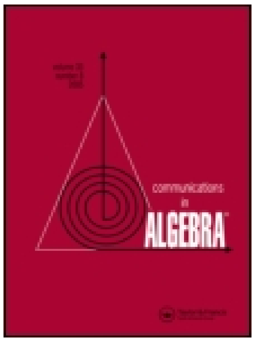

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COMPLETIONS OF UFDs WITH SEMI-LOCAL FORMAL FIBERS

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Let (T, M) be a complete local ring such that $|T/M| = |T|$. Given a finite set of incomparable nonmaximal prime ideals C of T, we provide necessary and sufficient conditions for T to be the completion of a local UFD A, whose generic formal fiber is semilocal with maximal ideals the elements of C. We also show that, given the T above, we can find necessary and sufficient conditions for T to be the completion of a UFD, whose formal fiber over a height one prime ideal is semilocal.

Key Words: Completion; Formal fiber; UFD.

Mathematics Subject Classification: 13H99.

1. INTRODUCTION

Because of Cohen's structure theorems, the structure of complete local rings is well known. The properties of the relationship between a local ring and its completion, however, are not as well understood. By making sense of this relationship, we can better comprehend local rings in general. A typical goal of research in this field is to characterize the completions of rings with certain special properties. In Lech (1986), it is shown that a complete local ring (T, M) is the completion of a local integral domain if and only if

1. unless $M = (0)$, M is not an associated prime ideal of T, and
2. no integer of T is a zerodivisor.

Similarly, in Heitmann (1993), the author provides necessary and sufficient conditions for a ring to be the completion of a unique factorization domain (UFD).

Theorem 1.1 (Heitmann, 1993). *Let T be a complete local ring. Then T is the completion of a UFD if and only if it is a field, a discrete valuation ring (DVR), or has depth at least two and no integer of T is a zero divisor.*

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Among the important aspects of the relationship between a local ring and its completion are the *formal fibers* of the ring. If A is a local ring with maximal ideal M and P is a prime ideal of A, we define the formal fiber of A at P to be $\operatorname{Spec}(\widehat{A} \otimes_A k(P))$, where $\widehat{A}$ is the M-adic completion of A, and $k(P) = A_P/PA_P$. Because there is a one-to-one correspondence between the elements of the formal fiber of A at P and the inverse image of P under the map $\operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$, we often think of the formal fiber of A at P as the inverse image of P under the map $\operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$. If A is an integral domain, we call the formal fiber of A at the zero ideal the *generic formal fiber* of A. If the ring $\widehat{A} \otimes_A k(P)$ is semilocal with maximal ideals $Q_1 \otimes_A k(P), \ldots, Q_n \otimes_A k(P)$, we say that the formal fiber of A at P is semilocal with maximal ideals $Q_1, \ldots, Q_n$. In Charters and Loepp (2004), the authors characterize those rings that are completions of integral domains with semilocal generic formal fiber. Specifically, they provide necessary and sufficient conditions on a complete local ring T and a set of prime ideals G of T with finitely many maximal elements for the existence of a local domain A that completes to T and has semilocal generic formal fiber G.

Theorem 1.2 (Charters and Loepp, 2004). *Let (T, M) be a complete local ring and $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. Then there exists a local domain A that completes to T and has generic formal fiber exactly G if and only if T is a field and $G = \{(0)\}$, or the following conditions hold:*

1. *$M \notin G$, and G contains all of the associated prime ideals of T;*
2. *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$;*
3. *If $Q \in G$ then the intersection of Q with the prime subring of T is (0).*

Notice that, if T satisfies Lech's conditions, then the set of associated primes of T satisfies the conditions on G in the theorem above. We therefore obtain the surprising result that every completion of an integral domain is the completion of an integral domain with semilocal generic formal fiber.

The main body of this article focuses on combining the two results above. Specifically, we want to know when the domain A in Theorem 1.2 can be forced to be a UFD. Some progress toward this goal has already been made. Loepp (1997) provides the following result.

Theorem 1.3 (Loepp, 1997). *Let (T, M) be a complete local domain of dimension at least two, satisfying Serre's (S2) condition and $|T/M| = |T|$. Let $\{P_1, \ldots, P_n\}$ be a set of nonzero prime ideals of T such that for every j, $P_j \neq M$, $P_j \cap$ the prime subring of $T = (0)$, and $P_i \not\subseteq P_j$ when $i \neq j$. Then there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is semilocal with maximal ideals $\{P_1, \ldots, P_n\}$.*

In Section 2, we prove the main result. This theorem characterizes those local rings (T, M) satisfying the condition that $|T/M| = |T|$ that are the completion of a UFD with semilocal generic formal fiber. Though the proof mainly follows the proof of Theorem 1.3, it is an improvement on this result because it does not require T to be an integral domain or that T satisfy Serre's (S2) condition. In particular, we prove the following theorem.

Theorem 1.4. *Let (T, M) be a complete local ring and $|T/M| = |T|$. Let $G \subseteq \operatorname{Spec} T$ such that G is nonempty and has a finite number of maximal elements. Then there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly G if and only if T is a field or DVR and $G = \{(0)\}$, or T has depth at least two and the following conditions hold*:

1. *$M \notin G$, and G contains all of the associated prime ideals of T;*
2. *If $Q \in G$ and $P \in \operatorname{Spec} T$ such that $P \subseteq Q$, then $P \in G$;*
3. *If $Q \in G$, then the intersection of Q with the prime subring of T is (0);*
4. *If $J \in \operatorname{Spec} T$ such that $ht(J) > depth(T_J) = 1$, then $J \in G$.*

We also investigate the question of which rings satisfy the conditions of the theorem above. In particular, we show that the set of prime ideals J described in condition (4) is contained in a finite set of nonmaximal prime ideals if and only if T has no embedded associated primes ideals. This gives us the result that, if T is a complete local ring such that $|T/M| = |T|$, then T is the completion of UFD with semilocal generic formal fiber if and only if it is the completion of a UFD and has no embedded associated primes.

In Section 3, we extend this result so that the height of the ideal in A that has semilocal formal fiber is one. In other words, instead of constructing our UFD A to have a specified semilocal *generic* formal fiber, we construct it so that it contains a *height one* prime ideal with specified semilocal formal fiber.

In this article, we use the term local ring to describe a Noetherian ring with one maximal ideal. A ring with one maximal ideal that is not necessarily Noetherian is called quasi-local.

2. SEMI-LOCAL GENERIC FORMAL FIBERS

In this section, we provide necessary and sufficient conditions on a complete local ring (T, M) with $|T/M| = |T|$ for it to be the completion of a UFD with semilocal generic formal fiber. To show that out conditions are necessary is not difficult. The sufficiency of our conditions can be proven by construction, following the construction in Loepp (1997). In fact, the proof follows, once it is shown that we can weaken the conditions on T in Lemma 12 of that article.

In order to ensure that the ring A we construct is a unique factorization domain, we need to build subrings of T with special properties. These rings are called N-subrings. The definition comes from Heitmann (1993).

Definition 1. Let (T, M) be a complete local ring and let $(R, M \cap R)$ be a quasi-local unique factorization domain contained in T, satisfying the following conditions:

1. $|R| \leq \sup(\aleph_0, |T/M|)$, with equality only if T/M is countable;
2. $Q \cap R = (0)$ for all $Q \in Ass\, T$;
3. if $t \in T$ is regular and $P \in Ass(T/tT)$, then $ht(P \cap R) \leq 1$.

Then R is called an N-subring of T.

Now we are ready to prove the analogous result to Lemma 12 in Loepp (1997) under our weaker assumptions.

Lemma 2.1. *Let (T, M) be a complete local ring of dimension at least two, and $Q \in \operatorname{Spec} T$. Let $G \subseteq \operatorname{Spec} T$ have maximal elements $\{P_1, P_2, \dots, P_n\}$ such that G satisfies the following conditions*:

1. *$M \notin G$, and G contains all of the associated prime ideals of T*
2. *If $P \in G$ and $I \in \operatorname{Spec} T$ such that $I \subseteq P$, then $I \in G$*
3. *If $J \in \operatorname{Spec} T$ such that $ht(J) > depth(T_J) = 1$, then $J \in G$.*

Suppose R is an N-subring such that $P \cap R = (0)$ for every $P \in G$. Then there exists an N-subring S such that $R \subseteq S \subseteq T$, $P \cap S = (0)$ for every $P \in G$, $|S| \leq \sup(\aleph_0, |R|)$, and $Q \notin G$ implies $Q \cap S \neq (0)$. Furthermore, prime elements in R remain prime in S.

Proof. Let $C = \{P_1, \dots, P_n\} \cup \{P \in Ass(T/rT) \mid 0 \neq r \in R\}$. Suppose $Q \in C$. Then we can take $S = R$, because, if $Q \in Ass(T/rT)$, then $r \in Q \cap R$, so $Q \cap R \neq (0)$, and we do not need to do anything if $Q \in G$. So, assume that $Q \notin C$. Note that if $P \in \{P \in Ass(T/rT) \mid 0 \neq r \in R\}$, then either $P \in G$ or $ht\, P = 1$. If $ht\, P = 1$ and $Q \subset P$ with $Q \neq P$, then $Q \in Ass\, T$, and by condition (1), $Q \in C$. If $P \in G$ and $Q \subset P$, then by condition (2), $Q \in C$. This contradicts our assumption, so we have that $Q \nsubseteq P$ for every $P \in C$.

Now, for $P \in \operatorname{Spec} T$, define $D_{(P)}$ to be a full set of coset representatives of the cosets $t + P$ which are algebraic over $R/(R \cap P)$ as an element of T/P. Note that as $|R/(R \cap P)| \leq |R|$, the algebraic closure of $R/(R \cap P)$ in T/P has cardinality at most $\sup(\aleph_0, |R|)$. Hence, $|D_{(P)}| \leq \sup(\aleph_0, |R|)$. Letting $D = \bigcup_{P \in C} D_{(P)}$, we use Lemma 2 in Loepp (1997) if R is countable, and Lemma 3 in Loepp (1997) if not, to find a $t \in Q$ with $t \notin \{P + r \mid P \in C, r \in D\}$. We claim that $S = R[t]_{R[t] \cap M}$ is the desired subring. Clearly, we have $S \cap Q \neq (0)$. Now, by the way t was chosen, $t + P$ is transcendental over $R/(R \cap P)$ for all $P \in C$. So, Lemma 11 in Loepp (1997) shows that S is an N-subring, $|S| = \sup(\aleph_0, |R|)$, and prime elements of R are prime in S. All that is left to show is that $S \cap P = (0)$ for all $P \in G$.

It suffices to show that $R[t] \cap P = (0)$ for P a maximal element of G. Suppose that $f = a_n t^n + \cdots + a_1 t + a_0 \in P$ with $a_i \in R$. Now, by the way t was chosen, $t + P$ is transcendental over $R/(R \cap P)$ for all $P \in G \subseteq C$. Hence, $a_i \in P$ for every i. So $a_i \in R \cap P = (0)$. Therefore $f = 0$, and we have that $R[t] \cap P = (0)$. This completes the proof. □

We may now employ Lemmas 13–16 in Loepp (1997) to construct our UFD A. It can be verified that the same construction works under these assumptions, and it is unnecessary to repeat the proof here.

Note that we need $|T/M| = |T|$ in order to use Lemma 15 in Loepp (1997). It is because of this that we need this condition in our final theorem. If it were not for this condition, the main result of this section would be a complete characterization of completions of UFDs with semilocal generic formal fiber. As it is, however, we can only characterize those rings for which $|T/M| = |T|$.

Here we prove the main theorem.

Theorem 2.2. *Let (T, M) be a complete local ring and $|T/M| = |T|$. Let $G \subseteq \operatorname{Spec} T$ such that G is nonempty and has a finite number of maximal elements. Then there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly G if and* *only if T is a field or DVR and $G = \{(0)\}$ or T has depth at least two and the following conditions hold*:

1. *$M \notin G$, and G contains all of the associated prime ideals of T*;
2. *If $Q \in G$ and $P \in \operatorname{Spec} T$ such that $P \subseteq Q$ then $P \in G$*;
3. *If $Q \in G$ then the intersection of Q with the prime subring of T is* (0);
4. *If $J \in \operatorname{Spec} T$ such that $ht(J) > depth(T_J) = 1$, then $J \in G$.*

Proof. First, we prove the forward direction. Suppose that there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly G. From Theorem 1.1 we know that T must be a field, a DVR, or a ring of depth at least two. If T is a field or DVR, we know from Theorem 1.2 that $G = \{(0)\}$. Otherwise, we know from Theorem 1.2 that conditions (1)–(3) must hold. So all we have to show is the necessity of condition (4).

So suppose that $J \in \operatorname{Spec} T$ such that $ht(J) > \operatorname{depth}(T_J) = 1$, and $J \notin G$. First, we will show that $J \cap A$ has height one.

Note that, since $J \notin G$, $J \cap A \neq (0)$. Let $a \in J \cap A$, $a \neq 0$. As T_J is depth one and a is regular, the ring $T_J/(aT_J)$ consists only of zero divisors and units. Hence, $JT_J \in Ass(T_J/aT_J)$. By the Corollary to Theorem 6.2 in Matsumura (1989), we have $J \in Ass(T/aT)$. Now, suppose that $J \cap A \notin Ass(A/aA)$. Note that, as A is a UFD, A/aA has no embedded associated primes, so J is not contained in any of the associated primes of A/aA. Then there exists an element $j \in J \cap A$, $j \notin aA$ such that j is regular in A/aA. This means that j is regular on T/aT, because completion preserves regular elements, so $J \notin Ass(T/aT)$. This is a contradiction, so we know that $J \cap A \in Ass(A/aA)$. It follows from Serre's (S2) condition, that $J \cap A$ is height one.

Specifically, $J \cap A$ must be principal, so $J \cap A = aA$ for some nonzero $a \in A$. Now, let $I \in \operatorname{Spec} T$. If $I \subseteq J$ and $I \notin G$, then $I \cap A = aA$ as well. We will therefore have a contradiction if we can prove the existence of a prime ideal I such that $I \subseteq J$, $I \notin G$, and $a \notin I$.

Let C be the set containing the minimal associated primes of aT and the maximal elements of G. Note that $J \nsubseteq Q$ for all $Q \in C$. So, by prime avoidance, there exists a $c \in J$ such that $c \notin Q$ for all $Q \in C$. Because $c \in J$, we know that J contains a height one prime containing c. We will call this ideal I.

We know that $I \subseteq J$. Because $c \in I$ and $c \notin Q$ for all $Q \in G$, we have that $I \notin G$. Also, because $c \in I$ and $c \notin Q$ for all minimal associated primes Q of aT, we have that I is not a minimal associated prime of aT. The minimal associated primes of aT are precisely the height one primes that contain a. I is height one but not a member of this set, so $a \notin I$. Thus, I is the desired ideal. This is a contradiction, so it follows that $J \in G$.

Now, the backward direction. If T is a field or a DVR and $G = \{(0)\}$, $A = T$ is the desired ring. Otherwise, if T has depth at least two, we use the Lemma 3.2 together with Lemmas 13–16 in Loepp (1997) to construct the desired UFD A. □

It is natural to ask what sorts of rings satisfy the conditions in the theorem above. Specifically, for what rings is the set $\{J \in \operatorname{Spec} T \mid ht(J) > \operatorname{depth}(T_J) = 1\}$ contained in a finite set of nonmaximal prime ideals? The following corollary provides the answer.

Corollary 2.3. *Let (T, M) be a complete local ring and $|T/M| = |T|$. Then T is the completion of a UFD A with semilocal generic formal fiber if and only if T is a field, a DVR, or has depth at least two, no integer of T is a zerodivisor, and T has no embedded associated prime ideals.*

Proof. Suppose that T is the completion of a UFD A with semilocal generic formal fiber G. From Heitmann (1993), we know that T must be a field, a DVR, or have depth at least two and no integer of T is a zerodivisor. Now, let $P \in Ass\, T$ with $ht(P) > 0$, and let Q be a prime ideal containing P such that $ht(Q) = ht(P) + 1$. Note that $\text{depth}(T_Q)$ is either zero or one. But if $\text{depth}(T_Q) = 0$, then $Q \in Ass\, T$, so by the theorem above, $Q \in G$. If $\text{depth}(T_Q) = 1$, then, since $ht(Q) = ht(P) + 1 > 1$, by the theorem above, $Q \in G$. It follows that every prime ideal containing P of height $ht(P) + 1$ is an element of G, but this set canot be contained in a finite set of nonmaximal prime ideals. Therefore, no such P can exist, so T has no embedded associated prime ideals.

Now, suppose that T is a field, a DVR, or has depth at least two, no integer of T is a zerodivisor, and T has no embedded associated prime ideals. Let $P \in \text{Spec}\, T$ such that $ht(P) > \text{depth}(T_P) = 1$. Then P is an element of the non-Cohen-Macaulay locus, $\{P \in \text{Spec}\, T \mid ht(P) > \text{depth}(T_P)\}$, so in particular, P contains a minimal member Q of the non-Cohen-Macaulay locus. There are no height zero primes in the non-Cohen Macaulay locus, so $ht(Q) \geq 1$. Since T has no embedded associated primes, there must therefore be a regular element $a \in Q$. Then, since $a \in P$ and $\text{depth}(T_P) = 1$, $P \in Ass(T/aT)$, a finite set. It follows that, for every minimal member Q of the non-Cohen-Macaulay locus, there are a finite number of prime ideals P containing Q with $\text{depth}(T_P) = 1$. Now, since T is a complete local ring and therefore excellent, the non-Cohen-Macaulay locus of T is closed, meaning that it has a finite number of minimal elements. We therefore have that $\{P \in \text{Spec}\, T \mid ht(P) > \text{depth}(T_P) = 1\}$ is a finite set. We now let $G = Ass\, T \cup \{Q \in \text{Spec}\, T \mid Q \subseteq P,$ where $P \in \text{Spec}\, T$ such that $ht(P) > \text{depth}(T_P) = 1\}$, which satisfies all the conditions on G in the preceding theorem. T is therefore the completion of a UFD A with semilocal generic formal fiber equal to G. □

This next corollary is the version of the above theorem when we want the generic formal fiber of A to be local (rather than semilocal).

Corollary 2.4. *Let (T, M) be a complete local ring and $|T/M| = |T|$. Let $P \in \text{Spec}\, T$. Then there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal P if and only if T is a field or DVR and $P = (0)$, or T has depth at least two and the following conditions hold*:

1. *P is nonmaximal and contains all of the associated prime ideals of T;*
2. *The intersection of P with the prime subring of $T = (0)$;*
3. *If $J \in \text{Spec}\, T$ such that $ht(J) > \text{depth}(T_J) = 1$, then $J \subseteq P$.*

The following example is due to our informal correspondence with Ray Heitmann. We thank him for suggesting it.

Example 2.5. Let $T = \mathbb{C}[[x^3, x^2, xy, y, z]]$ and $P = (x^3, x^2, xy, y)$. Does there exist a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal P?

First, note that T/M is isomorphic to $\mathbb{C}$, so $|T/M| = |T|$. Also, as $\{y, z\}$ is a regular sequence in T, T has depth two. Since T is an integral domain and therefore has no embedded associated primes, we know that T is the completion of a UFD with semi-local generic formal fiber, but in this case we can actually show more. We know that P is nonmaximal because $z \notin P$, and since T is an integral domain, P contains all of the associated prime ideals of T. The prime subring of T contains only units, so $P \cap$ the prime subring of $T = (0)$. All that remains to be shown is condition (3).

Note that P itself has height greater than one, because $(0) \subset (xy, y) \subset P$, and also that $P \in Ass(T/xyT)$. This is not a problem because P will be in the generic formal fiber of A.

It can be seen that, if $t \in P$, then all of the associated primes of T/tT will be contained in P. This is because t may be seen as an element of the ring $\mathbb{C}[[x^3, x^2, xy, y]]$, and adjoining the indeterminate z will not alter the associated primes of (t) in this ring.

Let a be a regular element in T with $a \notin P$, and let $J \in Ass(T/aT)$. We want to show that $ht(J) \leq 1$. Because T_J must have depth one, we know that J is nonmaximal. Also, because $a \in J$ and $a \notin P$, we know that $J \neq P$. As P and the maximal ideal are the only two prime ideals in T that contain all four of the elements x^3, x^2, xy, and y, it follows that one of these four is not an element of J. Now consider T_J. We will show that $x^3/x^2 = xy/y \in T_J$ in every case. If $x^3 \notin J$, then $x^4/x^3 = x^3/x^2 \in T_J$. If $x^2 \notin J$, then $x^3/x^2 \in T_J$. If $xy \notin J$, then $x^2y/xy = xy/y \in T_J$. If $y \notin J$, then $xy/y \in T_J$. Because of this, the map $h : \mathbb{C}[[x, y, z]] \to T_J$ given by $x \to x^3/x^2$, $y \to y/1$, $z \to z/1$ is well defined.

Now, letting $g : T \to \mathbb{C}[[x, y, z]]$ be the inclusion map, we satisfy the conditions of Theorem 4.3 in Matsumura (1989). Thus, $T_J = \mathbb{C}[[x, y, z]]_Q$, where $Q = JT_J \cap \mathbb{C}[[x, y, z]]$. This means that T_J is a regular local ring, and so the depth of T_J is equal to its dimension. As T_J has depth one, it follows that $ht(J) = 1$. Therefore, there exists a local UFD A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal P.

Using these corollaries, it is easy to determine whether a given ring is the completion of a UFD with semilocal generic formal fiber, but much harder to determine whether it is the completion of a UFD with generic formal fiber equal to a given set of prime ideals. It should be noted that, of all the conditions in the main theorem, the last one is by far the most difficult to check. Fortunately, in any ring that satisfies Serre's (S2) criterion, every choice of prime ideals G (or prime ideal P in the local case) will meet this condition trivially. The example above is one where the condition is non-trivial but still satisfied. It appears that, in such rings, our choice of the prime ideal P is extremely limited. (There was only one option for P in the example given.)

3. SEMI-LOCAL FORMAL FIBERS AT HEIGHT ONE PRIME IDEALS

In this section, we control the height of the ideal in our UFD A that has semilocal formal fiber. In particular, we will provide necessary and sufficient conditions on a complete local ring (T, M) with $|T/M| = |T|$ for it to be the completion of a UFD containing a *height one* prime ideal with semilocal formal fiber.

Although we again follow the construction in Loepp (1997), here there is more to show. Specifically, we must show that we can extend both Theorem 6 and Lemma 12 in that article to the height one case. The rest of the construction follows.

From Proposition 1 in Heitmann (1993), in order to make A complete to T, we must have $IT \cap A = I$ for every finitely generated ideal I of A. The following lemma helps us get this condition. It is an extension of Theorem 6 in Loepp (1997).

Lemma 3.1. *Let (T, M) be a complete local ring and G a finite set of non-maximal prime ideals of T. Let R be an N-subring of T such that $R \cap P = \alpha R$ for every $P \in G$. Let I be a finitely generated ideal of R with $c \in IT \cap R$. Then there exists an N-subring S of T such that $R \subseteq S \subseteq T$, $|S| = |R|$, $c \in IS$, and $S \cap P = \alpha S$ for every $P \in G$. Also, prime elements in R remain prime in S.*

Proof. Except for the fact that $S \cap P = \alpha S$ for every $P \in G$, Lemma 9 in Loepp (1997) proves this result. We must therefore simply verify that that this condition holds for the ring S constructed in Loepp (1997). Let m be the number of generators of I. Lemma 9 in Loepp (1997) is proven by induction on m. There it is shown that we can reduce to the case where I is not contained in a height one prime ideal of R, and so we may begin with the $m = 2$ case. Here, we let $I = (y_1, y_2)R$ and notice that, for every $P \in G$, either $y_1 \notin P$ or $y_2 \notin P$. We then find elements $x_1, x_2 \in T$ such that $x_1 + P$ is transcendental over $R/(R \cap P)$ for all $P \in G$ such that $y_2 \notin P$ and similarly, $x_2 + P$ is transcendental over $R/(R \cap P)$ for all $P \in G$ such that $y_1 \notin P$. We then let $S = (R[x_1, y_2^{-1}] \cap R[x_2, y_1^{-1}])_{(R[x_1, y_2^{-1}] \cap R[x_2, y_1^{-1}]) \cap M}$.

We must show that $S \cap P = \alpha S$ for every $P \in G$. Let $P \in G$. Without loss of generality, assume that $y_1 \notin P$. Now, let $f = r_n(x_2)^n + \cdots + r_1(x_2) + r_0 \in R[x_2] \cap P$. Because $x_2 + P$ is transcendental over $R/(R \cap P)$, it follows that $r_i \in R \cap P = \alpha R$ for all i. Hence, $f \in \alpha R[x_2]$, and so $R[x_2] \cap P \subseteq \alpha R[x_2]$. Furthermore, because $\alpha \in R[x_2] \cap P$, we have $R[x_2] \cap P = \alpha R[x_2]$. Clearly, then, we have $R[x_2, y_2^{-1}] \cap P = \alpha R[x_2, y_1^{-1}]$. Now let's look at $R[x_1, y_2^{-1}]$. If $y_2 \notin P$, then we have $x_1 + P$ is transcendental over $R/(R \cap P)$, and so $R[x_1, y_2^{-1}] \cap P = \alpha R[x_1, y_2^{-1}]$ for the same reason as above. If, however, $y_2 \in P$, then $y_2 \in \alpha R$, so $y_2 = \alpha r$ for some $r \in R$. Now, because $\alpha(ry_2^{-1}) = 1, \alpha$ is a unit in $R[x_1, y_2^{-1}]$, and so $\alpha R[x_1, y_2^{-1}] = R[x_1, y_2^{-1}]$. It follows that $P \cap (R[x_2, y_1^{-1}] \cap R[x_1, y_2^{-1}]) \subseteq \alpha R[x_2, y_1^{-1}] \cap \alpha R[x_1, y_2^{-1}] = \alpha(R[x_2, y_1^{-1}] \cap R[x_1, y_2^{-1}])$. And, since $\alpha R[x_2, y_1^{-1}] \subseteq P$. we have $P = \alpha(R[x_2, y_1^{-1}] \cap R[x_1, y_2^{-1}])$. Thus, after localization, we have $S \cap P = \alpha S$. So, if I is generated by 2 elements, the theorem holds.

Now, assume that $m > 2$. Lemma 9 in Loepp (1997) shows how to construct an N-subring R' with $R \subseteq R' \subseteq T$, containing an element $c^* \in R'$ and an $(m-1)$ generated ideal J of R' with $c^* \in JT$. We need to show that $R' \cap P = \alpha R'$ for all $P \in G$. By induction, there will therefore exist an N-subring S such that $R' \subseteq S \subseteq T$, $c^* \in JS$ and $S \cap P = \alpha S$ for all $P \in G$. Then, we will show that $c \in IS$.

Let $I = (y_1, \ldots, y_m)R$ and define $J = (y_1, \ldots, y_{m-1})R$. Lemma 9 in Loepp (1997) shows that there exists an element $t \in T$ such that $t + P$ is transcendental over $R/(R \cap P)$ as an element of T/P for every $P \in G$ with $\{y_1, \ldots, y_{m-1}\} \nsubseteq P$. Let $R' = R[t]_{M \cap R[t]}$.

Because $P \cap R$ is height one, we have that $\{y_1, \ldots, y_{m-1}\} \nsubseteq P$. This means that $t + P$ is transcendental over $R/(R \cap P)$ as an element of T/P for every $P \in G$. So, if $f = r_n t^n + \cdots + r_1 t + r_0 \in R[t] \cap P$, it follows that $r_i \in R \cap P = \alpha R$ for all i.

Thus, $f \in \alpha R[t]$, so $R[t] \cap P = \alpha R[t]$. This is enough to get $R' \cap P = \alpha R'$. The rest of the theorem follows from Loepp (1997). □

The following lemma allows us to adjoin elements from prime ideals not contained in Q_i without increasing the height of $Q_i \cap R$. It is an extension of Lemma 12 in Loepp (1997).

Lemma 3.2. *Let (T, M) be a complete local ring of dimension at least two and R an N-subring of T. Let $C = \{Q_1, \dots, Q_n\}$ be a finite set of incomparable prime ideals satisfying the following conditions:*

1. *$M \notin C$;*
2. *There exists a nonzero element $\alpha \in R$ such that $Q_i \cap R = \alpha R$ for all i;*
3. *If $P \in Ass(T/\alpha T)$, then $P \subseteq Q_i$ for some i.*

Let $I \in \operatorname{Spec} T$ with $\alpha \in I$ and such that $I \nsubseteq Q_i$ for all i. Then there exists an N-subring S such that $R \subseteq S \subseteq T$, $Q_i \cap S = \alpha S$ for all i, $|S| = \sup(\aleph_0, |R|)$, and $\alpha S \subset I \cap S$ where the containment is strict. Furthermore, prime elements in R remain prime in S.

Proof. To begin with, we will show that $I \nsubseteq J$ for all $J \in \{J \in Ass(T/rT) \mid 0 \neq r \in R\}$. Suppose that $I \subseteq J$ where J is in this set. Then $\alpha \in J$, but since $\operatorname{depth}(T_J) = 1$, $J \in Ass(T/rT)$ for all regular $r \in J$. It follows that $J \in Ass(T/\alpha T)$. This means that $J \subseteq Q_i$ for some i, but, since $I \subseteq J$ and $I \nsubseteq Q_i$ for any i, this is a contradiction. It follows that $I \nsubseteq J$ for all $J \in \{J \in Ass(T/rT) \mid 0 \neq r \in R\}$.

Now, for $P \in \operatorname{Spec} T$ we define $D_{(P)}$ to be a full set of coset representatives of $t + P$ which are algebraic over $R/(R \cap P)$ as an element of T/P. Note that as $|R/(R \cap P)| \leq |R|$, the algebraic closure of $R/(R \cap P)$ in T/P has cardinality at most $\sup(\aleph_0, |R|)$. Hence, $|D_{(P)}| \leq \sup(\aleph_0, |R|)$. Now, let $G = C \cup Ass\, T \cup \{J \in Ass(T/rT) \mid 0 \neq r \in R\}$. Note that, as $\alpha \in I$ and α is not a zerodivisor in T, $I \notin Ass\, T$. It follows that $I \nsubseteq P$ for every $P \in G$.

Letting $D = \bigcup_{P \in G} D_{(P)}$, we use Lemma 2 in Loepp (1997) if R is countable and Lemma 3 in Loepp (1997) if not to find a $t \in I$ with $t \notin \{P + r \mid P \in G, r \in D\}$. We claim that $S = R[t]_{R[t] \cap M}$ is the desired subring. Because $t \in S \cap I$ and $t \notin Q_i$ for all i, we have $\alpha S \subset I \cap S$ where the containment is strict. Now, by the way t was chosen, $t + P$ is transcendental over $R/(R \cap P)$ for every $P \in G$.

Now, by Lemma 11 in Loepp (1997), S is an N-subring, $|S| = \sup(\aleph_0, |R|)$, and prime elements in R remain prime in S. All that is left to show is that $S \cap Q_i = \alpha S$ for all i. Suppose that $f(t) = a_n t^n + \cdots + a_1 t + a_0 \in Q_i \cap R[t]$ with $a_j \in R$. Since $Q_i \in G$, $t + P$ is transcendental over $R/(R \cap Q_i)$, so $a_j \in Q_i$ for every j. So $a_j \in R \cap Q_i = \alpha R$, and therefore $f(t) = \alpha r_n t^n + \cdots + \alpha r_1 t + \alpha r_0 \in \alpha R[t]$, so $R[t] \cap Q_i = \alpha R[t]$. S is just a localization of R, so $S \cap Q_i = \alpha S$. This completes the proof. □

Again, we may use Lemmas 13–15 in Loepp (1997) to construct the desired UFD A. All we have to check is that $Q_i \cap R = \alpha R$ for each of the rings R used in this construction, but this is not difficult. Theorem 16 in Loepp (1997), however, requires the existence of an N-subring with certain nice properties. In that case, the prime subring of T localized at its intersection with M satisfies the desired properties, but here this subring will not work. The following lemma extends Theorem 16 in Loepp (1997).

Lemma 3.3. *Let (T, M) be a complete local ring of depth at least two such that $|T/M| = |T|$ and let S' denote the prime subring of T. Let $G = \{Q_1, Q_2, \ldots, Q_n\}$ be a finite set of incomparable prime ideals of T such that the following conditions hold:*

1. *$M \notin G$;*
2. *No integer of T is a zerodivisor;*
3. *There exists a regular, nonzero element $\alpha \in T$ satisfying the following conditions:*
   (a) *$\alpha \in \bigcap_{i=1}^n Q_i$;*
   (b) *if $P \in Ass(T/\alpha T)$, then $P \subseteq Q_i$ for some i;*
   (c) *In the case where $Q_i \cap S' = (0)$ for all i, if $J \in Ass(T/rT)$ for some r such that $0 \neq r \in S'$, then $\alpha \notin J$;*
   (d) *In the case where $Q_i \cap S' = pS'$ for any i, there exists a $\beta \in T$ such that $p = \alpha\beta$, where if $P \in Ass(T/\beta T)$, then $\alpha \notin P$ and $P \nsubseteq Q_i$ for all i.*

Then there exists a local UFD A and a height one prime ideal I of A such that $\widehat{A} = T$ and the formal fiber of I is semilocal with maximal ideals the elements of G. Furthermore, $I \cap A = \alpha t A$ for some unit $t \in T$.

Proof. If we can find an N-subring $R_o \subseteq T$ such that $R_0 \cap Q_i = \alpha R_0$ for all i, Theorem 16 in Loepp (1997) shows how to recursively define a family of N-subrings beginning with R_0 whose union is the desired UFD. We now work to define R_0.

Define S' to be the prime subring of T and S to be $S'_{S' \cap M}$. It is easy to see that S is an N-subring of T. Note that, in the case where $Q_i \cap S' = pS'$ for some i, we have $\alpha \in \bigcap_{i=1}^n Q_i$, so $p = \alpha\beta \in \bigcap_{i=1}^n Q_i$. Thus, either $Q_i \cap S' = (0)$ for all i, or $Q_i \cap S' = pS'$ for all i. Now suppose that $S \cap Q_i = (0)$ for every i. Let $C = \{P \in Ass(T/rT) \mid 0 \neq r \in S\} \cup Ass\, T$ and, if $P \in \operatorname{Spec} T$, define $D_{(P)}$ to be a full set of coset representatives that make $t + P$ algebraic over $S/(S \cap P)$. Use Lemma 4 in Loepp (1997) with $D = \bigcup_{P \in C} D_{(P)}$ to find a unit t such that $\alpha t \notin \cup\{P + r \mid P \in C, r \in D\}$, and let $x = \alpha t$. Note that $\alpha t T = \alpha T$, so if $P \in Ass(T/xT)$, then $P \subseteq Q_i$ for some i.

Now let $R_0 = S[x]_{S[x] \cap M}$. Note that, since T has depth at least two, $M \notin Ass\, T$, so by Lemma 11 from Loepp (1997), R_0 is an N-subring of T. Also, suppose that $f(x) = s_n x^n + \cdots + s_1 x + s_0 \in Q_i$ for some i. Because $f(x) - s_0 \in xT \subseteq Q_i$, it follows that $s_0 \in Q_i$. So $s_0 \in Q_i \cap S = (0)$, so $f(x) = s_n x^n + \cdots + s_1 x \in xS[x]$. Thus, $Q_i \cap R_0 = xR_0$.

Otherwise, suppose that $S \cap Q_i = pS$ for some i. We must define R_0 differently. Let $C_1 = Ass(T/\beta T) \cup Ass(T)$ and $C_2 = Ass(T/\alpha T) \cup Ass(T) \cup \{Q_i \mid 1 \leq i \leq n\}$. Note that, by assumption, $\alpha \notin P$ for all $P \in C_1$. Because, for all $P \in Ass(T/\beta T)$, $P \nsubseteq Q_i$ for all i, it follows that $\beta \notin Q_i$ for all i. Since for every $P \in Ass(T/\alpha T)$, $P \subseteq Q_i$, it follows that $\beta \notin P$ for all $P \in C_2$. Let $D_{(P)}$ be a full set of coset representatives of the cosets $t + P$ that are algebraic over $S/(S \cap P)$ as an element of S/P. Now, by Lemma 4 in Loepp (1997), letting $D_1 = \bigcup_{P \in C_1} D_{(P)}$ and $D_2 = \bigcup_{P \in C_2} D_{(P)}$ we can find a unit $t \in T$ such that $\alpha t \notin \bigcup\{P + r \mid P \in C_1, r \in D_1\}$ and $\beta t^{-1} \notin \bigcup\{P + r \mid P \in C_2, r \in D_2\}$.

Now let $R_0 = S[\alpha t, \beta t^{-1}]_{S[\alpha t, \beta t^{-1}] \cap M}$. R_0 clearly meets the cardinality condition for N-subrings. Now, let $P \in Ass\, T$ and assume that $f = a_n(\alpha t)^n + \cdots + a_1(\alpha t) + b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in P \cap S[\alpha t, \beta t^{-1}]$. Then $(\alpha t)^m f = a_n(\alpha t)^{n+m} + \cdots + a_1(\alpha t)^{m+1} + b_m p^m + \cdots + b_1 p(\alpha t)^{m-1} + c(\alpha t)^m \in P \cap S[\alpha t, \beta t^{-1}]$. But $\alpha t + P$ is transcendental over $S/(S \cap P)$, so $a_i, b_i p^i, c \in P \cap S = (0)$. It follows that $(\alpha t)^m f = 0$. αt is not a zerodivisor, so $f = 0$. Thus $P \cap S[\alpha t, \beta t^{-1}] = (0)$, and, as

R_0 is simply a localization of this, we have $P \cap R_0 = (0)$ for all $P \in Ass\, T$, so the second condition of N-subrings holds.

Now let's show that R_0 is a UFD. Note that $S[\alpha t, \beta t^{-1}] = S[\alpha t, p/(\alpha t)]$, and when we adjoin $1/(\alpha t)$, we get $S[\alpha t, 1/(\alpha t)]$, which is a UFD. So, if we can show that αt is prime in $S[\alpha t, p/(\alpha t)]$, then by Theorem 20.2 in Matsumura (1989), $S[\alpha t, p/(\alpha t)]$ will be a UFD. Let $f = a_n(\alpha t)^n + \cdots + a_1(\alpha t) + b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in Q_i \cap S[\alpha t, \beta t^{-1}]$ for some i. Then $\alpha t T \subseteq Q_i$, so $b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in Q_i \cap S[\alpha t, \beta t^{-1}]$. But $\beta t^{-1} + Q_i$ is transcendental over $S/(S \cap Q_i)$, so $b_i, c \in Q_i \cap S = pS = (\alpha t \beta t^{-1} S)$. It follows that $f \in \alpha t S[\alpha t, \beta t^{-1}]$, so $Q_i \cap S[\alpha t, \beta t^{-1}] \subseteq \alpha t S[\alpha t, \beta t^{-1}]$. It follows that $Q_i \cap S[\alpha t, \beta t^{-1}] = \alpha t S[\alpha t, \beta t^{-1}]$. So αt is prime in $S[\alpha t, \beta t^{-1}]$ as desired. Thus $S[\alpha t, \beta t^{-1}]$ is a UFD. It follows that R_0 is a UFD as well. Because R_0 is just a localization of $S[\alpha t, \beta t^{-1}]$, it follows that $Q_i \cap R_0 = \alpha t R_0$ for all $i = 1, 2, \ldots, n$.

To complete the proof that R_0 is an N-subring, we must show that $ht(J \cap R_0) \leq 1$ for all $J \in Ass(T/tT)$, where t is a regular element of T. Let J be such an ideal. Since S is an N-subring, $ht(J \cap S) \leq 1$. Suppose $ht(J \cap S) = 0$. Then, $J \cap S = (0)$, and $ht(J \cap S[\alpha t]) \leq 1$. Localizing cannot increase height, so $ht(J \cap S[\alpha t, 1/(\alpha t)]) \leq 1$. Now, if $\alpha t \in J$, then $p \in J \cap S$ and $ht(J \cap S) \neq 0$, so $\alpha t \notin J$. Hence, when we adjoin $1/\alpha t$, the height of J is unaffected, so we have $ht(S[\alpha t, p/(\alpha t)] \cap J) = ht(J \cap S[\alpha t, 1/(\alpha t)]) \leq 1$. So, in this case, $ht(J \cap R_0) \leq 1$.

Now suppose that $ht(J \cap S) = 1$. Then $S \cap J = pS$, so $p \in J$, and, since J is prime and $p = \alpha\beta$, either $\alpha \in J$ or $\beta \in J$. Since $depth(T_J) = 1$ and both α and β are regular elements, either $J \in Ass(T/\alpha T)$ or $J \in Ass(T/\beta T)$. We can now consider two cases, one where $J \in Ass(T/\alpha T)$, and one where $J \in Ass(T/\beta T)$.

Suppose $J \in Ass(T/\alpha T)$ and let $f = a_n(\alpha t)^n + \cdots + a_1(\alpha t) + b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in J \cap S[\alpha t, \beta t^{-1}]$. Then $\alpha t T \subseteq J$, so $b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in J \cap S[\alpha t, \beta t^{-1}]$. But $\beta t^{-1} + J$ is transcendental over $S/(S \cap J)$, so $b_i, c \in J \cap S = pS = (\alpha t \beta t^{-1} S)$. It follows that $f \in \alpha t S[\alpha t, \beta t^{-1}]$, so $J \cap S[\alpha t, \beta t^{-1}] \subseteq \alpha t S[\alpha t, \beta t^{-1}]$, and $J \cap S[\alpha t, \beta t^{-1}] = \alpha t S[\alpha t, \beta t^{-1}]$, so $ht(J \cap S[\alpha t, \beta t^{-1}]) = 1$. Therefore, $ht(J \cap R_0) = 1$.

Now suppose $J \in Ass(T/\beta T)$ and let $f = a_n(\alpha t)^n + \cdots + a_1(\alpha t) + b_m(\beta t^{-1})^m + \cdots + b_1(\beta t^{-1}) + c \in J \cap S[\alpha t, \beta t^{-1}]$. Then $\beta T \subseteq J$, so $a_n(\alpha t)^n + \cdots + a_1(\alpha t) + c \in J \cap S[\alpha t, \beta t^{-1}]$. But $\alpha t + J$ is transcendental over $S/(S \cap J)$, so $a_i, c \in J \cap S = pS = (\alpha t \beta t^{-1} S)$. It follows that $f \in \beta t^{-1} S[\alpha t, \beta t^{-1}]$, so $J \cap S[\alpha t, \beta t^{-1}] \subseteq \beta t^{-1} S[\alpha t, \beta t^{-1}]$, and $J \cap S[\alpha t, \beta t^{-1}] = \beta t^{-1} S[\alpha t, \beta t^{-1}]$, so $ht(J \cap S[\alpha t, \beta t^{-1}]) = 1$. Therefore, $ht(J \cap R_0) = 1$.

So, for all cases, we have constructed an N-subring R_0 such that $S \subseteq R_0 \subset T$ and there exists an element $x = \alpha t \in R_0$ for some unit $t \in T$ such that $Q_i \cap R_0 = xR_0$ for all i. Now, following the proof of Theorem 16 in Loepp (1997), we can use R_0 along with the previous lemma to construct the UFD A. □

Finally, we arrive at the main theorem.

Theorem 3.4. *Let (T, M) be a complete local ring such that $|T/M| = |T|$, and let S' denote the prime subring of T. Let $C = \{Q_1, Q_2, \ldots, Q_n\}$ be a finite set of incomparable prime ideals of T. Then there exists a local UFD A and a height one prime ideal I of A such that $\widehat{A} = T$ and the formal fiber of I is semilocal with maximal ideals the elements of C if and only if T is a DVR and $C = \{M\}$ or T has depth at least two and* *the following conditions hold*:

1. $M \notin C$;
2. *No integer of T is a zerodivisor*;
3. *There exists a regular, nonzero element $\alpha \in T$ satisfying the following conditions*:

   (a) $\alpha \in \bigcap_{i=1}^{n} Q_i$;
   (b) *if $P \in Ass(T/\alpha T)$, then $P \subseteq Q_i$ for some i*;
   (c) *In the case where $Q_i \cap S' = (0)$ for all i, if $J \in Ass(T/rT)$ for some r such that $0 \neq r \in S'$, then $\alpha \notin J$*;
   (d) *In the case where $Q_i \cap S' = pS'$ for any i, there exists a $\beta \in T$ such that $p = \alpha\beta$, where if $P \in Ass(T/\beta T)$, then $\alpha \notin P$ and $P \nsubseteq Q_i$ for all i.*

Proof. First, we prove the forward direction. Suppose that there exists a local UFD A and a height one prime ideal I of A such that $\widehat{A} = T$ and the formal fiber of I is semilocal with maximal ideals the elements of C. From Theorem 1.1 we know that no integer of T is a zerodivisor and that T must be a field, a DVR, or a ring of depth at least two. Note that $(0) \notin C$, because $(0) \cap A = (0)$, which is not height one. Because in a field (0) is the only prime ideal, this implies that T cannot be a field. Moreover, because the only prime ideals in a DVR are (0) and M, this implies that if T is a DVR, $C = \{M\}$.

Now, suppose that T has depth at least two. Then T has dimension at least two, and the dimension of A is the same as that of T. Because $M \cap A$ is the maximal ideal of A, $M \cap A$ cannot be height one, so $M \notin C$.

Because A is a UFD and I is height one, I must be principal, so $I = xA$ for some regular, non-zero element $x \in T$. Now, as we showed in the generic formal fiber case, if $P \in Ass(T/xT)$, then the height of $P \cap A$ is one, so $P \cap A = xA$. Thus, P is in the formal fiber of I, so $P \subseteq Q_i$ for some i. Also, because $Q_i \cap A = xA$ for all i, it follows that $x \in \bigcap_{i=1}^{n} Q_i$. Therefore, x meets conditions (a) and (b) of our desired element α. We now divide into two cases. We will show that, if $Q_i \cap S' = (0)$, we can choose $\alpha = x$. Otherwise, we will choose $\alpha = x^n$ for some n.

Suppose that $Q_i \cap S' = (0)$ for all i. Then $xA \cap S' = (0)$. Let $P \in Ass(T/rT)$ for some nonzero $r \in S'$. By similar reasoning to that above, $P \cap A$ must have height one. If $x \in P$, then, $P \cap A = xA$. But $S' \subset A$, so $r \in P \cap A = xA$. This contradicts the fact that $xA \cap S' = (0)$, so if $J \in Ass(T/rT)$ for some nonzero $r \in S'$, then $x \notin J$. Thus, x meets condition (c) of our desired element α.

Finally, suppose that $Q_i \cap S' = pS'$ for some i. Then $p \in xA$, so there exists a $y \in A$ such that $p = xy$. Note that, if $y \in xA$, then $y = xy_1$ for some $y_1 \in A$ and $p = x^2 y_1$. By the Krull Intersection Theorem, this process must stop, so that $p = x^n \beta$ where $\beta \notin xA$. Note that $Ass(T/x^n T) = Ass(T/xT)$, so x^n satisfies conditions (a) and (b) for our desired element α. Now, let $P \in Ass(T/\beta T)$, and suppose that $P \subseteq Q_i$ for some i. By the same argument as above, we know that $P \cap A$ must be height one, and we know that $\beta \in P \cap A$. But, since $P \subseteq Q_i$ for some i, $P \cap A = (0)$ or $P \cap A = xA$. Because $\beta \notin xA$, neither of these cases is possible, so $P \nsubseteq Q_i$ for all i. Now, suppose that $x^n \in P$. This means that $x \in P$. Again, we know that the height of $P \cap A$ is one, so $P \cap A = xA$, but $P \nsubseteq Q_i$ for all i, so P is not in the formal fiber of xA. This is a contradiction, so $x^n \notin P$. Thus, x^n satisfies condition (d) of our desired element α.

Now, the backward direction. If T is a DVR and $C = \{M\}$, then $A = T$ is the desired ring. If T has depth greater than one, then use Lemma 3.3 to construct the desired UFD A. $\square$

Notice that we have some control over the generator of $Q_i \cap A$. In the backward direction of the proof above, we construct our UFD A so that $Q_i \cap A = \alpha t A$ for all i where t is some unit in T. For this reason, if we are given an element $x \in T$ and we want to construct a UFD A such that $xt \in A$ for some unit $t \in T$ and the formal fiber of xtA is semilocal with maximal ideals the elements of C, we can do so if x meets the four conditions on α in the above theorem.

In the forward direction of the proof, we show that conditions (a)–(c) must hold for the generator of $Q_i \cap A$. In the case where $Q_i \cap S' \neq (0)$ for some i, however, condition (d) is not a necessary condition on the generator of $Q_i \cap A$. If A is a UFD with all the desired conditions and $Q_i \cap A = xtA$ for all i, then all that has to be true is that x^n meets condition (d) for some integer n. On the other hand, in order to use the construction given here, condition (d) must be true of x itself, not x^n. For example, one of the consequences of condition (d) is that if $Q_i \cap S' = pS'$, then p cannot be divisible in T by x^n for any $n \geq 2$. There is no reason to think that this is a necessary condition on the generator of $Q_i \cap A$, but it is necessary for our construction.

The following corollary is the local version of the above theorem.

Corollary 3.5. *Let (T, M) be a complete local ring such that $|T/M| = |T|$, and let S' denote the prime subring of T. Let P be a prime ideal of T. Then there exists a UFD A and a height one prime ideal I of A such that $\widehat{A} = T$ and the formal fiber of I is local with maximal ideal P if and only if T is a DVR and $P = M$ or T has depth at least two and the following conditions hold*:

1. $P \neq M$;
2. *No integer of T is a zerodivisor*;
3. *There exists a regular, nonzero element $\alpha \in P$, satisfying the following*:
   (a) *If $Q \in Ass(T/\alpha T)$, then $Q \subseteq P$*;
   (b) *In the case where $P \cap S' = (0)$, if $J \in Ass(T/rT)$ for some r such that $0 \neq r \in S'$, then $\alpha \notin J$*;
   (c) *In the case where $P \cap S' = pS'$, there exists a $\beta \in T$ such that $p = \alpha\beta$, where if $Q \in Ass(T/\beta T)$, then $\alpha \notin Q$ and $Q \nsubseteq P$.*

Here is an example.

Example 3.6. Consider the complete local ring $T = \mathbb{C}[[x, y, z, w]]/(x^2 - yz)$. Let $P = (x, y, z)$. Is there a local UFD A that completes to T and a height one prime ideal I of A such that $\widehat{A} = T$ and the formal fiber of I is local with maximal ideal P?

First note that $|T/M| = |T|$, no integer is a zerodivisor, and T has depth at least two. Now, since $M = (x, y, z, w)$, we have $P \neq M$. Now, look at $x \in P$. We have $Ass(T/xT) = \{(x, y), (x, z)\}$, and both of these ideals are contained in P. Note that the prime subring of T consists only of units, so condition (b) holds trivially. Therefore, there does exist a UFD A with all of the desired properties.

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