REFINED BRILL-NOETHER THEORY FOR COMPLETE GRAPHS

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ABSTRACT. The divisor theory of the complete graph K_n is in many ways similar to that of a plane curve of degree n. We compute the splitting types of all divisors on the complete graph K_n. We see that the possible splitting types of divisors on K_n exactly match the possible splitting types of line bundles on a smooth plane curve of degree n. This generalizes the earlier result of Cori and Le Borgne computing the ranks of all divisors on K_n, and the earlier work of Cools and Panizzut analyzing the possible ranks of divisors of fixed degree on K_n.

1. INTRODUCTION

Brill-Noether theory is the study of line bundles on algebraic curves. Two important invariants of a line bundle are its degree and its rank, and it is common to study the space of line bundles with fixed degree and rank on a given curve C. In a famous series of results from the 1980's, it was shown that, if C is sufficiently general, then these spaces of line bundles are smooth [Gie82] of the expected dimension [GH80], and irreducible when this dimension is positive [FL81] . When C is not special, however, the situation is more mysterious.

The Brill-Noether theory of plane curves has been studied since at least the late 19th century, when Max Noether considered the possible ranks of line bundles of fixed degree on a plane curve $C \subset \mathbb{P}^2$ of degree n [Noe82]. This problem was solved simultaneously by Hartshorne [Har86] and Ciliberto [Cil84] a century later. More recently, Larson and Vemulapalli studied a more refined invariant of line bundles on plane curves, known as the *splitting type* [LV24]. Given a curve C, a line bundle $\mathcal{L}$ on C, and a map $\pi\colon C \to \mathbb{P}^1$ of degree n, the pushforward $\pi_*\mathcal{L}$ is a vector bundle of rank n on $\mathbb{P}^1$. Since every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles, there exists a sequence of integers $(e_1, \ldots, e_n)$, unique up to permutation, such that $\pi_*\mathcal{L} \cong \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(e_i)$. The sequence $(e_1, \ldots, e_n)$, called the *splitting type* of the line bundle $\mathcal{L}$, has attracted a great deal of recent interest [Lar21, CPJ22a, CPJ22b, LLV25]. In the case of a plane curve, the map $\pi\colon C \to \mathbb{P}^1$ is given by projection from a point in $\mathbb{P}^2$ not on C, and the splitting type is independent of the choice of point. In [LV24], Larson and Vemulapalli show that, for a general plane curve C, the locus of line bundles with fixed splitting type is smooth, and compute its dimension.

A standard way to approach problems in Brill-Noether theory is via degeneration. One degenerates to a singular curve and analyzes the limiting behavior of line bundles. In [Bak08], Baker develops a theory of divisors on graphs that is useful in such degeneration arguments. Briefly, a *divisor* on a graph is a formal $\mathbb{Z}$-linear combination of vertices of G. Baker defines an equivalence relation on divisors and a notion of rank. He shows that if $\mathfrak{C}$ is a regular semistable model over a discrete valuation ring, with general fiber C and whose special fiber has dual graph G, then there is a specialization map from the Picard group of C to that of G, and the rank cannot increase under specialization.

When studying plane curves of degree n, there is a natural choice of singular curve – the union of n general lines in $\mathbb{P}^2$. The dual graph of this singular curve is the complete graph K_n, which is the graph with n vertices labeled $v_1, \ldots, v_n$, and where every pair of vertices is adjacent. In the special case where $\mathfrak{C}$ is a flat family of plane curves whose special fiber is a union of n general lines, the class of a line specializes to the divisor $L = v_1 + v_2 + \cdots + v_n$ on the complete graph K_n.

In [CLB16], Cori and Le Borgne provide a formula for the rank of any divisor on the complete graph K_n. In [CP17], Cools and Panizzut use this to compute the possible ranks of divisors of fixed degree on K_n, providing a new proof of the Ciliberto-Hartshorne result for general plane curves. The main result of this note is a formula for the splitting type of a divisor on K_n. Because the splitting type of a divisor contains strictly more information than the rank and degree, this result generalizes the earlier work of both Cori-Le Borgne and Cools-Panizzut. In order to state our result, we first need some terminology.

Definition 1.1. *A divisor $D = \sum_{i=1}^{n} a_i v_i$ on the complete graph K_n is* concentrated *if, for every $i \leq n$, we have $\#\{j \mid a_j - \min\{a_k\} \leq i-1\} \geq i$.*

As noted in [CLB16], concentrated divisors with fixed minimum coefficient $\min\{a_k\}$ are in bijection with parking functions. In their paper, Cori and Le Borgne refer to a divisor as *parking* if $\min\{a_k \mid k \leq n-1\} = 0$ and, for every $i \leq n-1$, we have $\#\{j \leq n-1 \mid a_j \leq i-1\} \geq i$. Note that there is no condition on the coefficient a_n. We use the new term "concentrated" to distinguish these divisors from the parking divisors of Cori and Le Borgne, emphasizing that the conditions hold for all coefficients, including a_n. Our first result is that every divisor on K_n can be put into a standard form.

Proposition 1.2. *Every divisor on the complete graph K_n is equivalent to a concentrated divisor.*

Indeed, in Section 3, we provide an algorithm for computing a concentrated divisor equivalent to a given divisor on K_n.

Cori and Le Borgne define a divisor $D = \sum_{i=1}^{n} a_i v_i$ on the complete graph K_n ito be *sorted* if $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. Note again that there is no condition on a_n. We say that D is *super sorted* if $a_1 \leq a_2 \leq \cdots \leq a_n$. Given a divisor D on K_n, we can always choose a permutation of the vertices so that D is sorted or super sorted. This simplifies many of our arguments. Note that a super sorted divisor $D = \sum_{i=1}^{n} a_i v_i$ on the complete graph K_n is concentrated if and only if $0 \leq a_i - a_1 \leq i-1$ for all i.

For divisors in this standard form, there is a simple formula for the splitting type.

Theorem 1.3. *Let $D = \sum_{i=1}^{n} a_i v_i$ be a super sorted, concentrated divisor on the complete graph K_n, and let $e_i = a_i - i + 1$. Then D has splitting type $(e_1, \ldots, e_n)$.*

As a consequence, we see that the possible splitting types of divisors on K_n exactly match the possible splitting types of line bundles on a smooth plane curve of degree n.

Corollary 1.4. *There exists a divisor of splitting type $(e_1, \ldots, e_n)$ on K_n if and only if, when arranged in decreasing order $e_1 \geq e_2 \geq \cdots \geq e_n$, we have $e_i \leq e_{i+1} + 1$ for all i.*

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2. Preliminaries

2.1. **Divisors on Graphs.** Recall from the introduction that a *divisor* on a graph G is a formal $\mathbb{Z}$-linear combination of vertices of G. We think of a divisor $D = \sum_{i=1}^{n} a_i v_i$ as a configuration of poker chips on the vertices of the graph, where the vertex v_i has a_i chips. The *degree* of a divisor $D = \sum_{i=1}^{n} a_i v_i$ is the integer $\deg(D) = \sum_{i=1}^{n} a_i$. In other words, the degree of D is the total number of chips on the graph. Given a divisor D and a vertex v, we may *fire v* to obtain a new divisor D'. The chip firing move sends one chip from v to each of its neighbors. Thus, the divisor D' has one more chip on each vertex adjacent to v, and $\deg(v)$ fewer chips at v. In the particular case of the complete graph K_n, the divisor D' has one more chip on every vertex other than v, and $n-1$ fewer chips at v. We say that two divisors on a graph G are *equivalent* if one can be obtained from the other by a sequence of chip-firing moves. Note that the divisor L on K_n from the introduction is equivalent to the divisor nv_j for all j.

Given a subset A of the vertices of G, if we fire each of the vertices in A in any order, this results in sending one chip along each edge from A to its complement. We refer to this as firing the set A. On the complete graph K_n, if the set A has size j, then when fire A, each vertex in A loses $n-j$ chips, and each vertex not in A gains j chips.

2.2. **Reduced Divisors.** A divisor $D = \sum_{i=1}^{n} a_i v_i$ on a graph G is *effective* if $a_i \geq 0$ for all i. We say that D is *effective away from* a vertex v_j if $a_i \geq 0$ for all $i \neq j$. The divisor D is *v_j-reduced* if D is effective away from v_j, and firing any subset A not containing v_j results in a divisor that is not effective away from v_j. On the complete graph K_n, there is a simple characterization of v_n-reduced divisors.

Lemma 2.1. [CP17, Lemma 5] *A divisor $D = \sum_{i=1}^{n} a_i v_i$ on K_n is v_n-reduced if and only if it is a parking divisor in the sense of* [CLB16].

For any choice of vertex v, every divisor on G is equivalent to a unique v-reduced divisor. Given a divisor D that is effective away from v, there is an algorithm for computing the v-reduced divisor equivalent to D, known as *Dhar's Burining Algorithm*. For the vertex v_n on the complete graph K_n, this algorithm is simple to describe. First, find the maximum value of i such that $\#\{j \leq n-1 \mid a_i \leq i-1\} \leq i-1$. If no such i exists, then by Lemma 2.1, the divisor D is v_n-reduced and we are done. Otherwise, fire the set $A = \{v_j \mid j \leq n-1 \text{ and } a_j \geq i\}$. Each vertex in A loses $n-|A|$ chips and each vertex not in a gains $|A|$ chips. Since each vertex in A has at least i chips and $n-|A| \leq i$, the resulting divisor D' is still effective away from v_n. We then replace D with D' and iterate this procedue until the resulting divisor is v_n-reduced. Note that, each time we fire the set A, the total number of chips on vertices other than v_n decreases. Thus, this procedure terminates after finitely many steps.

2.3. **Ranks and Splitting Types.** In [BN07], Baker and Norine define the rank of a divisor on a graph.

Definition 2.2. *Let D be a divisor on a graph G. If D is not equivalent to an effective divisor, we say that it has* rank -1. *Otherwise, we define the* rank *of D to be the maximum integer r such that, for all effective divisors E of degree r, $D-E$ is equivalent to an effective divisor.*

The *canonical divisor* of a graph G is the divisor $K_G = \sum_{i=i}^{n}(\mathrm{val}(v_i)-2)v_i$. On the complete graph K_n, the canonical divisor K is equal to $(n-3)L$. In [BN07], Baker and Norine prove an analogue of the Riemann-Roch Theorem for graphs.

Theorem 2.3. [BN07, Theorem 1.12] *Let G be a graph with first Betti number g, and let D be a divisor on G. Then*

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1.$$

As a consequence, note that if $D - K_G$ is effective and nontrivial, then $\mathrm{rk}(D) = \deg(D) - g$.

The main result of [CLB16] is a formula for the rank of a divisor on a complete graph.

Theorem 2.4. [CLB16, Theorem 12][1] *Let D be a sorted, v_n-reduced divisor on the complete graph K_n. Let q and r be the unique integers such that $a_n + 1 = q(n-1) + r$ and $0 \leq r \leq n-2$. We define $\chi(P)$ to be 1 if the proposition P is true and 0 if it is false. Then*

$$\mathrm{rk}(D) = \Big(\sum_{i=1}^{n-1} \max\{0, q-i+1+a_i+\chi(i \leq r)\}\Big) - 1.$$

The goal of this paper is to compute the splitting type of divisors on the complete graph. Note that the splitting type of a line bundle $\mathcal{L}$ is completely determined by the ranks of the line bundles $\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^2}(k)$ for all integers k. More precisely,

$$\begin{aligned}
h^0(C, \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^2}(k)) &= h^0(\mathbb{P}^1, \pi_*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \\
&= h^0(\mathbb{P}^1, \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(e_i + k)) \\
&= \sum_{i=1}^{n} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e_i + k)) \\
&= \sum_{i=1}^{n} \max\{0, e_i + k + 1\}.
\end{aligned}$$

For this reason, we define the splitting type of a divisor on K_n as follows.

Definition 2.5. *Let $e_1, \ldots, e_n$ be integers. A divisor D on the complete graph K_n has* splitting type $(e_1, \ldots, e_n)$ *if*

$$\mathrm{rk}(D + kL) = \Big(\sum_{i=1}^{n} \max\{0, e_i + k + 1\}\Big) - 1 \textit{ for all } k \in \mathbb{Z}.$$

[1] Note that there is a typo in the formula appearing on page 3 of the published version of [CLB16]. The correct formula for the rank can be found on page 19.

3. Canonical Representatives of Divisors on Complete Graphs

We now prove Proposition 1.2.

Proof of Proposition 1.2. Let D be a divisor on the complete graph K_n. We provide an algorithm for producing a concentrated divisor equivalent to D. First, by running Dhar's Burning Algorithm, we may reduce to the case where D is v_n-reduced. It follows that $\min\{a_i \mid i \leq n-1\} = 0$ and, for every $i \leq n-1$, we have $\#\{j \leq n-1 \mid a_j \leq i-1\} \geq i$. Now, let m be the maximum integer such that $a_n - mn \geq 0$, and let $D' = D - mnv_n$. Note that $D' + mL$ is equivalent to D. Our goal is to show that $D' + mL$ is concentrated. Note that $D' + mL$ is concentrated if and only if D' is, so it suffices to show that D' is concentrated. By construction, $0 \leq a_n \leq n-1$, hence $\#\{j \mid a_j \leq n-1\} = n$. For $i \leq n-1$, we have

$$\#\{j \mid a_j \leq i-1\} \geq \#\{j \leq n-1 \mid a_j \leq i-1\} \geq i,$$

hence D' is concentrated. □

Note that the concentrated divisor equivalent to D is not unique. For example, the divisors $D = (n-1)v_n$ and $D' = v_1 + \cdots + v_{n-1}$ are equivalent, and both are concentrated.

Example 3.1. Figure 1 depicts the algorithm described in the proof of Proposition 1.2. In the upper left, we see the complete graph K_5, with its vertices labeled $v_1, \ldots, v_5$. In the middle figure of the top row, we see a divisor D on this graph. This divisor is not concentrated, since $6 - 0 = 6 > 4$. This divisor is also not v_5-reduced, because there are 6 chips on v_4. Applying Dhar's Burning Algorithm, we first fire v_4 to obtain the divisor on the top right. (Note that this divisor is concentrated, but the algorithm described in the proof of Proposition 1.2 has not yet terminated.) This divisor is still not v_5-reduced, because there is at least 1 chip on every vertex. We then fire the set $A = \{v_1, v_2, v_3, v_4\}$ to obtain the divisor on the bottom left, which is v_5-reduced by Lemma 2.1. This divisor has 6 chips at v_5, thus $m = 1$ is the largest integer such that $6 - 5m \geq 0$. The divisor $D - 1 \cdot L$ is equivalent to the divisor in the middle of the bottom row. Finally, adding the divisor L to this, we obtain the concentrated divisor on the bottom right. Thus we see that D is equivalent to a concentrated divisor.

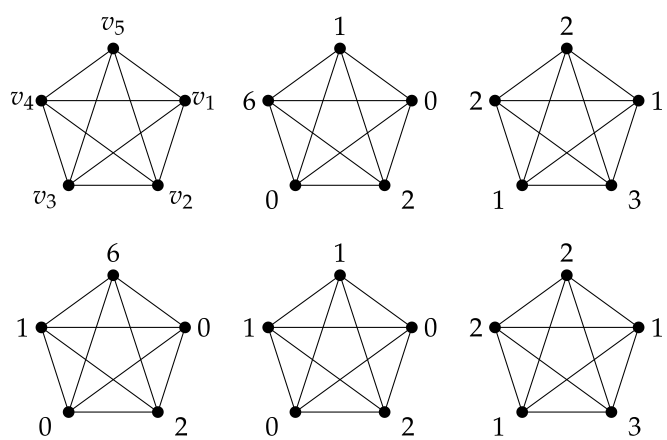

Figure 1. Finding a concentrated divisor.

4. Splitting Types of Divisors on Complete Graphs

We now prove the main theorem. The proof is essentially a calculation using Theorem 2.4, which is a bit lengthy due to the number of cases.

Proof of Theorem 1.3. Note that, if D has splitting type $(e_1, \ldots, e_n)$, then $D + mL$ has splitting type $(e_1 + m, \ldots, e_n + m)$. We may therefore further reduce to the case where $a_1 = 0$. In this case, by Lemma 2.1,

$D + knv_n$ is v_n-reduced for all integers k. We prove, by induction on k, that

$$\mathrm{rk}(D + knv_n) = \Big(\sum_{i=1}^{n} \max\{0, a_i - i + k + 2\}\Big) - 1 \tag{1}$$

for all integers k. For the base case, suppose that $k \leq -1$. Since $D + knv_n$ is v_n-reduced and has a negative number of chips at v_n, we see that $\mathrm{rk}(D + knv_n) = -1$. On the other hand, since $a_i \leq i - 1$ for all i, we have $a_i - i + k + 2 \leq 0$ for all i, hence (1) holds.

For the inductive step, it suffices to prove that

$$\mathrm{rk}(D + knv_n) = \mathrm{rk}(D + (k-1)nv_n) + \#\{i \mid a_i - i + 1 \geq -k\}. \tag{2}$$

The divisor $D + knv_n$ is sorted and v_n-reduced for all integers k, so we may apply Theorem 2.4. We let q, q', r, r' be the unique integers such that

$$\begin{aligned} a_n + (k-1)n + 1 &= q(n-1) + r \\ a_n + kn + 1 &= q'(n-1) + r' \\ 0 \leq r, r' &\leq n-2. \end{aligned}$$

By Theorem 2.4, we have

$$\begin{aligned} \mathrm{rk}(D + knv_n) - \mathrm{rk}(D + (k-1)nv_n) = \sum_{i=1}^{n-1} \Big(&\max\{0, q' - i + 1 + a_i + \chi(i \leq r')\} \\ &- \max\{0, q - i + 1 + a_i + \chi(i \leq r)\}\Big). \end{aligned} \tag{3}$$

We break this into several cases.

Case 1: First, consider the case where $0 \leq k \leq n - a_n - 3$. Note that the value n does not contribute to the sum (2). In this case, $q' = k$, $r = a_n + k$, $r' = r + 1$, and $q' = q + 1$. Thus, (3) becomes

$$\begin{aligned} \mathrm{rk}(D + knv_n) - \mathrm{rk}(D + (k-1)nv_n) = \sum_{i=1}^{n-1} \Big(&\max\{0, a_i - i + 1 + k + \chi(i \leq r + 1)\} \\ &- \max\{0, a_i - i + k + \chi(i \leq r)\}\Big). \end{aligned}$$

If $i \leq r$, then

$$\begin{aligned} \max\{0, a_i - i + 1 + k + \chi(i \leq r+1)\} &- \max\{0, a_i - i + k + \chi(i \leq r)\} \\ &= \max\{0, a_i - i + 2 + k\} - \max\{0, a_i - i + 1 + k\} \\ &= \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k - 1 \\ 1 & \text{if } a_i - i + 1 \geq -k. \end{cases} \end{aligned}$$

Thus, the value i contributes 1 to the sum (3) if and only if $a_i - i + 1 \geq -k$, if and only if it contributes 1 to the sum (2).

If $i \geq r + 2$, then since D is super sorted, we have $a_i - i + 1 \leq a_n - i + 1 \leq a_n - r - 1 = -k - 1$. Thus, the value i does not contribute to the sum (2). Now, we have

$$\begin{aligned} \max\{0, a_i - i + 1 + k + \chi(i \leq r+1)\} &- \max\{0, a_i - i + k + \chi(i \leq r)\} \\ &= \max\{0, a_i - i + 1 + k\} - \max\{0, a_i - i + k\} \\ &= \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k \\ 1 & \text{if } a_i - i + 1 \geq -k + 1. \end{cases} \end{aligned}$$

Thus, the value i does not contribute to the sum (3).

If $i = r+1$, then since D is super sorted, we have $a_i - i + 1 = a_i - r \leq a_n - r = -k$. Thus, the value i contributes 1 to the sum (2) if and only if $a_i - i + 1 = -k$. Now, we have

$$\begin{aligned}\max\{0, a_i - i + 1 + k + \chi(i \leq r+1)\} - \max\{0, a_i - i + k + \chi(i \leq r)\} \\ = \max\{0, a_i - i + 2 + k\} - \max\{0, a_i - i + k\} \\ = \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k-1 \\ 1 & \text{if } a_i - i + 1 = -k \\ 2 & \text{if } a_i - i + 1 \geq -k+1. \end{cases}\end{aligned}$$

Thus, the value i contributes 1 to the sum (3) if and only if $a_i - i + 1 = -k$. Putting this all together, we see that (2) holds in this case.

Case 2: Next, consider the case where $k = n - a_n - 2 \geq 0$. As before, the value n does not contribute to the sum (2). In this case, $q = k-1$, $q' = k+1$, $r = n-2$, and $r' = 0$. Thus, (3) becomes

$$\begin{aligned}\mathrm{rk}(D + knv_n) - \mathrm{rk}(D + (k-1)nv_n) = \sum_{i=1}^{n-1} \max\{0, k - i + 2 + a_i\} \\ - \sum_{i=1}^{n-2} \Big(\max\{0, a_i - i + 1 + k\} \Big) - \max\{0, a_{n-1} - n + 1 + k\}.\end{aligned}$$

If $i \leq n-2$, we have

$$\begin{aligned}\max\{0, a_i - i + 2 + k\} - \max\{0, a_i - i + 1 + k\} \\ = \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k-1 \\ 1 & \text{if } a_i - i + 1 \geq -k. \end{cases}\end{aligned}$$

Thus, the value i contriubtes 1 to the sum (3) if and only if $a_i - i + 1 \geq -k$, if and only if it contributes 1 to the sum (2).

If $i = n-1$, then since D is super sorted, we have $a_{n-1} - (n-1) + 1 \leq a_n - n + 2 = -k$. We have

$$\begin{aligned}\max\{0, a_{n-1} - (n-1) + 2 + k\} - \max\{0, a_{n-1} - n + 1 + k\} \\ = \begin{cases} 0 & \text{if } a_{n-1} - (n-1) + 1 \leq -k-1 \\ 1 & \text{if } a_{n-1} - (n-1) + 1 = -k \\ 2 & \text{if } a_{n-1} - (n-1) + 1 \geq -k+1. \end{cases}\end{aligned}$$

Thus, the value $i = n-1$ contributes 1 to the sum (3) if and only if $a_{n-1} - (n-1) + 1 = -k$, if and only if it contributes 1 to the sum (2). Putting this all together, we see that (2) holds in this case.

Case 3: Next, consider the case where $0 < n - a_n - 1 \leq k \leq n-3$. Here, the value n contributes 1 to the sum (2). In this case, $q' = k+1$, $r = a_n + k - (n-1)$, $r' = r+1$, and $q' = q+1$. Thus, (3) becomes

$$\begin{aligned}\mathrm{rk}(D + knv_n) - \mathrm{rk}(D + (k-1)nv_n) = \sum_{i=1}^{n-1} \Big(\max\{0, a_i - i + 2 + k + \chi(i \leq r+1)\} \\ - \max\{0, a_i - i + 1 + k + \chi(i \leq r)\} \Big).\end{aligned}$$

If $i \leq r$, then $a_i - i + 1 \geq -i + 1 \geq -r + 1 = n - k - a_n \geq -k + 1$. Thus, i contributes 1 to the sum (2). We have

$$\begin{aligned}\max\{0, a_i - i + 2 + k + \chi(i \leq r+1)\} - \max\{0, a_i - i + 1 + k + \chi(i \leq r)\} \\ = \max\{0, a_i - i + 3 + k\} - \max\{0, a_i - i + 2 + k\} \\ = \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k-2 \\ 1 & \text{if } a_i - i + 1 \geq -k-1. \end{cases}\end{aligned}$$

Thus, the value i contributes 1 to the sum (3) as well.

If $i \geq r+2$, then we have

$$\begin{aligned}\max\{0, a_i - i + 2 + k + \chi(i \leq r+1)\} - \max\{0, a_i - i + 1 + k + \chi(i \leq r)\} \\ = \max\{0, a_i - i + 2 + k\} - \max\{0, a_i - i + 1 + k\} \\ = \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k - 1 \\ 1 & \text{if } a_i - i + 1 \geq -k. \end{cases}\end{aligned}$$

Thus, the value i contributes to the sum (3) if and only if $a_i - i + 1 \geq -k$, if and only if it contributes 1 to the sum (2).

If $i = r+1$, then $a_i - i + 1 = a_i - r \geq -r = n - k - a_n - 1 \geq -k$. Thus, i contributes 1 to the sum (2). We have

$$\begin{aligned}\max\{0, a_i - i + 2 + k + \chi(i \leq r+1)\} - \max\{0, a_i - i + 1 + k + \chi(i \leq r)\} \\ = \max\{0, a_i - i + 3 + k\} - \max\{0, a_i - i + 1 + k\} \\ = \begin{cases} 0 & \text{if } a_i - i + 1 \leq -k - 2 \\ 1 & \text{if } a_i - i + 1 = -k - 1 \\ 2 & \text{if } a_i - i + 1 \geq -k. \end{cases}\end{aligned}$$

Thus, i contributes 2 to the sum (3). Since both i and n contribute 1 to the sum (2), we see that (2) holds in this case.

Case 4: Finally, consider the case where $k \geq n-2$. Note that the canonical divisor K is equal to $(n-3)L$. Since D is effective, we see that $D + kL - K$ is effective and nontrivial. It follows from Riemann-Roch that

$$\begin{aligned}\mathrm{rk}(D + knv_n) = kn - \binom{n-1}{2} + \sum_{i=1}^{n} a_i = \Big(\sum_{i=1}^{n}(a_i - i + k + 2)\Big) - 1 \\ = \Big(\sum_{i=1}^{n} \max\{0, a_i - i + k + 2\}\Big) - 1.\end{aligned}$$

□

Example 4.1. Consider the divisor D pictured on the left in Figure 2. In Example 3.1, we showed that D is equivalent to the concentrated divisor D' pictured on the right in Figure 2. By Theorem 1.3, we have

$$\begin{aligned}e_1 &= 1 - 1 + 1 = 1 \\ e_2 &= 1 - 2 + 1 = 0 \\ e_3 &= 2 - 3 + 1 = 0 \\ e_4 &= 2 - 4 + 1 = -1 \\ e_5 &= 3 - 5 + 1 = -1.\end{aligned}$$

Thus, the splitting type of the divisor D is $(1, 0, 0, -1, -1)$. Since all of the terms in the splitting type are greater than -2, the divisor D is nonspecial. In other words, the rank of D is 3, which is equal to that of a general divisor of degree 9 on a plane quintic. The splitting type of D, however, is not equal to that of a general divisor of degree 9 on a plane quintic, which is $(0, 0, 0, 0, -1)$. This can be seen from the fact that $D - L$ is effective, which is false for a general divisor of degree 9.

FIGURE 2. A divisor D (left) and its associated concentrated divisor D' (right).

Finally, we prove Corollary 1.4.

Proof of Corollary 1.4. First, let $D = \sum_{i=1}^{n} a_i v_i$ be a divisor on the complete graph, and let $(e_1, \ldots, e_n)$ be its splitting type. We show that, when arranged in decreasing order $e_1 \geq \cdots \geq e_n$, we have $e_i \leq e_{i+1} + 1$ for all i. By Proposition 1.2, we may assume that D is concentrated, and by permuting the vertices, we may assume that D is super sorted. Then $e_i = a_i - i + 1$ for all i. Since D is concentrated and super sorted, $e_i \leq e_1$ for all i. Because the sequence $(e_1, \ldots, e_n)$ obtains its maximum at e_1, it suffices to show that if $e_i < e_{i-1}$, then $e_{i-1} = e_i + 1$. But since $a_i \geq a_{i-1}$, if $e_i < e_{i-1}$, we see that $a_i = a_{i-1}$ and $e_{i-1} = e_i + 1$.

For the converse, let $(e_1, \ldots, e_n)$ be a splitting type with $e_i \geq e_{i+1} \geq e_i - 1$ for all i. Now, let $a_i = e_i + i - 1$ for all i, and let $D = \sum_{i=1}^{n} a_i v_i$. By assumption, we have $e_{i+1} \geq e_i - 1$ for all i, so $a_{i+1} \geq a_i$ for all i, hence D is super sorted. Also by assumption, we have $e_i \leq e_1$ for all i, so $a_i - a_1 \leq i - 1$ for all i, hence D is concentrated. By Theorem 1.3, it follows that the splitting type of D is $(e_1, \ldots, e_n)$. □

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