investigations and the discussion on page 86 and 87 of B&D, give a function for $f(y)$.

**Solution:** Something like $f(y) = -y(y/40 - 1)(y/80 - 1)$
Section 2.6

7. (Problems 1,5,6,8 from B&D) For the following problems, determine whether or not the equations are exact. You do not need to solve the equations.

(a) \((2x + 3) + (2y - 2)y' = 0\)
(b) \(\frac{dy}{dx} = -\frac{ax + by}{bx + cy}\)
(c) \(\frac{dy}{dx} = -\frac{ax - by}{bx - cy}\)
(d) \((e^x \sin(y) + 3y)dx - (3x - e^x \sin(y))dy = 0\)

Solution: We have proven that showing an equation is exact is equivalent to showing that \(\frac{d}{dy}M(x, y) = \frac{d}{dx}N(x, y)\)

(a) \(\frac{d}{dy}(2x + 3) = 0 = \frac{d}{dx}(2y - 2)\) so it is exact

(b) We must first put this in the standard form:
\[
\frac{dy}{dx} = -\frac{ax + by}{bx + cy} \implies (bx + cy)y' = -(ax + by) \implies (ax + by) + (bx + cy)y' = 0
\]

Then \(\frac{d}{dy}(ax + by) = b = \frac{d}{dx}(bx + cy)\) so it is exact.

(c) Putting this in the standard form gives
\[
\frac{dy}{dx} = -\frac{ax - by}{bx - cy} \implies (bx - cy)y' = -(ax - by) \implies (ax - by) + (bx - cy)y' = 0
\]

Then \(\frac{d}{dy}(ax - by) = -b\) and \(\frac{d}{dx}(bx - cy) = b\) so it is only exact if \(b = 0\).

(d) \(\frac{d}{dy}(e^x \sin(y) + 3y) = e^x \cos(y) + 3 \neq 3 + e^x \sin(y) = \frac{d}{dx}(-3x + e^x \sin(y))\) so it is not exact.
8. Show that the following equations are exact and use this to find an implicit solution for $y$.

(a) $(2xy + 5y^3) + (x^2 + 15xy^2 + 15)y' = 0$

(b) $(-\sin(x)y^2 + 30xy^2 + e^x) + (2y\cos(x) + 30x^2y)y' = 0$

**Solution:** We leave it to the reader to check that the equations are in fact exact. (To see the method for this, reference problem 1.)

(a) We want a function $\psi = \int M(x, y)dx + h(y) = \int 2xy + 5y^3 dx + h(y)$. Integrating gives $\psi = x^2 y + 5xy^3 + h(y)$. However we also know that $\frac{\partial}{\partial y} \psi = N(x, y)$ which means $x^2 + 15xy^2 + h'(y) = x^2 + 15xy^2 + 15$. Thus $h'(y) = 15$ meaning $h(y) = 15y + C_1$. Putting this all together gives us

$$\psi = x^2 y + 5xy^3 + 15y + C_1 = C_2$$

$$\implies x^2 y + 5xy^3 + 15y = C_3$$

(b) Let’s solve this one in a slightly different manner. This time we will integrate $N(x, y)$ with respect to $y$. We let $\psi = \int N(x, y)dy + g(x) = \int 2y\cos(x) + 30x^2y)dy + g(x) = y^2 \cos(x) + 15x^2y^2 + g(x)$. However, we want it to be true that $\frac{\partial}{\partial x} \psi = M(x, y)$ meaning $-\sin(x)y^2 + 30xy^2 + g'(x) = -\sin(x)y^2 + 30xy^2 + e^x$. It follows that $g'(x) = e^x$ implying $g(x) = e^x + C_1$. Thus

$$\psi = y^2 \cos(x) + 15x^2y^2 + e^x + C_1 = C_2$$

$$y^2 \cos(x) + 15x^2y^2 + e^x = C_3$$

9. Put the equation

$$y' = e^{2x} + y - 1$$

into the standard form for exact equations and show that the equation is not exact. Determine which of the following integration factors make the equation exact. You don’t need to solve the equation.

(a) $\mu(t) = e^y$

(b) $\mu(t) = e^{-x}$

(c) $\mu(t) = y$
Solution: Subtracting everything over we get \((1 - y - e^{2x}) + y' = 0\) which is standard form. It is easy to check that the equation is not exact. From here, it is simply a matter of multiplying through by each and checking to see which result in an exact equation. We see that the correct answer is (b).

10. (Problems 25 and 27 from B&D) For each of the following, find an integrating factor and solve the given equation.

(a) \((3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0\)
(b) \(dx + \left(\frac{x}{y} - \sin(y)\right)dy = 0\)

Solution: We have learned methods for attempting to find integrating factors only in terms of \(x\), namely \(\mu_x = e^{\int \frac{M_y - N_x}{N} dx}\), or only in terms of \(y\), namely \(\mu_y = e^{\int \frac{N_x - M_y}{M} dy}\). Since the numerator in the integral only changes by a negative sign, we are essential picking whether to divide by \(M\) or \(N\). Thus it makes sense to first try the simpler one.

(a) Here we have \(M_y = 3x^2 + 2x + 3y^2\) and \(N_x = 2x\). Then \(\mu_x = e^{\int \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} dx} = e^{\int 3dx} = e^{3x}\). Multiplying through we get
\[e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0\]

Thus we want to set
\[\psi = \int e^{3x}(x^2 + y^2)dy + g(x)\]
\[= e^{3x}(x^2y + \frac{1}{3}y^3) + g(x)\]

We then set \(\frac{\partial}{\partial x} \psi = M(x, y)\) implying
\[\frac{\partial}{\partial x} \psi = 3e^{3x}x^2y + e^{3x}y^3 + e^{3x}2xy + g'(x) = e^{3x}(3x^2y + 2xy + y^3) + g'(x)\]
\[= e^{3x}(3x^2y + 2xy + y^3) = M(x, y)\]

Thus \(g'(x) = 0\) meaning \(g(x) = C_1\). Thus
\[\psi = e^{3x}(x^2y + \frac{1}{3}y^3) + C_1 = C_2\]
\[\implies e^{3x}(x^2y + \frac{1}{3}y^3) = C_3\]
11. Consider the differential equation

\[(3x^2y + 10e^x + 4y^2) + (x^3 + \cos(y) + 8xy)y' = 0\]

with initial condition \(y(0) = y_0\).

(a) Show that the equation is exact.

**Solution:** Noting that \(M(x, y) = 3x^2y + 10e^x + 4y^2\) and \(N(x, y) = x^3 + \cos(y) + 8xy\), we have that \(M_y = N_x = 3x^2 + 8y\).

(b) Solve the equation in the following ways:

i. Integrate \(M(x, y)\) with respect to \(x\) and find the function \(h(y)\) such that \(\psi = \int M(x, y)\,dx + h(y)\).

**Solution:** Letting

\[\psi = \int M(x, y)\,dx + h(y) = x^3y + 10e^x + 4y^2x + h(y)\]

and taking the partial of \(\psi\) with respect to \(y\) we have

\[\psi_y = x^3 + 8yx + h'(y).\]
Setting \( \psi_y = N(x, y) \)

we can conclude that \( h'(y) = \cos(y) \) and \( h(y) = \sin(y) + c \). This leaves us with a final solution of

\[
x^3 y + 10e^x + 4y^2 x + \sin(y) = c.
\]

ii. Integrate \( N(x, y) \) with respect to \( y \) and find the function \( g(x) \) such that \( \psi = \int N(x, y) dy + g(x) \).

**Solution:** Now let

\[
\psi = \int N(x, y) \ dy + g(x) = yx^3 + \sin(y) + 4xy^2 + g(x).
\]

Then

\[
\psi_x = 3yx^2 + 4y^2 + g'(x).
\]

Setting

\[
\psi_x = M(x, y)
\]

we can conclude that \( g'(x) = 10e^x \) and \( g = 10e^x + c \). This leaves us with the same final solution of

\[
yx^3 + \sin(y) + 4xy^2 + 10e^x = c.
\]

12. Consider the differential equation \( y \, dx + (2xy - e^{-2y}) \, dy = 0 \). Show that it is not exact, find an integrating factor, and use this to find an implicit solution to the equation.

**Solution:** Noting that \( M(x, y) = y \) and \( N(x, y) = 2xy - e^{-2y} \), we have

\[
M_y = 1 = 2y = N_x.
\]

It follows that the equation is not exact. Noticing that

\[
\frac{N_x - M_y}{M} = \frac{2y - 1}{y}
\]

we see that there is an integrating factor, \( \mu \), in terms of \( y \) alone. It follows that

\[
\mu = \exp \left( \int \frac{2y - 1}{y} \ dy \right) = e^{2y - \ln|y|} = \frac{e^{2y}}{y}.
\]
We dropped the absolute values here because (as you will see below) the integrating factor still works without them. Multiplying through by \( \mu \), we have the exact equation

\[
e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)y' = 0.
\]

Using the ‘new’ \( M \) and \( N \) and letting

\[
\psi = \int M(x, y) \, dx + h(y) = xe^{2y} + h(y)
\]

and setting

\[
\psi_y = 2xe^{2y} + h'(y) = N(x, y)
\]

we can conclude that \( h'(y) = -\frac{1}{y} \) and \( h(y) = -\ln|y| + c \). This leaves us with a final solution of

\[
xe^{2y} - \ln|y| = c
\]

Section 2.7

13. (Problem 1 from B&D) Consider the initial value problem

\[
y' = 3 + t - y \quad y(0) = 1
\]

(a) Find approximate values of the solution of the given initial value problem at \( t = 0.1, 0.2, 0.3, \) and 0.4 using the Euler method with \( h = 0.1 \).

(b) Repeat part (a) with \( h = 0.05 \). Compare the results with those found in (a).

(c) Find the solution \( y = \phi(t) \) of the given problem and evaluate \( \phi(t) \) at \( t = 0.1, 0.2, 0.3, \) and 0.4. Compare these values with the results of (a) and (b).

Solution:

(a) We know the formula \( y_n = hf(t_{n-1}, y_{n-1}) + y_{n-1} \) where here \( f(t, y) = 3 + t - y \).

Thus \( y_n = 0.3 + 0.1t_{n-1} + 0.9y_{n-1} \). This allows us to build the following table inductively.

<table>
<thead>
<tr>
<th>i</th>
<th>( t_i )</th>
<th>( y_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>1.2</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>1.39</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>1.571</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>1.7439</td>
</tr>
</tbody>
</table>
(b) We now use $h = 0.05$ and proceed similarly to get the table

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>1.1</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>1.1975</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>1.2926</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>1.3855</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>1.4762</td>
</tr>
<tr>
<td>6</td>
<td>0.30</td>
<td>1.5649</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>1.6517</td>
</tr>
<tr>
<td>8</td>
<td>0.40</td>
<td>1.7366</td>
</tr>
</tbody>
</table>

While the solutions do not differ drastically, those produced using $h = 0.05$ are all slightly less than those produced using $h = 0.1$.

(c) To solve the equation $y' = 3 + t - y$, we note that it is a first order linear equation. Putting it in the standard form, we have $y' + y = t + 3$ Then we have $\mu(t) = e^{\int dt} = e^{t}$ so the equation becomes $y'e^{t} + ye^{t} = (t + 3)e^{t}$. However the left side is just $\frac{d}{dt} ye^{t}$. Therefore integrating both sides with respect to $t$ gives

$$ye^{t} = \int (t + 3)e^{t} dt$$

Integration by parts gives

$$\int (t + 3)e^{t} dt = (t + 3)e^{t} - \int e^{t} dt = (t + 2)e^{t} + C$$

Therefore we get

$$y = t + 2 + Ce^{-t}$$

Applying our initial condition that $y(0) = 1$, we see that $C = -1$. Therefore we have our solution to the initial value problem

$$\phi(t) = t + 2 - e^{-t}$$

Making a final table we have

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$\phi(t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.998729</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.994829</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>0.988166</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>0.978597</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>0.965975</td>
</tr>
<tr>
<td>6</td>
<td>0.30</td>
<td>0.950141</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>0.930932</td>
</tr>
<tr>
<td>8</td>
<td>0.40</td>
<td>0.908175</td>
</tr>
</tbody>
</table>
Obviously, these results differ quite a bit from our approximations we got using $h = 0.1$ and $h = 0.05$. It appears we would need to use much smaller values of $h$ to approximate this curve well.

14. (Problem 12 from B&D) Consider the initial value problem

$$y' = y(3 - ty) \quad y(0) = 0.5$$

Use Euler’s method to find approximate values of the solution at $t = 0.5, 1.0, 1.5, 2.0$

(a) with $h = 0.5$

(b) with $h = 0.25$

Solution:

(a) Again, we make a table

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.25</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.7344</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>3.0975</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>0.5478</td>
</tr>
</tbody>
</table>

(b) With $h = 0.25$ we have the following table

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.8750</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td><strong>1.4834</strong></td>
</tr>
<tr>
<td>3</td>
<td>0.75</td>
<td>2.3209</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td><strong>3.0516</strong></td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>3.0122</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td><strong>2.4359</strong></td>
</tr>
<tr>
<td>7</td>
<td>1.75</td>
<td>2.0377</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td><strong>1.7494</strong></td>
</tr>
</tbody>
</table>
15. **Optional Hard Problem** Solve the differential equation

\[(3xy + y^2) + (x^2 + xy)y' = 0\]

using the integrating factor \(\mu(x, y) = [xy(2x + y)]^{-1}\). Verify that the solution is the same as that obtained using a different integrating factor in the lecture notes (example 4 from B&D).

**Solution:**

(a) Multiplying by the integrating factor gives

\[\frac{3xy + y^2}{xy(2x + y)} + \frac{x^2 + xy}{xy(2x + y)} y' = 0\]

We need to find

\[\psi = \int \frac{3xy + y^2}{xy(2x + y)} \, dx + h(y)\]

To do this we need to use partial fractions as follows:

\[\frac{3xy + y^2}{xy(2x + y)} = \frac{A}{xy} + \frac{B}{2x + y} = \frac{2Ax + Ay + Bxy}{xy(2x + y)}\]

Thus we must have \(A = y\) which necessitates that \(B = 1\). We now have

\[\int \frac{3xy + y^2}{xy(2x + y)} \, dx = \int \frac{y}{xy} + \frac{1}{2x + y} \, dx = \int \frac{1}{x} + \frac{1}{2x + y} \, dx\]

so \(\psi = \ln |x| + \frac{1}{2} \ln |2x + y| + h(y)\)

Then we consider

\[\frac{\partial \psi}{\partial y} = \frac{1}{2} \cdot \frac{1}{2x + y} + h'(y) = \frac{x^2 + xy}{xy(2x + y)}\]

Thus

\[h'(y) = \frac{x^2 + xy}{xy(2x + y)} - \frac{1}{2} \cdot \frac{1}{2x + y} = \frac{x^2 + xy}{xy(2x + y)} - \frac{\frac{1}{2}xy}{xy(2x + y)}\]

\[= \frac{x^2 + \frac{1}{2}xy}{xy(2x + y)} = \frac{x + \frac{1}{2}y}{y(2x + y)} = \frac{1}{2y}\]
so $h(y) = \frac{1}{2} \ln |y|$. Therefore $\psi = \ln |x| + \frac{1}{2} \ln |2x + y| + \frac{1}{2} \ln |y| = C$. We can simply this somewhat by multiply through by 2 then using the log properties to get

$$2 \ln |x| + \ln |2x + y| + \ln |y| = C_1$$

$$\Rightarrow \ln |x^2| + \ln |2x + y| + \ln |y| = C_1$$

$$\Rightarrow \ln |x^2(2x + y)| = C_1$$

The putting both sides as the exponent of $e$ we get

$$|x^2(2x + y)| = e^{C_1}$$

$$\Rightarrow 2x^3y + \frac{1}{2} x^2 y^2 = C_2$$

$$\Rightarrow x^3y + \frac{1}{2} x^2 y^2 = C_3$$

which was the solution we found using the other integrating factor.

(Note that we dropped the absolute values sense we previously had to decide a domain for $x$ and $y$ as the integrating factor had both in the denominator. Thus we can account for the sign which cannot change with the constant.)

16. (Problem 2 from B&D) Consider the initial value problem

$$y' = 2y - 1 \quad y(0) = 1$$

(a) Find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3, \text{ and } 0.4$ using the Euler method with $h = 0.1$.

\begin{center}
\begin{tabular}{c|c|c}
$i$ & $t_i$ & $y_i$
\hline
0 & 0 & 1 \\
1 & .1 & 1.1 \\
2 & .2 & 1.22 \\
3 & .3 & 1.364 \\
4 & .4 & 1.5368 \\
\end{tabular}
\end{center}

(b) Repeat part (a) with $h = 0.05$. Compare the results with those found in (a).
Solution:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$t_i$</th>
<th>$y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>1.05</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>1.105</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>1.1650</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>1.2320500</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>1.3052500</td>
</tr>
<tr>
<td>6</td>
<td>0.30</td>
<td>1.385780500</td>
</tr>
<tr>
<td>7</td>
<td>0.35</td>
<td>1.474358550</td>
</tr>
<tr>
<td>8</td>
<td>0.40</td>
<td>1.571794405</td>
</tr>
</tbody>
</table>

(c) Find the solution $y = \phi(t)$ of the given problem and evaluate $\phi(t)$ at $t = 0.1, 0.2, 0.3,$ and $0.4$. Compare these values with the results of (a) and (b).

Solution: The differential equation

$$y' = 2y - 1$$

is both separable and linear. We will solve it as a linear equation. Rearranging yields

$$y' - 2y = -1.$$  

Then the integrating factor $\mu = e^{-2t}$ and we have

$$\mu y = \int -\mu \, dt = \int -e^{-2t} \, dt = \frac{1}{2} e^{-2t} + c.$$  

The general solution is then

$$y = \frac{1}{2} + ce^{2t}.$$  

Plugging the initial conditions in gives

$$1 = y(0) = \frac{1}{2} + ce^{2 \cdot 0} = \frac{1}{2} + c.$$  

So $c = \frac{1}{2}$ and the final solution is

$$y = \frac{1}{2}(1 + e^{2t}).$$  

Plugging numbers in we have

<table>
<thead>
<tr>
<th>$t$</th>
<th>$y(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>1.110701379</td>
</tr>
<tr>
<td>.2</td>
<td>1.245912349</td>
</tr>
<tr>
<td>.3</td>
<td>1.411059400</td>
</tr>
<tr>
<td>.4</td>
<td>1.612770464</td>
</tr>
</tbody>
</table>
17. Consider the autonomous equation \( \frac{dy}{dt} = -ry \) where \( r > 0 \). If we apply Euler’s method with step size \( h \), we notice that the equation \( y_n = h \cdot f(t_{n-1}, y_{n-1}) + y_{n-1} \) becomes \( y_n = h \cdot f(y_{n-1}) + y_{n-1} \). Suppose we are given the initial condition \( y(t_0) = y_0 \).

(a) Show that \( y_i = y_{i-1}(1 - rh) \).

**Solution:** From Euler’s method we know that
\[
y_i = y_{i-1} + hf(t_{i-1}, y_{i-1}) = y_{i-1}(1 + hr).
\]

(b) Use (a) to show \( y_n = y_0(1 - rh)^n \).

**Solution:** Letting \( i = 1 \) in our equation from part (a) we have
\[
y_1 = y_0(1 - rh)
\]
and the formula is true in the case that \( n = 1 \). Now if we assume that it is true for \( n - 1 \) and show that this implies that the formula holds in the \( n \) case, we are done. Why is this? We would then have that the formula holds for \( n = 0 \), which implies that it holds for \( n = 1 \), which implies that it holds for \( n = 2 \), etc...

So assume \( y_{n-1} = y_0(1 - rh)^{n-1} \). From part (a) we have
\[
y_n = y_{n-1}(1 - rh)
\]

Plugging in our above equation we have
\[
y_n = y_0(1 - rh)^{n-1}(1 - rh) = y_0(1 - rh)^n.
\]

(c) **Optional Hard Problem** We say a family of solution converges if as \( t \to \infty \), the solutions approach the same value regardless of initial condition.

i. For what values of \( h \) do the approximations given by Euler’s method converge?
Solution: From our above solution, we have that the approximation converges if and only if
\[
\lim_{n \to \infty} y_0 (1 - rh)^n \neq \pm \infty.
\]
This happens if and only if \(|(1 - rh)| < 1\). Assuming \(h\) is positive we have that this happens if and only if \(0 < rh < 2\) if and only if \(0 < h < \frac{2}{r}\).

ii. Solve the differential equation explicitly. You should note that all its solutions converge.

Solution: The equation \(y' = -ry\) is linear (and separable). So we have
\[
y' + ry = 0.
\]
Using the integrating factor \(\mu = e^{rt}\) our solution is
\[
y = ce^{-rt}.
\]

iii. Why is this not true for the approximations given by Euler’s method if \(h\) is too large? (It may be helpful to consider the slope fields of the equation.)

Solution: If \(h\) is too large, successive iterations of Euler’s method produce points that are farther and farther away from the equilibrium solution (alternating positive and negative \(y\) values). For large values of \(y\) the slope of the tangent lines to the solution curves become larger and larger in magnitude and therefore successive iterations get ‘thrown’ increasing farther. This is why the method fails.

18. See problem 7
19. See problem 8
20. See problem 9
21. See problem 10
Section 3.1

22. Find the general solution to the differential equation \( y'' + y' - 30y = 0 \).

**Solution:** The characteristic equation for the above differential equation is

\[ r^2 + r - 30 = (r + 6)(r - 5). \]

This corresponds to the two solutions \( e^{5t} \) and \( e^{-6t} \). Our general solution then becomes

\[ y = c_1 e^{5t} + c_2 e^{-6t}. \]

23. Find the solution to the following initial value problems:

(a) \( y'' + 4y' + 3y = 0; \quad y(0) = 2, \quad y'(0) = 0 \)

**Solution:** The characteristic equation is

\[ r^2 + 4r + 3 = (r + 3)(r + 1) \]

and the general solution is

\[ y = c_1 e^{-3t} + c_2 e^{-t}. \]

Plugging in the initial conditions we have

\[ c_1 + c_2 = 2 \]
\[ -3c_1 - c_2 = 0. \]

Solving the system we have \( c_1 = -1 \) and \( c_2 = 3 \) and our final solution is

\[ y = -e^{-3t} + 3e^{-t}. \]

(b) \( y'' - 6y' + 8y = 0; \quad y(0) = 3, \quad y'(0) = -2 \)

**Solution:** Here we have a characteristic equation on

\[ r^2 - 6r + 8 = (r - 2)(r - 4). \]

Then the general solution is

\[ y = c_1 e^{2t} + c_2 e^{4t}. \]
and our initial conditions leave us with
\[ c_1 + c_2 = 3 \]
\[ 2c_1 + 4c_2 = -2. \]
Solving this gives that \( c_1 = 7, c_2 = -4 \) so our final solution is
\[ y = 7e^{2t} - 4e^{4t}. \]

24. Find a second order linear equation with constant coefficients whose general solution is:

(a) \( C_1e^{2t} + C_2e^{-5t} \)

Solution: This corresponds to the characteristic equation
\[ (r - 2)(r + 5) = r^2 + 3r - 10, \]
which corresponds to the differential equation
\[ y'' + 3y' - 10y = 0. \]

(b) \( C_1e^{3t} + C_2e^t \)

Solution: This corresponds to a characteristic equation
\[ (r - 3)(r - 1) = r^2 - 4r + 3, \]
which corresponds to the differential equation
\[ y'' - 4y' + 3y = 0. \]