# IBL CHIP FIRING NOTES

These notes are intended for use in a class taught in an inquiry-based learning format. They contain very few proofs. Instead, the reader is expected to provide proofs of the stated theorems, propositions, and exercises on their own. Everything is presented in an order that guides the reader through the process of discovery. There is a "teacher's guide" containing proofs, available upon request.

## 1. Divisors on Graphs

### 1.1. **Basic Theory.**

Throughout these notes, all graphs are assumed to be connected and loopless, though possibly with multi-edges. Given a graph $G$, we write $V(G)$ for the set of vertices and $E(G)$ for the set of edges. Our main object of study is divisors on graphs.

**Definition 1.1.** *A* divisor $D$ *(or* chip configuration *or* sandpile*) on a graph $G$ is a formal $\mathbb{Z}$-linear combination of vertices of $G$,*

$$D = \sum_{v \in V(G)} D(v) \cdot v$$

*with $D(v) \in \mathbb{Z}$.*

For example, Figure 4 depicts a divisor on the wedge of two triangles. Here and elsewhere, a vertex with coefficient zero is undecorated.

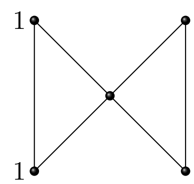


Figure 1. A divisor.

Divisors on graphs have been studied in combinatorics, computer science, and dynamics long before algebraic geometers got interested in them. In these disciplines it is more common to refer to divisors on graphs as *chip configurations* or *abelian sandpiles*. The term "chip configuration" comes from thinking of a divisor as a stack of poker chips on each vertex of the graph. Here we use the term *divisor* to emphasize the analogy with divisors on algebraic curves. Note that the divisors on a graph $G$ form an abelian group, which we denote $\mathrm{Div}(G) = \mathbb{Z}^{V(G)}$.

The chip-firing game is a game played with divisors on graphs, in which there is only one move. Starting with a divisor, we may "fire" a vertex, which results in that vertex giving a chip to each of its neighbors. More concretely, we have the following definition.

**Definition 1.2.** *The* chip-firing move *at a vertex $v$ takes a divisor $D$ to $D'$ where*

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v \\ D(v) + \#\text{ of edges between } w \text{ and } v & \text{if } w \neq v. \end{cases}$$

In our example, if we fire the top left vertex, we get the divisor pictured in Figure 2.

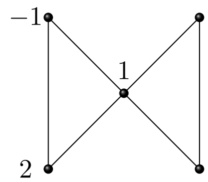


FIGURE 2. The result of firing a vertex.

**Exercise 1.3.** Let $D$ be a divisor on a graph $G$, and let $v, w \in V(G)$. Show that the divisor obtained from $D$ by first firing $v$ and then $w$ is the same as the divisor obtained from $D$ by first firing $w$ and then $v$.

**Exercise 1.4.** Let $D$ be a divisor on a graph $G$, and let $A \subseteq V(G)$. What is the effect of firing each of the vertices in $A$ exactly once (in any order)?

Firing each vertex in the subset $A$ is sometimes referred to as the *cluster-fire* of $A$.

**Definition 1.5.** *Two divisors $D, D'$ are* linearly equivalent*, and we write $D \sim D'$, if $D'$ can be obtained from $D$ by a sequence of chip-firing moves.*

**Lemma 1.6.** *Linear equivalence of divisors is an equivalence relation.*

**Lemma 1.7.** *If $D_1 \sim D_2$ then, for any divisor $E$, we have $D_1 + E \sim D_2 + E$.*

**Definition 1.8.** *A divisor that is equivalent to 0 is called a* principal divisor*. We denote the set of principal divisors by* $\mathrm{Prin}(G)$*.*

*The* Picard group *of a graph $G$ is the group of linear equivalence classes of divisors on $G$. That is,*

$$\mathrm{Pic}(G) = \mathrm{Div}(G)/\mathrm{Prin}(G).$$

1.2. **The Graph Laplacian.** To compute the Picard group of a graph $G$, it helps to have an algebraic description of the principal divisors. This is accomplished by way of the Laplacian matrix.

**Definition 1.9.** *The* graph Laplacian *of a graph $G$ is the square matrix with rows and columns indexed by the vertices of $G$, and whose $(i, j)$th entry is*

$$\Delta_{i,j} = \begin{cases} \mathrm{val}(v_i) & \text{if } i = j \\ -\#\text{ of edges between } v_i \text{ and } v_j & \text{if } i \neq j. \end{cases}$$

*That is, $\Delta$ is the difference of the valency matrix and the adjacency matrix.*

**Exercise 1.10.** Compute the graph Laplacian of the graph pictured in Figure 3[1].

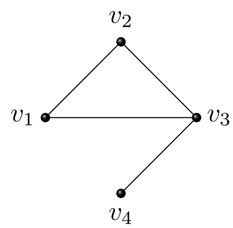


FIGURE 3. A simple graph.

The graph Laplacian defines a map:

$$\mathbb{Z}^{V(G)} \xrightarrow{\Delta} \mathbb{Z}^{V(G)} = \mathrm{Div}(G).$$

**Theorem 1.11.** *Let $D$ be a divisor on a graph $G$. The following are equivalent:*

(1) *$D$ is a principal divisor,*
(2) *$D \in \mathrm{Im}(\Delta)$, and*
(3) *There exists a function $f\colon V(G) \to \mathbb{Z}$ such that $D(v) = \sum_{e=vw}(f(w) - f(v))$ for all $v$, where the sum is over all edges with one endpoint $v$.*

**Corollary 1.12.** $\mathrm{Pic}(G) = \mathbb{Z}^{V(G)}/\mathrm{Im}(\Delta)$.

**Exercise 1.13.** Let $G$ be the graph from Exercise 1.10.

(1) Compute the Smith normal form of the matrix $\Delta$.
(2) Show that $\mathrm{Pic}(G) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

1.3. **The degree of a divisor.** We now consider a fundamental invariant of divisors on graphs.

**Definition 1.14.** *The* degree *of a divisor $D = \sum_{v \in V(G)} D(v)v$ is the integer*

$$\deg(D) = \sum_{v \in V(G)} D(v).$$

**Exercise 1.15.** The degree of a divisor is invariant under chip-firing.

**Lemma 1.16.** *The degree is a surjective group homomorphism from* $\mathrm{Pic}(G)$ *to* $\mathbb{Z}$.

**Definition 1.17.** *The* Jacobian $\mathrm{Jac}(G)$ *of a graph $G$ is the group of linear equivalence classes of divisors of degree 0 on $G$.*

The Jacobian is also known as the *sandpile group* or *critical group*. Again, we use the term *Jacobian* to emphasize the connection with algebraic curves.

**Proposition 1.18.** *For any graph $G$, we have* $\mathrm{Pic}(G) \cong \mathbb{Z} \oplus \mathrm{Jac}(G)$.

[1]In the film "Good Will Hunting", the first of the two problems to appear on the blackboard is a four-parter, the first part of which is to compute the Laplacian of this graph.

For an integer $d$, let $\mathrm{Div}^d(G)$ denote the set of divisors on $G$ of degree $d$, and let $\mathrm{Pic}^d(G)$ denote the set of equivalence classes of divisors on $G$ of degree $d$.

**Lemma 1.19.** *The group* $\mathrm{Jac}(G)$ *acts freely and transitively on* $\mathrm{Pic}^d(G)$ *by addition. (In other words,* $\mathrm{Pic}^d(G)$ *is a* $\mathrm{Jac}(G)$*-torsor.)*

**Exercise 1.20.** Let $T$ be a tree. Show that any two adjacent vertices of $T$ are linearly equivalent. Conclude that *any* two vertices of $T$ are linearly equivalent, and the Jacobian of $T$ is trivial.

**Exercise 1.21.** Let $G$ be a cycle with $n$ vertices. Label the vertices of $G$ counter-clockwise with the elements of $\mathbb{Z}/n\mathbb{Z}$. Show that the map $\mathrm{Div}(G) \to \mathbb{Z}/n\mathbb{Z}$ given by

$$\sum_{i=0}^{n-1} a_i v_i \mapsto \sum_{i=0}^{n-1} a_i i \pmod n$$

is invariant under linear equivalence. Use this to prove that $\mathrm{Jac}(G) \cong \mathbb{Z}/n\mathbb{Z}$.

**Exercise 1.22.** Let $G_1$ and $G_2$ be graphs, and let $G$ be the graph obtained by connecting a single vertex of $G_1$ to a single vertex of $G_2$ by an edge. Show that

$$\mathrm{Jac}(G) \cong \mathrm{Jac}(G_1) \times \mathrm{Jac}(G_2).$$

1.4. **Finiteness of the graph Jacobian.** The following series of exercises shows that the Jacobian of a graph is a finite group. Later, in Corollary 2.20, we will compute the order of this group.

**Lemma 1.23.** *Let $G$ be a graph, $\Delta$ its graph Laplacian, and $\mathbf{x} \in \mathbb{R}^{V(G)}$. Then*

$$\mathbf{x}^T \Delta \mathbf{x} = \sum_{i,j \in E(G)} (\mathbf{x}_i - \mathbf{x}_j)^2.$$

**Corollary 1.24.** *The graph Laplacian is a positive semi-definite matrix.*

Recall that we assume throughout that our graphs are connected.

**Lemma 1.25.** *Let $G$ be a graph and $\Delta$ its Laplacian. The rank of $\Delta$ is $|V(G)| - 1$.*

**Corollary 1.26.** *The Jacobian of a graph is a finite group.*

The following corollary will be useful in the next subsection.

**Corollary 1.27.** *For all $D \in \mathrm{Div}^0(G)$, there exists $\mathbf{x} \in \mathbb{Q}^{V(G)}$ such that $\Delta \mathbf{x} = D$.*

1.5. **The Energy and Monodromy Pairings.** Lemma 1.25 shows that the graph Laplacian $\Delta$ is not invertible. However, every matrix has a generalized inverse.

**Definition 1.28.** *A* generalized inverse *of a matrix $\Delta$ is a matrix $L$ such that*

$$\Delta L \Delta = \Delta.$$

The next lemma shows that a generalized inverse of the graph Laplacian exists.

**Lemma 1.29.** *Let $G$ be a graph, $\Delta$ its graph Laplacian, and $\Delta_j$ the matrix obtained by deleting the $j$th row and $j$th column of $\Delta$. The matrix $\Delta_j$ is invertible, and the matrix $L_j$ obtained by adding a zero row and zero column to $\Delta_j^{-1}$ is a generalized inverse of $\Delta$.*

Given a generalized inverse $L$ of $\Delta$, we define the *energy pairing*

$$\langle , \rangle \colon \operatorname{Div}^0(G) \times \operatorname{Div}^0(G) \to \mathbb{Q}$$

by

$$\langle D_1, D_2 \rangle = D_1^T L D_2.$$

**Lemma 1.30.** *The energy pairing is independent of the choice of generalized inverse.*

**Lemma 1.31.** *Let $D_1, D_2, E \in \operatorname{Div}^0(G)$. If $D_1 \sim D_2$, then*

$$\langle D_1, E \rangle = \langle D_2, E \rangle \pmod{\mathbb{Z}}.$$

By Lemma 1.31, the energy pairing descends to a pairing

$$\overline{\langle , \rangle} \colon \operatorname{Jac}(G) \times \operatorname{Jac}(G) \to \mathbb{Q}/\mathbb{Z}$$

called the *monodromy pairing*.

**Lemma 1.32.** *The monodromy pairing is nondegenerate. That is, if $\overline{\langle \cdot, D \rangle}$ is identically zero in $\mathbb{Q}/\mathbb{Z}$, then $D$ is principal.*

## 2. Effective and Reduced Divisors

2.1. **Effective Divisors.** The main topic of this section is the theory of $v$-reduced divisors, which are canonical representatives of divisor classes on graphs, depending only on the choice of a base vertex $v$. We will prove the existence and uniqueness of $v$-reduced divisors, along with some of the fundamental properties that make them essential tools in this subject.

**Definition 2.1.** *A divisor $D = \sum_{v \in V(G)} D(v)v$ is* effective *if $D(v) \geq 0$ for all $v \in V(G)$.*

Not every divisor on a graph is equivalent to an effective divisor. For example, a divisor of negative degree cannot be effective. A divisor of degree 0 is effective if and only if it is the zero divisor. To determine whether a given divisor is equivalent to an effective divisor, we use the theory of $v$-reduced divisors. These are canonical representatives of divisor classes in $\operatorname{Pic}(G)$, depending only on the choice of a base vertex $v$.

**Definition 2.2.** *Let $G$ be a graph and let $v \in V(G)$. A divisor $D = \sum_{w \in V(G)} D(w)w$ is* effective away from $v$ *if $D(w) \geq 0$ for all $w \neq v$.*

**Lemma 2.3.** *Every divisor is equivalent to one that is effective away from $v$. (Hint: use Corollary 1.26.)*

2.2. **Reduced Divisors.** For a subset $A \subseteq V(G)$, let $\chi_A$ denote the characteristic function. Note that $\Delta\chi_A$ is the divisor obtained from 0 by firing all vertices of $A$ (see Exercise 1.4). The following is the key definition.

**Definition 2.4.** *A divisor $D$ is $v$-*reduced *if it is effective away from $v$, and for any subset $A \subseteq V(G) \smallsetminus \{v\}$, $D + \Delta\chi_A$ is not effective away from $v$.*

The primary goal of this section is to prove that every divisor is equivalent to a unique $v$-reduced divisor. We break this into separate steps.

**Lemma 2.5.** *Let $D$ and $E$ be divisors on a graph $G$, both of degree zero. Let $A \subseteq V(G)$. If $E = D + \Delta\chi_A$, then*

$$\langle E, E\rangle = \langle D, D\rangle + \sum_{v\in A}(D+E)(v).$$

**Lemma 2.6.** *Let $v \in V(G)$, and let $D$ be a divisor of degree zero that is effective away from $v$. If $D$ is not $v$-reduced, then there exists $E \sim D$, also effective away from $v$, such that $\langle E, E\rangle > \langle D, D\rangle$.*

**Proposition 2.7.** *Let $G$ be a graph, $v \in V(G)$. Every divisor is equivalent to a $v$-reduced divisor.*

Theorem 2.9 below shows that every divisor is equivalent to a *unique* $v$-reduced divisor. A useful tool for proving this is the following, which Baker and Shokrieh call the Maximum Principle.

**Lemma 2.8.** *Let $\mathbf{x} \in \mathbb{Z}^{V(G)}$, and let $A \subseteq V(G)$ be the set of vertices where $\mathbf{x}$ obtains its maximum. Then*

$$\Delta\mathbf{x}(v) \geq \operatorname{outdeg}_v(A)$$

*for all $v \in A$.*

**Theorem 2.9.** *Let $G$ be a graph, $v \in V(G)$. Every divisor is equivalent to a* unique *$v$-reduced divisor.*

2.3. **Dhar's Burning Algorithm.** In the previous section, we saw that every divisor is equivalent to a unique $v$-reduced divisor. In this section, we describe a procedure for computing this $v$-reduced divisor. This is known as Dhar's Burning Algorithm.

Let $G$ be a graph, $v \in V(G)$, and $D$ a divisor on $G$ that is effective away from $v$. Dhar's Burning Algorithm proceeds as follows:

(1) Start a fire at $v$.
(2) Burn every edge of the graph that is adjacent to a burnt vertex.
(3) If the number of edges adjacent to a vertex $w$ exceeds the number of chips $D(w)$ at $w$, burn $w$. If no such vertex exists, proceed to step (4). Otherwise, return to step (2).
(4) Let $A$ be the set of unburnt vertices. If $A$ is non-empty, then $D + \Delta\chi_A$ is effective away from $v$. Replace $D$ with $D + \Delta\chi_A$ and return to step (1). Otherwise, if $A$ is empty, then $D$ is $v$-reduced.

An example of Dhar's Burning Algorithm appears on the next page.

**Example 2.10.** We run Dhar's Burning Algorithm to compute the $v$-reduced divisor equivalent to the divisor pictured in Figure 4, where $v$ is the lower right vertex.

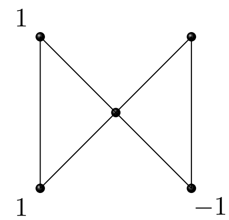


FIGURE 4. A divisor that is effective away from $v$

After burning vertices and edges until there are none left to burn, we see that the two vertices on the left remain unburnt. Therefore, the divisor pictured in Figure 5 is not $v$-reduced.

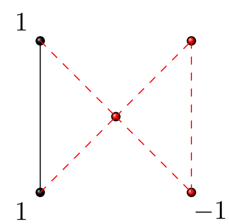


FIGURE 5. First iteration of Dhar's Burning Algorithm

Firing the two vertices on the left, we obtain the divisor pictured in Figure 6.

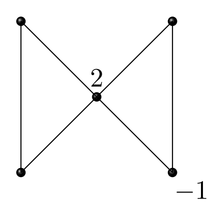


FIGURE 6. Result of firing the left two vertices

Running a second iteration of Dhar's Burning algorithm, we see that the three left vertices are unburnt, as in Figure 7.

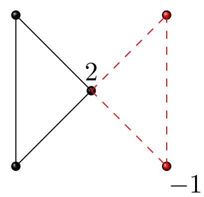


Figure 7. Second iteration of Dhar's Burning Algorithm

Firing these three vertices, we obtain the divisor pictured in Figure 8.

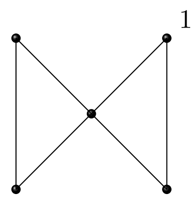


Figure 8. Result of firing the left three vertices

Running Dhar's Burning Algorithm a final time, we see that the whole graph burns. Therefore, the divisor depicted in Figure 8 is $v$-reduced.

**Exercise 2.11.** Let $v$ be the top right vertex in Figure 9. Find the $v$-reduced divisor equivalent to the divisor depicted below.

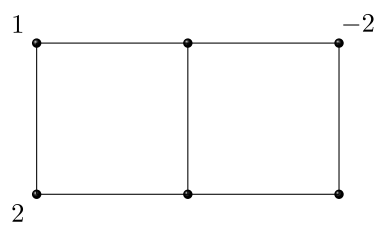


Figure 9

**Exercise 2.12.** Prove that Dhar's Burning Algorithm works. That is, in step (4), the divisor is $v$-reduced if and only if the set $A$ of unburnt vertices is empty.

**Proposition 2.13.** *Let $G$ be a graph and $v \in V(G)$. A divisor $D$ is equivalent to an effective divisor if and only if its $v$-reduced representative is effective.*

**Exercise 2.14.** Is the divisor depicted in Figure 9 equivalent to an effective divisor? Why or why not?

2.4. **Cori-Le Borgne Algorithm.** In the previous section, we described Dhar's Burning Algorithm as burning multiple edges at once. In the Cori-Le Borgne Algorithm, we instead fix a total order on the set $E(G)$ of edges. Now we run Dhar's Burning Algorithm, but burn edges one at a time. Whenever there are multiple edges that are eligible to burn, we burn the smallest one first. Whenever a vertex $v$ burns, we mark the edge along which the fire travled just prior to burning $v$.

**Lemma 2.15.** *Let $D$ be a $v$-reduced divisor. The set of marked edges under the Cori-Le Borgne Algorithm is a spanning tree.*

**Example 2.16.** Consider the divisor pictured in Figure 10, with edges labeled from 1 to 6. If $v$ is the bottom right vertex, then this divisor is $v$-reduced.

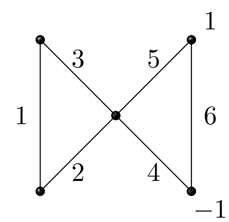


FIGURE 10. A $v$-reduced divisor of degree 0

Running the Cori-Le Borgne Algorithm, the edge 4 burns first, and then the central vertex, so we mark edge 4. The edge 2 burns next, and then the bottom left vertex, so we mark edge 2. The edge 1 burns next, and then the top left vertex, so we mark edge 1. The edge 3 burns, followed by edge 5, followed by edge 6. Note that the top right vertex does not burn until edge 6 does, so we mark edge 6. The resulting spanning tree is pictured in Figure 11.

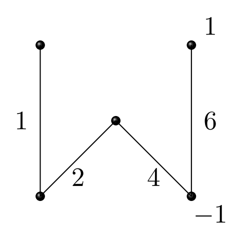


FIGURE 11. The corresponding spanning tree

The key insight of Cori and Le Borgne is that this procedure produces a bijection between $v$-reduced divisors of degree zero and spanning trees. To see this, we describe an inverse algorithm, which takes as input a spanning tree $T$, and returns a $v$-reduced divisor of degree zero:

(1) Start a fire at $v$.

(2) As before, burn every edge of the graph that is adjacent to a burnt vertex one at a time, starting with the smallest one.
(3) At the moment that vertex $w$ is adjacent to an burnt edge in $T$, set $D(w)$ to be one less than the number of burnt vertices adjacent to $w$, and then burn $w$. If all of the vertices are burnt, proceed to step (4). Otherwise, return to step (2).
(4) Once all the vertices are burnt, set $D(v) = -\sum_{w\neq v} D(w)$.

**Example 2.17.** Consider the spanning tree in Figure 11 from Example 2.16. Running the inverse Cori-LeBorgne Algorithm, the edges burn in the order 4, 2, 1, 3, 5, 6. All vertices $w$ but the top right are adjacent to only 1 burnt edge at the moment they burn, so we set $D(w) = 0$. The top right vertex $x$ does not burn until edge 6 burns, at which point it is adjacent to 2 burnt edges, so we set $D(x) = 1$. Finally, we set $D(v) = 0$, so that the total degree is zero.

**Lemma 2.18.** *The output of the inverse Cori-Le Borgne Algorithm is a $v$-reduced divisor.*

**Proposition 2.19.** *The two algorithms defined by Cori and Le Borgne are inverses.*

**Corollary 2.20.** *For any graph $G$, $|Jac(G)|$ is equal to the number of spanning trees in $G$.*

**Exercise 2.21.** Use Corollary 2.20 to give short solutions to Exercises 1.20 and 1.21.

## 3. Orientable Divisors and Break Divisors

In this section, we introduce divisors corresponding to graph orientations. This was an essential ingredient in the Baker-Norine proof of the Riemann Roch theorem for graphs. Readers who wish to skip the proof of Riemann-Roch may choose to read Subsection 3.1 and then skip the rest of this section.

### 3.1. The Canonical Divisor.

**Definition 3.1.** *The* genus *of a graph $G$ is*

$$g = |E(G)| - |V(G)| + 1.$$

The genus of a graph is its first Betti number. In other words, it is the rank of $H^1(G, \mathbb{Z})$. We use the term genus to emphasize the analogy between divisors on graphs and divisors on algebraic curves. This should not be confused with the other common graph invariant known as the genus, which is the minimal genus of a surface in which the graph can be embedded without crossings.

Another invariant of a graph is its canonical divisor, defined as follows.

**Definition 3.2.** *The canonical divisor of a graph $G$ is the divisor*

$$K_G = \sum_{v\in V(G)} (\mathrm{val}(v) - 2)v.$$

The degree of the canonical divisor is computed in Corollary 3.6 below. While this computation follows directly from the theory of orientable divisors, this theory is not necessary to prove Corollary 3.6.

3.2. **Orientable Divisors.** Much of the theory we have developed concerning divisors on graphs can also be formulated in terms of graph orientations. The connection between divisors and graph orientations begins with orientable divisors.

**Definition 3.3.** *Let $G$ be a graph and $\mathcal{O}$ an orientation of $G$. The corresponding* orientable divisor *is*

$$D_{\mathcal{O}} := \sum_{v \in V(G)} (\mathrm{indeg}_{\mathcal{O}}(v) - 1)v.$$

**Lemma 3.4.** *If $\mathcal{O}$ is an orientation and $\overline{\mathcal{O}}$ is the reverse orientation, then*

$$D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K_G.$$

Another simple fact about orientable divisors is that they all have the same degree.

**Lemma 3.5.** *Let $G$ be a graph of genus $g$. Every orientable divisor on $G$ has degree $g-1$.*

**Corollary 3.6.** *If $G$ is a graph of genus $g$, then* $\deg(K_G) = 2g - 2$.

A key connection between chip firing and graph orientations is the following observation.

**Proposition 3.7.** *Let $G$ be a graph and $\mathcal{O}$ an orientation of $G$. Let $A \subseteq V(G)$ be a subset of the vertices with the property that all edges in the cut $(A, A^c)$ are directed toward $A$, and let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by reversing this directed cut. Then*

$$D_{\mathcal{O}'} = D_{\mathcal{O}} + \Delta\chi_A.$$

*In particular, $D_{\mathcal{O}'}$ is equivalent to $D_{\mathcal{O}}$.*

**Definition 3.8.** *Let $G$ be a graph and $v \in V(G)$. An orientation $\mathcal{O}$ of $G$ is called* $v$-connected *if, for every vertex $w$ in $G$, there is a directed path in $\mathcal{O}$ from $v$ to $w$.*

**Corollary 3.9.** *Let $G$ be a graph and $\mathcal{O}$ an orientation of $G$. There exists a $v$-connected orientation $\mathcal{O}'$ such that $D_{\mathcal{O}} \sim D_{\mathcal{O}'}$. (Hint: let $A \subseteq V(G)$ be the set of vertices that can be reached from $v$ by a directed path in $\mathcal{O}$, and use Proposition 3.7.)*

Our next goal is to show that every divisor of degree $g-1$ on a graph of genus $g$ is equivalent to an orientable divisor. To see this, we present an algorithm that starts with a divisor $D$ of degree $g-1$ and an orientation $\mathcal{O}$. At each step, it either modifies the orientation $\mathcal{O}$ or replaces $D$ with an equivalent divisor. When the algorithm terminates, we will have $D = D_{\mathcal{O}}$. The algorithm proceeds as follows:

(1) Define

$$A^+ = \{v \in V(G) \mid D(v) > D_{\mathcal{O}}(v)\}$$
$$A^- = \{v \in V(G) \mid D(v) < D_{\mathcal{O}}(v)\}$$
$$A = \{v \in V(G) \mid \text{ there exists a directed path from a vertex in } A^+ \text{ to } v\}.$$

(2) If $A^+ = \emptyset$, then $D = D_{\mathcal{O}}$ and the algorithm terminates.
(3) Otherwise, if $A \cap A^- \neq \emptyset$, then there is an oriented path in $\mathcal{O}$ from a vertex $v \in A^+$ to a vertex $w \in A^-$. Let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by reversing this directed path, and replace $\mathcal{O}$ with $\mathcal{O}'$. Return to step (1).

(4) Otherwise, if $A \cap A^- = \emptyset$, then by definition, the cut $(A, A^c)$ is directed away from $A$. Let $\mathcal{O}'$ be the orientation obtained by reversing this directed cut. Replace $\mathcal{O}$ with $\mathcal{O}'$ and $D$ with $D + \Delta\chi_A$. Return to step (1).

**Theorem 3.10.** *This algorithm terminates. As a consequence, if $G$ is a graph of genus $g$, then every divisor on $G$ of degree $g-1$ is equivalent to an orientable divisor.*

Theorem 3.10 has the following interesting consequence.

**Corollary 3.11.** *Let $G$ be a graph of genus $g$. Every divisor of degree at least $g$ on $G$ is equivalent to an effective divisor.*

The bound in Corollary 3.11 is sharp. In other words, on any graph of genus $g$, there exist divisors of degree $g-1$ that are not equivalent to an effective divisor.

**Lemma 3.12.** *Let $\mathcal{O}$ be an acyclic orientation of a graph $G$. Then $D_{\mathcal{O}}$ is not equivalent to an effective divisor. (Hint: use Lemma 2.8.)*

**Corollary 3.13.** *Let $G$ be a graph of genus $g$. There exists a divisor of degree $g-1$ on $G$ that is not equivalent to an effective divisor.*

In fact, acyclic orientations can be used to distinguish between divisors that are equivalent to effective divisors and those that are not.

**Lemma 3.14.** *Let $G$ be a graph, $v \in V(G)$, and let $D$ be a $v$-reduced divisor. Define an orientation $\mathcal{O}$ on $G$ by running Dhar's Burning Algorithm and orienting each edge from the first endpoint that burns to the second endpoint that burns. Then $D(w) \leq D_{\mathcal{O}}(w)$ for all $w \neq v$.*

**Corollary 3.15.** *For any divisor $D$ on a graph $G$, either $D$ is equivalent to an effective divisor or there is an acyclic orientation $\mathcal{O}$ such that $D_{\mathcal{O}} - D$ is equivalent to an effective divisor (but not both).*

**Corollary 3.16.** *Let $G$ be a graph of genus $g$. A divisor $D$ on $G$ of degree $g-1$ is equivalent to an effective divisor if and only if $K_G - D$ is equivalent to an effective divisor.*

By Theorem 3.10, every divisor of degree $g-1$ on a graph of genus $g$ is equivalent to a $v$-connected orientable divisor. We will now show that this $v$-connected orientable divisor is unique.

**Theorem 3.17.** *Let $\mathcal{O}$ and $\mathcal{O}'$ be orientations of a graph $G$, and let $v$ be a vertex of $G$. If $D_{\mathcal{O}}$ and $D_{\mathcal{O}'}$ are equivalent but not equal, then at most one of them is $v$-connected. (Hint: let $D_{\mathcal{O}'} = D_{\mathcal{O}} + \Delta\mathbf{x}$, let $A$ be the set where $\mathbf{x}$ achieves its maximum, and consider $\deg(D_{\mathcal{O}'}|_A)$.)*

3.3. **Break Divisors.** In previous sections, we have studied canonical representatives of divisor classes on graphs. More specifically, given a vertex $v$ on a graph of genus $g$, every divisor class contains a unique $v$-reduced representative, and every divisor class of degree $g-1$ contains a unique $v$-connected orientable representative. A drawback of these theories is that these representatives depend on the choice of vertex $v$. In this lecture, we discuss canonical representatives of degree $g$ divisors that do not depend on any choices.

**Definition 3.18.** *Let $G$ be a graph, and let $T$ be a spanning tree of $G$. For each edge $e$ not in $T$, let $v_e$ be one of its endpoints. A divisor of the form*

$$D = \sum_{e \notin T} v_e$$

*is called a* break divisor.

By definition, a break divisor is effective of degree $g$.

**Exercise 3.19.** Find all break divisors in the graph pictured in Figure 12.

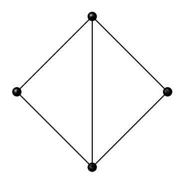

FIGURE 12. A graph of genus 2

There is a direct connection between break divisors and orientable divisors.

**Proposition 3.20.** *Let $G$ be a graph and $v \in V(G)$. A divisor $D$ on $G$ is a break divisor if and only if $D - v$ is a $v$-connected orientable divisor.*

**Corollary 3.21.** *Let $G$ be a graph of genus $g$. Every divisor of degree $g$ on $G$ is equivalent to a unique break divisor.*

## 4. The Rank of a Divisor

### 4.1. Basics on Rank.

A key invariant of a divisor is its (Baker-Norine) *rank*. It is this invariant that powers the connection between chip firing and algebraic geometry.

**Definition 4.1.** *Let $D$ be a divisor on a graph. If $D$ is not equivalent to an effective divisor, we say that $D$ has* rank $-1$. *Otherwise, we define the* rank *of $D$ to be the largest integer $r$ such that, for all effective divisors $E$ of degree $r$, $D - E$ is equivalent to an effective divisor.*

Computing the rank of a divisor can be thought of as a game, in which our opponent is allowed to "steal" $r$ chips from wherever they like, and our task is to perform a sequence of chip firing moves that eliminates the debt created by our opponent. If we can win this game regardless of which $r$ chips our opponent chooses to steal, then the divisor has rank at least $r$.

**Exercise 4.2.** A divisor has nonnegative rank if and only if it is equivalent to an effective divisor.

**Exercise 4.3.** Compute the ranks of the two divisors of degree 2 depicted in Figure 13.

**Exercise 4.4.** Let $T$ be a tree, and let $D$ be a divisor on $T$ of nonnegative degree. Show that $\mathrm{rk}(D) = \deg(D)$.

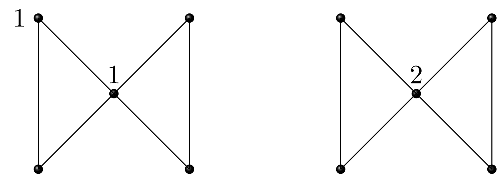

Figure 13. Two divisors of the same degree on a graph of genus 2

**Exercise 4.5.** Let $G$ be a cycle with $n$ vertices, and let $D$ be a divisor on $G$ of positive degree. Show that $\mathrm{rk}(D) = \deg(D) - 1$.

We record a few other observations about ranks of divisors.

**Lemma 4.6.** *Let $D$ be a divisor on a graph of genus $g$. Then* $\mathrm{rk}(D) \geq \deg(D) - g$.

**Lemma 4.7.** *Let $D_1, D_2$ be divisors of nonnegative rank on a graph $G$. Then*

$$\mathrm{rk}(D_1 + D_2) \geq \mathrm{rk}(D_1) + \mathrm{rk}(D_2).$$

**Exercise 4.8.** Let $G$ be a simple bipartite graph, and let $D$ be the sum of the vertices of a single color. Show that $\mathrm{rk}(D) \geq 1$.

**Lemma 4.9.** *A divisor $D$ has rank at least 1 if and only if, for all $v \in V(G)$, the $v$-reduced divisor $D_v$ equivalent to $D$ satisfies $D_v(v) \geq 1$.*

4.2. **Riemann-Roch.** In this section, we prove possibly the most important result about ranks of divisors on graphs, the Riemann-Roch Theorem. While we have developed all the necessary background to prove this theorem, the results in this section are nevertheless still quite difficult. We begin with the following definition.

**Definition 4.10.** *Let $D$ be a divisor on a graph $G$. We define*

$$\deg^+(D) = \sum_{v \in V(G), D(v) > 0} D(v).$$

Baker and Norine give an alternate characterization of the rank.

**Proposition 4.11.** *Let $D$ be a divisor on a graph $G$. Then*

$$\mathrm{rk}(D) = \min_{\substack{D' \sim D \\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - 1.$$

*(Hint: prove two inequalities, one using Lemma 3.12 and the other using Lemma 3.14.)*

**Theorem 4.12** (Riemann-Roch Theorem)**.** *Let $D$ be a divisor on a graph $G$ of genus $g$. Then*

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1.$$

4.3. **Consequences of Riemann-Roch.** We now explore some consequences of the Riemann-Roch theorem.

**Corollary 4.13.** *Let $D$ be a divisor on a graph of genus $g$. If* $\deg(D) > 2g - 2$, *then* $\mathrm{rk}(D) = \deg(D) - g$.

By Corollary 4.13, if a divisor has large degree, then its rank is completely determined by its degree. Similarly, if a divisor has negative degree, then it has rank $-1$. It follows that, on a given graph $G$, there are only finitely many divisors whose rank is not determined by their degree. In the edge cases, when the degree of a divisor is $0$ or $2g-2$, there are two possibilities.

**Corollary 4.14.** *Let $D$ be a divisor on a graph of genus $g$. If $\deg(D) = 2g-2$, then*

$$\mathrm{rk}(D) = \begin{cases} g-1 & \text{if } D \sim K_G \\ g-2 & \text{otherwise.} \end{cases}$$

Our next consequence of Riemann-Roch is usually referred to as the Clifford bound.

**Theorem 4.15.** *Let $D$ be a divisor on a graph $G$, and suppose that both $D$ and $K_G - D$ have nonnegative rank. Then*

$$\mathrm{rk}(D) \leq \frac{1}{2}\deg(D).$$

**Exercise 4.16.** Let $G$ be a graph of genus $g$, and let $D$ be a divisor on $G$ of degree 2 and rank 1. Compute the rank of $mD$ for all integers $m$. Conclude that $K_G \sim (g-1)D$.

As mentioned above, there are only finitely many divisors on a given graph $G$ whose rank is not determined by their degree. For each such divisor, we see that

$$\max\{-1, \deg(D) - g\} \leq \mathrm{rk}(D) \leq \frac{1}{2}\deg(D).$$

There are therefore only finitely many possibilities for the rank of such a divisor. The possible pairs $(d, r)$, where $d$ and $r$ are the degree and rank, respectively, of a divisor, are illustrated in Figure 14.

Of particular interest are the divisors that have larger than expected rank. These divisors correspond to lattice points in Figure 14 that lie above the lower bound.

**Definition 4.17.** *A divisor $D$ on a graph of genus $g$ is called* special *if*

$$\mathrm{rk}(D) > \max\{-1, \deg(D) - g\}.$$

**Exercise 4.18.** Use Riemann-Roch to give short proofs of Exercises 4.4 and 4.5.

**Exercise 4.19.** Find two graphs, each of genus 3, one of which has a divisor of degree 2 and rank 1 and the other of which does not.

### 4.4. **Ramification.**

**Definition 4.20.** *Let $D$ be a divisor of rank $r$ on a graph $G$, and let $v$ be a vertex of $G$. The sequence*

$$a_0 < a_1 < \cdots < a_r$$

*defined by*

$$a_i := \max\{m \mid \mathrm{rk}(D - mv) \geq r - i\}$$

*is called the* ramification sequence *of $D$ at $v$. We say that $v$ is a* ramification point *of $D$ if the ramification sequence of $D$ at $v$ is anything other than $0 < 1 < \cdots < r$.*

**Exercise 4.21.** Show that a divisor on a tree has no ramification points.

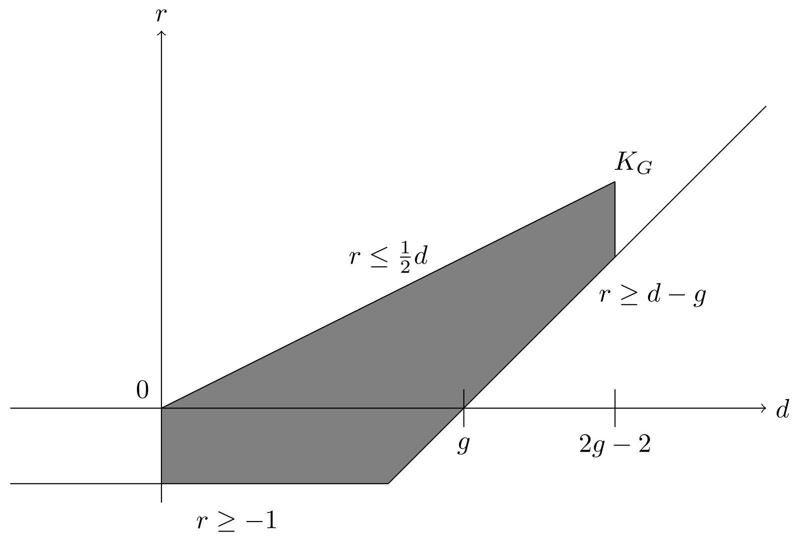


FIGURE 14. Possibilities for the degree and rank of a divisor.

**Exercise 4.22.** Let $D$ be a divisor of degree $d > 0$ on a cycle, and let $v$ be a vertex of the cycle. Show that $v$ is a ramification point of $D$ if and only if $D \sim dv$.

**Exercise 4.23.** In Figure 15, which of the vertices $v$ and $w$ is a ramification point of the canonical divisor? Ramification points of the canonical divisor are typically referred to as *Weierstrass points*.

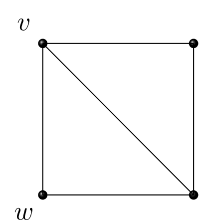


FIGURE 15. A graph of genus 2.

It is traditional to express the ramification sequence using partitions. In what follows, we identify the boxes in the Ferrers diagram of a partition with lattice points in $\mathbb{Z}^2_{>0}$[2].

[2]We have chosen to depict partitions in the English style, which has the unfortunately reflects the $y$-axis from the standard coordinate system. In particular, the box $(1, 1)$ appears in the *upper* left of the Ferrers diagram, and the box $(1, 2)$ appears *below* it.

**Definition 4.24.** *Let $D$ be a divisor on a graph $G$ of genus $g$, and let $v$ be a vertex of $G$. We define the* Weierstrass partition *of $D$ at $v$ to be the partition*

$$\lambda_{G,v}(D) := \{(r+1, g-d+r) | \operatorname{rk}(D - (\deg(D) - d)v) \geq r\}.$$

**Lemma 4.25.** *The Weierstrass partition is a partiion. That is, if $(x,y) \in \lambda_{G,v}(D)$, then:*

(1) *either $x = 1$ or $(x-1, y) \in \lambda_{G,v}(D)$, and*
(2) *either $y = 1$ or $(x, y-1) \in \lambda_{G,v}(D)$.*

We now record several other simple facts about Weierstrass partitions.

**Lemma 4.26.** *Let $G$ be a graph of genus $g$ and let $v$ be any vertex of $G$. A divisor $D$ on $g$ has rank at least $r$ if and only if*

$$(r+1, g - \deg(D) + r) \in \lambda_{G,v}(D).$$

One nice aspect of this definition is that the Weierstrass partition is invariant under addition of the vertex $v$.

**Lemma 4.27.** *Let $G$ be a graph and let $D$ be a divisor on $G$. For any vertex $v$ of $G$, we have $\lambda_{G,v}(D) = \lambda_{G,v}(D+v)$.*

The term $g - d + r$ in the definition of the Weierstrass partition may appear mysterious, but it is motivated by Riemann-Roch. There is a natural involution on the set of partitions given by the transpose. There is also a natural involution on the set of divisors given by mapping a divisor $D$ to $K_G - D$. The Weierstrass partition is defined so that these two involutions agree.

**Proposition 4.28.** *Let $G$ be a graph and $v$ a vertex of $G$. For any divisor $D$ on $G$, we have $\lambda_{G,v}(K_G - D) = \lambda^T_{G,v}(D)$.*

**Exercise 4.29.** Let $G$ be the graph in Figure 15. Compute the Weierstrass partitions $\lambda_{G,w}(K_G)$ and $\lambda_{G,v}(K_G)$.

4.5. **Extended Example: A Chain of Loops.** In this section, we compute Weierstrass partitions (and therefore ranks) of divisors on a certain family of graphs. To begin, we consider the graph $G$ pictured in Figure 16. Specifically, we let $G'$ be a graph of genus $g-1$, and $v$ a vertex of $G$. We let $C$ be a cycle with $m$ vertices, labeled counterclockwise by $v_0, \ldots, v_{m-1}$. We let $G$ be the graph obtained by connecting the vertex $v$ of $G'$ to the vertex $v_0$ of $C$. Our goal is to compute the ranks of divisors on this graph $G$.

**Lemma 4.30.** *Let $D$ be a divisor on $G'$. Then $\operatorname{rk}_G(D + v_i) \geq r$ if and only if:*

(1) *$\operatorname{rk}_{G'}(D) \geq r$ when $i \neq 0$, or*
(2) *$\operatorname{rk}_{G'}(D+v) \geq r$ and $\operatorname{rk}(D-v) \geq r-1$, when $i = 0$.*

We now translate Lemma 4.30 into the language of Weierstrass partitions.

**Proposition 4.31.** *Let $D$ be a divisor on $G'$. Then $\lambda_{G',v}(D) \subseteq \lambda_{G,v_j}(D+v_i)$. Moreover, a box $(x,y) \notin \lambda_{G',v}(D)$ is contained in the Weierstrass partition $\lambda_{G,v_j}(D+v_i)$ if and only if the following conditions hold:*

(1) *either $x = 1$ or $(x-1, y) \in \lambda_{G',v}(D)$,*

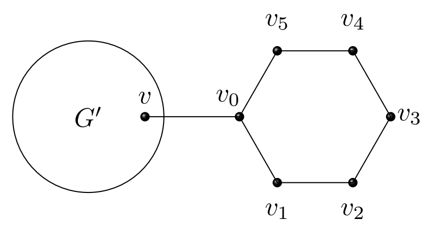


FIGURE 16. A graph with an attached cycle.

(2) *either* $y = 1$ *or* $(x, y-1) \in \lambda_{G',v}(D)$, *and*
(3) $i \equiv (\deg(D) - g - x + y)j \pmod{m}$.

We now use Proposition 4.31 to compute the ranks of divisors on the graph pictured in Figure 17. These graphs were studied heavily by Cools, Draisma, Payne and Robeva in their tropical proof of the Brill-Noether theorem. They were later studied in more depth by Pflueger, and our analysis follows his closely. We assume that the bottom part of each cycle is a single edge, while the top part of the $k$th cycle consists of $m_k - 1$ edges. (So the total number of edges in the $k$th cycle is $m_k$.) We define $\vec{m} = (m_1, \ldots, m_g)$, and we refer to this graph as the chain of $g$ loops with torsion profile $\vec{m}$.

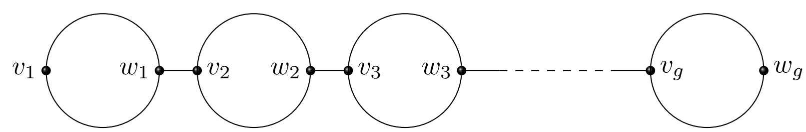


FIGURE 17. A chain of loops.

Let $D$ be a divisor on this graph. By Lemma 4.27, we have

$$\lambda_{G,w_g}(D) = \lambda_{G,w_g}(D + (g - \deg(D))w_g),$$

so we may assume that $D$ has degree $g$. By Corollary 3.21, every divisor of degree $g$ is equivalent to a unique break divisor, so we may assume that $D$ is a break divisor. In other words, $D$ has exactly 1 "chip" on each cycle of $G$. That is, the restriction of $D$ to any individual cycle in $G$ has degree 1.

Let $G_k$ be the union of the first $k$ cycles of $G$, and for ease of notation, let

$$\lambda_k = \lambda_{G_k,w_k}(D|_{G_k}).$$

By Proposition 4.31, we have

$$\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_g.$$

Moreover, a box $(x, y) \notin \lambda_{k-1}$ is contained in $\lambda_k$ if and only if:

(1) either $x = 1$ or $(x-1, y) \in \lambda_{k-1}$,
(2) either $y = 1$ or $(x, y-1) \in \lambda_{k-1}$, and
(3) the distance from $w_k$ to the chip of $D$ on $\gamma_k$, in the counterclockwise direction, is equivalent to $y - x \pmod{m_k}$.

This sequence of partitions defines a tableau $t$ on the partition $\lambda_g$, defined by

$$t(x,y) = k \text{ if } (x,y) \in \lambda_k \smallsetminus \lambda_{k-1}.$$

This tableau has the property that, if $t(x,y) = t(x',y') = k$, then $y - x \equiv y' - x' \pmod{m_k}$. Equivalently, the lattice distance between the boxes $(x,y)$ and $(x',y')$ is divisible by $m_k$. We say that a tableau with this property is an *$\vec{m}$-displacement tableau*.

Conversely, given an $\vec{m}$-displacement tableau $t$, we define

$$\lambda_k = \{(x,y) | t(x,y) \leq k\}.$$

We may then construct a break divisor $D$ such that $\lambda_k \subseteq \lambda_{G_k,w_k}(D|_{G_k})$ for all $k$, as follows. If $t(x,y) = k$, then we place a chip on the $k$th loop, at a distance of $y - x \pmod{m_k}$ from $w_k$, in the counterclockwise direction. This is well-defined by the definition of $\vec{m}$-displacement tableaux. If the symbol $k$ does not appear in the tableau $t$, then we place a chip at any vertex of the $k$th loop.

**Lemma 4.32.** *There exists a divisor of degree $d$ and rank at least $r$ on the chain of $g$ loops with torsion profile $\vec{m}$ if and only if there exists an $\vec{m}$-displacement tableau with alphabet $\{1,\ldots,g\}$ on the rectangular partition with $r+1$ columns and $g-d+r$ rows.*

**Corollary 4.33.** *Suppose that $m_k > g$ for all $k$. Then there exists a divisor of degree $d$ and rank $r$ on the chain of $g$ loops with torsion profile $\vec{m}$ if and only if*

$$g - (r+1)(g-d+r) \geq 0.$$

The number $\rho(g,r,d) := g - (r+1)(g-d+r)$ in Corollary 4.33 is known as the Brill-Noether number. The fact that these graphs possess a divisor of degree $d$ and rank $r$ if and only if the Brill-Noether number is nonnegative is a key step in the tropical proof of the Brill-Noether theorem.

**Exercise 4.34.** Let $G$ be a chain of 4 loops, and suppose that $m_k \neq 2$ for all $k$. Show that there are exactly two divisor classes of degree 3 and rank 1 on $G$.

**Exercise 4.35.** Let $G$ be a chain of 4 loops. Show that $G$ has a divisor of degree 2 and rank 1 if and only if $m_2 = m_3 = 2$.

**Exercise 4.36.** Let $G$ be a chain of 6 loops, and suppose that $m_k > 3$ for all $k$. Show that there are exactly five divisor classes of degree 4 and rank 1 on $G$.

**Exercise 4.37.** Let $G$ be a chain of $2d-2$ loops, and suppose that $m_k \geq d$ for all $k$. Show that the number of divisor classes of degree $d$ and rank 1 on $G$ is equal to the Catalan number $C_{d-1}$.

## 5. Metric Graphs

5.1. **Divisors on Metric Graphs.** We now turn from discrete graphs to metric graphs.

**Definition 5.1.** *A* metric graph *is a compact, connected metric space $\Gamma$ obtained by identifying the edges of a graph $G$ with line segments of fixed positive real length. The graph $G$ is called a* model *for $\Gamma$*

**Example 5.2.** If we assign lengths to the edges of a cycle, we obtain a circle. Thus, the circle is a metric graph.

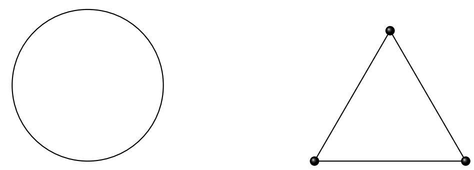

FIGURE 18. A metric graph and one of its models

A metric graph $\Gamma$ does not have a unique model. Two graphs are models for the same metric graph if and only if they admit a common refinement.

**Definition 5.3.** *The* divisor group $\mathrm{Div}(\Gamma)$ *of a metric graph* $\Gamma$ *is the free abelian group on points of the metric space* $\Gamma$*.*

Note that a divisor on a metric graph $\Gamma$ is supported on *points* of $\Gamma$, not *vertices* of a model. In particular, a divisor can have a chip at any point on the interior of an edge. Many properties of divisors can be defined in a way that is completely analogous to the discrete graph case.

**Definition 5.4.** *A divisor* $D = \sum a_i v_i$ *on a metric graph is* effective *if* $a_i \geq 0$ *for all* $i$*. Its* degree *is defined to be*

$$\deg(D) := \sum a_i.$$

As in the case of discrete graphs, we want to talk about equivalence of divisors. For this, we need a notion of rational functions on metric graphs.

**Definition 5.5.** *A* rational function *on a metric graph* $\Gamma$ *is a continuous, piecewise linear function* $\varphi : \Gamma \to \mathbb{R}$ *with integer slopes. We write* $\mathrm{PL}(\Gamma)$ *for the group of rational functions on* $\Gamma$*.*

**Example 5.6.** Figure 19 indicates the domains of linearity and slopes of a rational function $\varphi$ on a circle. It therefore determines the rational function up to translation. Note that, in order for the function to be continuous, the two regions on which the function has slope 1 must be of equal length.

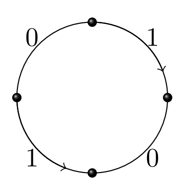


FIGURE 19. The domains of linearity and slopes of $\varphi$

**Definition 5.7.** *Given* $\varphi \in \mathrm{PL}(\Gamma)$ *and* $v \in \Gamma$*, we define the* order of vanishing *of* $\varphi$ *at* $v$*,* $\mathrm{ord}_v(\varphi)$*, to be the sum of the incoming slopes of* $\varphi$ *at* $v$*. Note that* $\mathrm{ord}_v(\varphi)$ *is nonzero for only finitely many points* $v \in \Gamma$*. We define the divisor associated to* $\varphi$ *to be*

$$\mathrm{div}(\varphi) = \sum_{v \in \Gamma} \mathrm{ord}_v(\varphi) \cdot v.$$

*Divisors of the form* $\mathrm{div}(\varphi)$ *are called* principal*.*

**Exercise 5.8.** Let $\varphi$ be the rational function of Example 5.6. Find $\mathrm{div}(\varphi)$.

Note that $\mathrm{div}(\varphi)$ is equal to $\mathrm{div}(\varphi + c)$ for any real number $c$. This is analogous to the fact that, on an algebraic curve, the divisor associated to a rational function is invariant under scaling the function by a non-zero constant. Indeed, we have the following.

**Lemma 5.9.** *Let* $\Gamma$ *be a metric graph and let* $\varphi, \psi \in \mathrm{PL}(\Gamma)$*. We have* $\mathrm{div}(\varphi) = \mathrm{div}(\psi)$ *if and only if there exists a constant* $c$ *such that* $\varphi = \psi + c$*.*

**Example 5.10.** On any metric graph $\Gamma$, given a point $v \in \Gamma$, let $\epsilon \in \mathbb{R}$ be sufficiently small so that the open ball $B_\epsilon(v)$ contains no points of valence greater than 2 other than possibly $v$. Let $\chi$ be the rational function that takes the value $\epsilon$ on $\Gamma \smallsetminus B_\epsilon(v)$, the value 0 at $v$, and has slope 1 on the edges in $B_\epsilon(v)$ emanating from $v$. Then $\chi$ has order of vanishing $-\mathrm{val}(v)$ at $v$ and 1 at each of the boundary points of $B_\epsilon(v)$. In this way, we can view addition of $\mathrm{div}(\chi)$ as a continuous version of chip firing, where we specify not only the vertex $v$ that we fire from, but also the distance $\epsilon$ that we fire the chips.

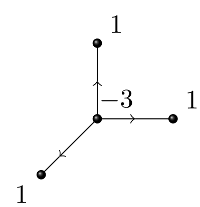


FIGURE 20. A rational function that is constant outside a local neighborhood, and its associated divisor
bottom

We note the following.

**Lemma 5.11.** *The degree of a principal divisor is zero.*

**Lemma 5.12.** *The map* $\mathrm{div}\colon \mathrm{PL}(\Gamma) \to \mathrm{Div}(\Gamma)$ *is a group homomorphism.*

Now that we have a notion of principal divisors on metric graphs, we can use it to define equivalence of divisors.

**Definition 5.13.** *We say that two divisors $D$ and $D'$ on a metric graph $\Gamma$ are* equivalent *if $D - D'$ is principal. We define the* Picard group *of $\Gamma$ to be the group of equivalence classes of divisors on $\Gamma$. That is,*

$$\mathrm{Pic}(\Gamma) = \mathrm{Div}(\Gamma)/\operatorname{div}(\mathrm{PL}(\Gamma)).$$

*The* Jacobian $\mathrm{Jac}(\Gamma)$ *of $\Gamma$ is the group of equivalence classes of divisors of degree zero.*

The Jacobian of a metric graph is related to the Jacobian of a discrete graph.

**Lemma 5.14.** *Let $G$ be a graph, let $\Gamma$ be the associated metric graph with all edge lengths 1, and let $D \in \mathrm{Div}(G)$. Then $D$ is principal on $G$ if and only if it is principal on $\Gamma$.*

**Corollary 5.15.** *Let $G$ be a graph, let $\Gamma$ be the associated metric graph with all edge lengths 1, and let $D_1, D_2 \in \mathrm{Div}(G)$. Then $D_1$ and $D_2$ are equivalent divisors on $G$ if and only if they are equivalent divisors on $\Gamma$.*

**Corollary 5.16.** *Let $G$ be a graph and let $\Gamma$ be the associated metric graph with all edge lengths 1. Then $\mathrm{Jac}(G)$ is a subgroup of $\mathrm{Jac}(\Gamma)$.*

5.2. **Reduced Divisors on Metric Graphs.** We now develop the theory of reduced divisors on metric graphs, analogous to the corresponding theory for discrete graphs. We begin with the definition.

**Definition 5.17.** *Let $\Gamma$ be a metric graph and $v \in \Gamma$. A divisor $D$ on $\Gamma$ is* $v$-reduced *if*

(1) *$D$ is effective away from $v$ and*
(2) *every closed connected subset $A \subseteq \Gamma \setminus \{v\}$ contains a point $x$ with $D(x) < \mathrm{outdeg}_A(x)$.*

This is reminiscent of the definition for discrete graphs, but with a set of vertices replaced with a connected closed set. As in the discrete case, we will show that every divisor on a metric graph is equivalent to a unique $v$-reduced divisor. In this way, $v$-reduced divisors give a natural choice of representatives for divisors classes on metric graphs. We begin by showing that $v$-reduced divisors, if they exist, are unique.

**Theorem 5.18.** *If $D \sim D'$ are $v$-reduced, then $D = D'$. (Hint: mimic the proof in the discrete case.)*

To prove the existence of $v$-reduced divisors, we first need the following lemma.

**Lemma 5.19.** *Let $D \in \mathrm{Div}(\Gamma)$, $v \in \Gamma$, and let $G$ be a model for $\Gamma$ with vertex set containing $\{v\} \cup \mathrm{supp}(D)$. Then $D$ is $v$-reduced if and only if the corresponding divisor $D$ on $G$ is $v$-reduced.*

**Remark 5.20.** Note that Lemma 5.19 does not imply that the $v$-reduced divisor on $\Gamma$ corresponds to the $v$-reduced divisor on $G$. In particular, since our choice of model $G$ depends on the divisor $D$, replacing $D$ with an equivalent divisor may force us to change the model.

We prove the existence of a $v$-reduced divisor equivalent to a given divisor $D$ in two parts, starting with the case where $D$ is effective away from $v$. Let $G$ be a model for $\Gamma$ with vertex set containing $v$. Choose an ordering of $V(G) \cup E(G)$ such that every edge is adjacent to a vertex that precedes it, and every vertex other than $v$ is adjacent to an edge that precedes it. This induces a quasi-order[3] on points of $\Gamma$, which in turn induces the lexicographic quasi-order on divisors of $\Gamma$.

**Lemma 5.21.** *Let $D$ be a divisor that is effective away from $v$. If $D$ is not $v$-reduced, then there is a divisor equivalent to $D$, also effective away from $v$, that is strictly larger in the lexicographic quasi-order.*

**Theorem 5.22.** *If $D$ is effective away from $v$, then $D$ is equivalent to a $v$-reduced divisor.*

To handle the case where $D$ is not effective, we will introduce the rank of divisors on metric graphs. This is defined in exactly the same way as on a discrete graph.

**Definition 5.23.** *Given $D \in \mathrm{Div}(\Gamma)$, the* complete linear series *of $D$ is*

$$|D| := \{D' \sim D \mid D' \geq 0\}.$$

*The* rank *of $D$ is the largest integer $r$ such that $|D - E| \neq \emptyset$ for all effective divisors $E$ of degree $r$.*

**Lemma 5.24.** *Any effective divisor of degree at least $g + r$ on a metric graph $\Gamma$ of genus $g$ has rank at least $r$. (Hint: induct on $r$, and us Lemma 5.19.)*

We now complete the proof that every divisor on a metric graph is equivalent to a unique $v$-reduced divisor. Recall that, in the discrete case, we used the fact that the Jacobian is finite, and hence every element of the Jacobian is torsion, to complete the proof. This argument will not work in the case of metric graphs, where the Jacobian has non-torsion elements.

**Corollary 5.25.** *Every divisor on $\Gamma$ is equivalent to a divisor that is effective away from $v$.*

**Corollary 5.26.** *Let $\Gamma$ be a metric graph, $v \in \Gamma$. Every divisor on $\Gamma$ is equivalent to a unique $v$-reduced divisor.*

5.3. **Riemann-Roch for Metric Graphs.** In this section, we prove the Riemann-Roch theorem for metric graphs. In their seminal paper, Baker and Norine describe a general strategy for proving theorems of Riemann-Roch type. Specifically, let $X$ be a non-empty set and let $\mathrm{Div}(X)$ be the free abelian group on $X$. As usual, we define the degree of a divisor to be the sum of its coefficients, and we say that a divisor is effective if all of its coefficients are nonnegative. Let $\sim$ be an equivalence relation on $\mathrm{Div}(X)$ satisfying:

(1) if $D \sim D'$, then $\deg(D) = \deg(D')$, and
(2) if $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_2 \sim D_1' + D_2'$.

[3] A quasi-order is a reflexive, transitive binary relation such that, for any $v, w$, either $v \leq w$ or $w \leq v$ (or possibly both). It is like a total order, but distinct elements may have the same order.

We can define a rank function on $\mathrm{Div}(X)$ by declaring $\mathrm{rk}(D)$ to be $-1$ if $D$ is not equivalent to an effective divisor, and otherwise declaring $\mathrm{rk}(D)$ to be the largest integer $r$ such that $D-E$ is equivalent to an effective divisor for all effective divisors $E$ of degree $r$. Finally, let $g$ be an integer, and let $K \in \mathrm{Div}(X)$ be a divisor of degree $2g-2$. Baker and Norine prove the following.

**Theorem 5.27.** *The Riemann-Roch formula*

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1$$

*holds for every $D \in \mathrm{Div}(X)$ if and only if the following 2 conditions hold:*

(1) *For every $D \in \mathrm{Div}(X)$, either $\mathrm{rk}(D) \geq 0$, or there exists a divisor $D'$ of degree $g-1$ and rank $-1$ such that $\mathrm{rk}(D'-D) \geq 0$.*
(2) *For every $D \in \mathrm{Div}(X)$ of degree $g-1$, if $\mathrm{rk}(D) = -1$, then $\mathrm{rk}(K-D) = -1$.*

We now show that the 2 conditions hold for divisors on metric graphs. For the first condition, we must identify a suitably large collection of divisors of degree $g-1$ and rank $-1$. In the case of discrete graphs, these were the orientable divisors.

**Definition 5.28.** *An* orientation *on a metric graph $\Gamma$ is an orientation of some model for $\Gamma$. As in the discrete case, given an orientation $\mathcal{O}$, define*

$$D_{\mathcal{O}} = \sum_{v \in \Gamma} (\mathrm{indeg}_{\mathcal{O}}(v) - 1)v.$$

**Lemma 5.29.** *If $\mathcal{O}$ is an acyclic orientation, then $D_{\mathcal{O}}$ is not equivalent to an effective divisor.*

We now show that the first condition holds.

**Lemma 5.30.** *For any $D \in \mathrm{Div}(\Gamma)$, either $D$ is equivalent to an effective divisor, or there exists an acyclic orientation $\mathcal{O}$ such that $D_{\mathcal{O}} - D$ is equivalent to an effective divisor. Moreover, for any $v \in \Gamma$, $\mathcal{O}$ can be taken to have unique source $v$. (Hint: choose an appropriate model for $\Gamma$, and then use Corollary 3.15.)*

The second condition follows from the previous two lemmas.

**Lemma 5.31.** *Let $D \in \mathrm{Div}(\Gamma)$ be a divisor of degree $g-1$. If $D$ is not equivalent to an effective divisor, then $K_\Gamma - D$ is not equivalent to an effective divisor.*

Combining Lemmas 5.30 and 5.31 with Theorem 5.27, we obtain the Riemann-Roch theorem for metric graphs.

**Theorem 5.32** (Riemann-Roch for Metric Graphs)**.** *Let $\Gamma$ be a metric graph of genus $g$. For any $D \in \mathrm{Div}(\Gamma)$,*

$$r(D) - r(K_\Gamma - D) = \deg(D) - g + 1.$$

5.4. **Rank Determining Sets.** On a discrete graph $G$, the rank of a divisor $D$ can be computed as follows. Choose a vertex $v$ of $G$. For each effective divisor $E$ of degree $r$, run Dhar's Burning Algorithm to compute the $v$-reduced divisor equivalent to $D-E$. The divisor $D$ has rank at least $r$ if and only if this $v$-reduced is effective for all $E$.

On a metric graph, however, this procedure is impossible to implement because for $r > 0$ there are infinitely many effective divisors of degree $r$. The goal of this section is to show that there exists a finite set of "test" divisors $E$ such that, if $|D - E|$ is nonempty for all $E$ in this finite set, then the divisor $D$ has rank at least $r$. This will make it feasible to compute the ranks of divisors on metric graphs. This idea is made precise by the notion of rank determining sets. We first make the following definition.

**Definition 5.33.** *Let $D$ be a divisor on a metric graph $\Gamma$. We define the* support *of the complete linear series $|D|$ to be*

$$\operatorname{supp}(|D|) := \{v \in \Gamma \mid D'(v) > 0 \text{ for some } D' \in |D|\}.$$

We say that a divisor $D$ has support in $A$ if $\operatorname{supp}(D)$ is contained in $A$.

**Definition 5.34.** *Let $\Gamma$ be a graph, and let $A$ be a subset of $\Gamma$.*

1. *The $A$-rank $r_A(D)$ of a divisor $D$ is the largest integer $r$ such that $|D - E|$ is nonempty for all effective divisors $E$ of degree $r$ with support in $A$.*
2. *The set $A$ is* rank determining *if $r_A(D) = r(D)$ for all $D \in \operatorname{Div}(G)$.*

**Lemma 5.35.** *For any subset $A \subseteq \Gamma$ and any divisor $D$, we have $r_A(D) \geq r(D)$.*

**Definition 5.36.** *Let $A \subseteq \Gamma$ be a subset. We define $\mathcal{L}(A)$ to be*

$$\mathcal{L}(A) = \bigcap_{\operatorname{supp}|D| \supseteq A} \operatorname{supp}|D|.$$

**Proposition 5.37.** *Let $A$ be a nonempty subset of $\Gamma$. The following are equivalent:*

1. *$\mathcal{L}(A) = \Gamma$.*
2. *If $D$ is a divisor with $r_A(D) \geq 1$, then $r(D) \geq 1$.*
3. *$A$ is a rank-determining set.*

*(Hint: to show that (2) implies (3), use induction on $r_A(D)$.)*

We now provide a topological condition to determine when $\mathcal{L}(A) = \Gamma$. To do this we define a YL set. These sets are named after Ye Luo, whose work this section is based upon.

**Definition 5.38.** *Let $\Gamma$ be a metric graph and $U \subseteq \Gamma$ a connected open subset. We call $U$ a* YL set *if every connected component $X$ of the complement $\Gamma \smallsetminus U$ contains a point $v$ such that* $\operatorname{outdeg}_X(v) > 1$.

We can characterize YL sets in terms of divisor theory.

**Lemma 5.39.** *Let $U \subseteq \Gamma$ be a nonempty connected open subset. Then $U$ is a YL set if and only if $D = \sum_{v \in \partial U} v$ is $w$-reduced for any $w \in U$.*

Given a divisor $D$ on $\Gamma$, we may use Lemma 5.39 to find YL sets that are disjoint from the support of $|D|$.

**Lemma 5.40.** *For $v \in \Gamma$, let $D$ be a $v$-reduced divisor, and let $U$ be the set of vertices that can be reached from $v$ by a path that does not pass through* $\operatorname{supp}(D) \smallsetminus \{v\}$. *Then $U$ is a YL set. Moreover, if $v \notin \operatorname{supp}(D)$, then $U$ is disjoint from* $\operatorname{supp}|D|$.

The following consequence of Lemma 5.40 is not necessary for our other results on rank-determining sets, but may be of independent interest.

**Corollary 5.41.** *Let $D$ be a divisor on $\Gamma$. Then* $(\mathrm{supp}|D|)^c$ *is a disjoint union of YL sets.*

We now turn to the main theorem of this lecture, which gives a sufficient condition for subsets of the vertices to be rank-determining.

**Theorem 5.42.** *Let $A \subseteq \Gamma$ be a nonempty subset. Then*

$$\mathcal{L}(A) \supseteq \bigcap_{\substack{U \text{ is YL} \\ A \cap U = \emptyset}} U^c.$$

*Moreover, if all YL sets intersect $A$, then $A$ is a rank determining set.*

**Remark 5.43.** Luo proves the stronger result that the containment of Theorem 5.42 is in fact an equality, from which he derives that this sufficient condition for subsets to be rank-determining is also necessary. For our purposes, we will only need the fact that this condition is sufficient.

We note the following interesting property of YL sets.

**Lemma 5.44.** *If $\Gamma$ is a metric graph of genus $g$ and $U$ is a YL set in $\Gamma$, then the closure $\overline{U}$ has genus at least 1. As a consequence, a collection of disjoint YL sets in $\Gamma$ can contain at most $g$ elements.*

The condition for rank determining sets provided in Theorem 5.42 is useful for many reasons. An important consequence of this result is that every metric graph contains a finite rank determining set.

**Theorem 5.45.** *Let $\Gamma$ be a metric graph of genus $g$, let $G$ be a model for $\Gamma$, let $T$ be a spanning tree in $G$, and let $e_1, \ldots, e_g$ be the edges of $G$ not in $T$. Choose a point $v_0 \in T$, and a point $v_i$ in the interior of $e_i$ for each $i$. Then $A = \{v_0, v_1, \ldots, v_g\}$ is a rank determining set. In particular, there exists a rank-determining set of cardinality $g + 1$.*

A metric graph of genus $g$ may have a rank determining set of cardinality less than $g + 1$.

**Exercise 5.46.** Let $\Gamma$ be a metric graph with model the complete graph $K_4$. Then $\Gamma$ has a rank determining set of cardinality 3.

**Exercise 5.47.** Let $\Gamma$ be a metric graph with model the complete graph $K_4$, where the edges have arbitrary lengths. Let $v_1, v_2, v_3 \in V(K_4)$ be distinct vertices. Show that $D = v_1 + v_2 + v_3$ has rank 1.

**Theorem 5.48.** *Let $G$ be a graph, let $\Gamma$ be the associated metric graph with all edge lengths 1, and let $D \in \mathrm{Div}(G)$. Then $\mathrm{rk}_G(D) = \mathrm{rk}_\Gamma(D)$.*