Components of Brill–Noether Loci for Curves with Fixed Gonality

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ABSTRACT. We describe a conjectural stratification of the Brill–Noether variety for general curves of fixed genus and gonality. As evidence for this conjecture, we show that this Brill–Noether variety has at least as many irreducible components as predicted by the conjecture and that each of these components has the expected dimension. Our proof uses combinatorial and tropical techniques. Specifically, we analyze containment relations between the various strata of tropical Brill–Noether loci identified by Pflueger in his classification of special divisors on chains of loops.

1. Introduction

Given a curve C over the complex numbers, the *Brill–Noether variety* $W_d^r(C)$ parameterizes line bundles of degree d and rank at least r on C. Brill–Noether varieties encode a significant amount of geometric information and consequently are among the most well-studied objects in the theory of algebraic curves. A series of results in the 1980s concern the geometry of $W_d^r(C)$ when C is general in the moduli space $\mathcal{M}_g$. In this case the locally closed stratum $W_d^r(C) \smallsetminus W_d^{r+1}(C)$ is smooth [Gie82] of dimension

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \quad [\text{GH80}]$$

and irreducible when $\rho(g, r, d)$ is positive [FL81].

More recent work has focused on the situation where C is general in the Hurwitz space $\mathcal{H}_{k,g}$ parameterizing branched covers of the projective line of degree k and genus g. The Hurwitz space $\mathcal{H}_{k,g}$ admits a natural map to the moduli space $\mathcal{M}_g$, given by forgetting the data of the map to $\mathbb{P}^1$. When $k \geq \lfloor \frac{g+3}{2} \rfloor$, this map is dominant, and there is nothing new to show, so we restrict our attention to the case where k is smaller than $\lfloor \frac{g+3}{2} \rfloor$. We refer to a general point in the Hurwitz space $\mathcal{H}_{k,g}$ as a *general curve of genus g and gonality k*. Our main result is the following:

THEOREM 1.1. *Let C be a general curve of genus g and gonality $k \geq 2$. Then there exists an irreducible component of $W_d^r(C)$ of dimension*

$$\rho(g, \alpha - 1, d) - (r + 1 - \alpha)k,$$

as long as this number is nonnegative, for every positive integer $\alpha \leq \min\{r + 1, k - 1\}$ satisfying either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$.

We strongly suspect that Theorem 1.1 identifies *all* irreducible components of $W_d^r(C)$ for a reason that we will explain in Section 1.1. Theorem 1.1 is a generalization of several previous results. Pflueger [Pfl17a] shows that the dimension of $W_d^r(C)$ is at most

$$\rho_k(g, r, d) := \max_{\alpha} \rho(g, \alpha - 1, d) - (r + 1 - \alpha)k$$

and asks whether every component has dimension $\rho(g, \alpha - 1, d) - (r + 1 - \alpha)k$ for some value of α. In [JR17], Ranganathan and the second author show that the maximal dimensional component has dimension exactly $\rho_k(g, r, d)$. Coppens and Martens [CM99] exhibit components of dimension $\rho(g, \alpha - 1, d) - (r + 1 - \alpha)k$ for α equal to 1, r, and $r + 1$. They further expand on this result in [CM02], constructing components of dimension $\rho(g, \alpha - 1, d) - (r + 1 - \alpha)k$ for all α dividing r or $r + 1$.

1.1. The Splitting Type Stratification

Let $\pi : C \to \mathbb{P}^1$ be a branched cover of degree k. Given a line bundle L on C, its pushforward $\pi_* L$ is a vector bundle of rank k on $\mathbb{P}^1$. Every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles

$$\pi_* L \cong \mathcal{O}(\mu_1) \oplus \cdots \oplus \mathcal{O}(\mu_k)$$

for some integers $\mu_1, \ldots, \mu_k$, which are unique up to permutation.

The vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ is known as the *splitting type* of the vector bundle, and we write $\pi_* L \cong \mathcal{O}(\boldsymbol{\mu})$ for ease of notation. We write $W^{\boldsymbol{\mu}}(C)$ for the locally closed subscheme parameterizing line bundles on C whose pushforward has splitting type $\boldsymbol{\mu}$:

$$W^{\boldsymbol{\mu}}(C) := \{L \in \operatorname{Pic}(C) | \pi_* L \cong \mathcal{O}(\boldsymbol{\mu})\}.$$

The splitting type of $\pi_* L$ determines not only the degree and rank of the line bundle L, but also the rank of $L \otimes \pi^* \mathcal{O}(m)$ for all integers m (see Section 2). In this way the varieties $W^{\boldsymbol{\mu}}(C)$ stratify $W_d^r(C)$. The number of irreducible components of $W_d^r(C)$ and the dimensions of these components are predicted by the following conjecture. We refer the reader to Definition 2.1 for the partial order on splitting types and to Definition 2.3 for the magnitude of a splitting type.

CONJECTURE 1.2. *Let C be a general curve of genus g and gonality $k \geq 2$. Then:*

(1) $W^{\boldsymbol{\mu}}(C)$ *is contained in the closure of $W^{\boldsymbol{\lambda}}(C)$ if and only if $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$;*
(2) $W^{\boldsymbol{\mu}}(C)$ *is smooth;*
(3) $W^{\boldsymbol{\mu}}(C)$ *has dimension $g - |\boldsymbol{\mu}|$ if $g \geq |\boldsymbol{\mu}|$ and is empty otherwise;*
(4) $W^{\boldsymbol{\mu}}(C)$ *is irreducible if $g > |\boldsymbol{\mu}|$.*

At the time of writing, we learned of a simultaneous and independent proof of parts (1)–(3) of Conjecture 1.2, due to Larson [Lar19].

As evidence for Conjecture 1.2, we consider the strata $W^{\mu}(C)$ that the conjecture predicts to be maximal in $W_d^r(C)$. For a given rank r and degree d, the maximal elements of the poset of splitting types are in correspondence with positive integers $\alpha \leq \min\{r+1, k-1\}$ satisfying either $\alpha \geq k - (g-d+r)$ or $\alpha = r+1$. (See Definition 2.5 and Proposition 2.11 for details.) Let $\boldsymbol{\mu}_\alpha$ denote the splitting type corresponding to the integer α. Conjecture 1.2 predicts that the irreducible components of $W_d^r(C)$ are precisely the closures of the strata $W^{\boldsymbol{\mu}_\alpha}(C)$. We prove the following stronger version of Theorem 1.1.

THEOREM 1.3. *Let C be a general curve of genus g and gonality $k \geq 2$. If $g \geq |\boldsymbol{\mu}_\alpha|$, then $W^{\boldsymbol{\mu}_\alpha}(C)$ has an irreducible component of dimension $g - |\boldsymbol{\mu}_\alpha|$. The closure of this component is an irreducible component of $W_d^r(C)$.*

1.2. Approach and Techniques

Our approach is based on tropical techniques developed in [CDPR12; Pfl17a; Pfl17b; JR17]. Each of these papers establishes results about Brill–Noether varieties by studying the divisor theory of a particular family of metric graphs, known as the chains of loops. The first of these papers [CDPR12] provides a new proof of the Brill–Noether theorem. Key to this argument is the classification of special divisors on chains of loops Γ with generic edge lengths. Specifically, [CDPR12] shows that $W_d^r(\Gamma)$ is a union of tori $\mathbb{T}(t)$, where the tori are indexed by standard Young tableaux t.

Pflueger [Pfl17b] generalizes this result to chains of loops with arbitrary edge lengths. In this case, $W_d^r(\Gamma)$ is still a union of tori, but here the tori are indexed by a more general type of tableaux, known as *displacement tableaux*. (See Definition 3.3 and Theorem 3.5.) Pflueger [Pfl17a] computes the dimension of the largest of these tori and thus obtains his bound on the dimensions of Brill–Noether loci for general k-gonal curves.

Instead of studying the tori of maximum dimension, in this paper, we study the tori that are maximal with respect to containment. The tableaux corresponding to maximal-dimensional tori belong to a larger family, known as *scrollar tableaux;* see Definition 3.8. There is a natural partition of scrollar tableaux into types, where the types are indexed by positive integers $\alpha \leq \min\{r+1, k-1\}$ satisfying either $\alpha \geq k - (g-d+r)$ or $\alpha = r+1$. It is shown in [JR17] that, under certain mild hypotheses, divisor classes corresponding to scrollar tableaux lift to divisor classes on k-gonal curves in families of the expected dimension.

Our main combinatorial result is the following:

THEOREM 1.4. *Let Γ be a k-gonal chain of loops of genus g, and let t be a k-uniform displacement tableau on $[r+1] \times [g-d+r]$. The torus $\mathbb{T}(t)$ is max-*

imal with respect to containment in $W^r_d(\Gamma)$ if and only if t is scrollar. In other words,

$$W^r_d(\Gamma) = \bigcup_{t \text{ scrollar}} \mathbb{T}(t).$$

1.3. Outline of the Paper

Sections 2 and 3 contain preliminary material. In Section 2, we review the basic theory of splitting types and identify those that are maximal with respect to the dominance order. In Section 3, we review the classification of special divisor classes on chains of loops from [Pfl17a; Pfl17b] and the necessary results on scrollar tableaux from [JR17]. In Section 4, we discuss the relation between our combinatorial and geometric results and in particular show that Theorem 1.4 implies Theorem 1.3. In the final two sections, which are purely combinatorial, we prove Theorem 1.4. In Section 5, we show that if t is a scrollar tableau, then $\mathbb{T}(t)$ is maximal, and in Section 6, we establish the converse.

2. Splitting Types

2.1. Preliminary Definitions

In this section, we review the definition of splitting types and discuss some of their basic properties. Let $\pi : C \to \mathbb{P}^1$ be a branched cover of degree k and genus g, and let L be a line bundle on C. As explained in Section 1.1, the pushforward π_*L is a vector bundle of rank k on $\mathbb{P}^1$, and every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles

$$\pi_*L \cong \mathcal{O}(\mu_1) \oplus \cdots \oplus \mathcal{O}(\mu_k).$$

The integers $\mu_1, \ldots, \mu_k$ are unique up to permutation. We will assume throughout that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k.$$

The vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ is known as the *splitting type* of the vector bundle, and we write $\pi_*(L) \cong \mathcal{O}(\boldsymbol{\mu})$ for ease of notation. It is helpful to think of a splitting type $\boldsymbol{\mu}$ as a partition with possibly negative parts. This is because, for any ℓ, the sum of the ℓ smallest entries of $\boldsymbol{\mu}$ is a lower semicontinuous invariant. It is therefore natural to endow the set of splitting types with a partial order, extending the dominance order on partitions.

DEFINITION 2.1. We define the *dominance order* on splitting types as follows. Let $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ be splitting types satisfying $\sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i$. We say that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ if and only if

$$\mu_1 + \cdots + \mu_\ell \leq \lambda_1 + \cdots + \lambda_\ell \quad \text{for all } \ell \leq k.$$

The splitting type of π_*L determines the rank and degree of the line bundle L, as well as the rank of all its twists by line bundles pulled back from the $\mathbb{P}^1$. This can

be seen by the projection formula as follows:

$$h^0(C, L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(m)) = h^0(\mathbb{P}^1, \pi_* L \otimes \mathcal{O}_{\mathbb{P}^1}(m))$$

$$= \sum_{i=1}^{k} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mu_i + m))$$

$$= \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\}. \qquad (\star)$$

In particular, we have

$$h^0(L) = \sum_{i=1}^{k} \max\{0, \mu_i + 1\} \quad \text{and}$$

$$\deg L = g + k - 1 + \sum_{i=1}^{k} \mu_i.$$

This suggests the following definition.

DEFINITION 2.2. Let $W^\mu(C)$ denote the locally closed subscheme parameterizing line bundles on C whose pushforward has splitting type $\boldsymbol{\mu}$:

$$W^\mu(C) := \{L \in \mathrm{Pic}(C) | \pi_* L \cong \mathcal{O}(\boldsymbol{\mu})\}.$$

The expected codimension of $W^\mu(C)$ in $\mathrm{Pic}^d(C)$ is given by the *magnitude* of $\boldsymbol{\mu}$.

DEFINITION 2.3. The *magnitude* of a splitting type $\boldsymbol{\mu}$ is

$$|\boldsymbol{\mu}| := \sum_{i<j} \max\{0, \mu_j - \mu_i - 1\}.$$

EXAMPLE 2.4. Let C be a trigonal curve of genus 5. We will show that $W_4^1(C)$ has two irreducible components, both isomorphic to C. First, there is a one-dimensional family of rank 1 divisor classes obtained by adding a basepoint to the g_3^1. If $D \in W_4^1(C)$ is not in this one-dimensional family, then $D - g_3^1$ is not effective. It follows from the basepoint free pencil trick that the multiplication map

$$\nu : H^0(D) \otimes H^0(g_3^1) \to H^0(D + g_3^1)$$

is injective. The divisor class $D + g_3^1$ is therefore special. From this we see that the Serre dual $K_C - D$ is a divisor class in $W_4^1(C)$ with the property that $(K_C - D) - g_3^1$ is effective.

Therefore we see that $W_4^1(C)$ has two components, both isomorphic to C, as pictured in Figure 1. One of these components consists of divisor classes D such that $D - g_3^1$ is effective, and the other component consists of the Serre duals of classes in the first component. Since $K_C - 2g_3^1$ is effective of degree 2, we see that these two components intersect in two points.
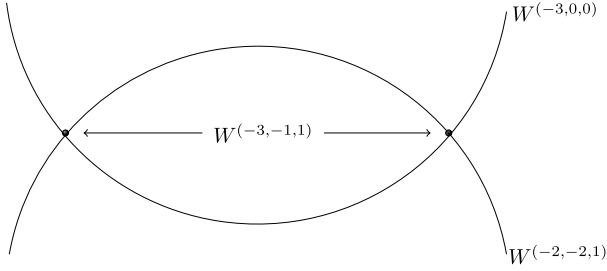
Figure 1   Stratification of W_4^1 for a general curve of genus 5 and gonality 3.

Alternatively, this analysis can be carried out by examining the splitting type stratification of $W_4^1(C)$. By (⋆) we see that line bundles in the first component, in the complement of the two intersection points, have the splitting type $(-2, -2, 1)$. Similarly, line bundles in the second component, in the complement of the two intersection points, have the splitting type $(-3, 0, 0)$. Finally, the two line bundles in the intersection have the splitting type $(-3, -1, 1)$. Notice that this third splitting type is smaller than each of the previous two in the dominance order and that the codimension of each stratum in $\text{Pic}^4(C)$ is the magnitude of the splitting type.

2.2. *Maximal Splitting Types*

For the remainder of this section, we fix positive integers g, r, d, and k such that $r > d - g$. Among the possible splitting types of line bundles of degree d and rank at least r on a k-gonal curve of genus g, we identify those that are maximal with respect to the dominance order.

DEFINITION 2.5. Let $\alpha \leq \min\{r + 1, k - 1\}$ be a positive integer. By the division algorithm there exists a unique pair of integers q, β such that

$$r + 1 = q\alpha + \beta, \quad 0 \leq \beta < \alpha.$$

Similarly, there exists a unique pair of integers q', β' such that

$$g - d + r = q'(k - \alpha) + \beta', \quad 0 \leq \beta' < k - \alpha.$$

We define the splitting type $\boldsymbol{\mu}_\alpha$ as follows:

$$\mu_{\alpha,i} := \begin{cases} -q' - 2 & \text{if } 0 < i \leq \beta', \\ -q' - 1 & \text{if } \beta' < i \leq k - \alpha, \\ q - 1 & \text{if } k - \alpha < i \leq k - \beta, \\ q & \text{if } k - \beta < i \leq k. \end{cases}$$

Heuristically, $\boldsymbol{\mu}_\alpha$ is the "most balanced" splitting type of degree d and rank r, subject to the constraint that precisely α of its entries are nonnegative. We show that the expected codimension of $W^{\boldsymbol{\mu}_\alpha}(C)$ coincides with the dimensions of irreducible components of $W_d^r(C)$ predicted by [Pfl17a, Question 1.12].

LEMMA 2.6. *For any integer α, we have*

$$g - |\boldsymbol{\mu}_\alpha| = \rho(g, \alpha - 1, d) - (r + 1 - \alpha)k.$$

Proof. First, recall that

$$|\boldsymbol{\mu}_\alpha| = \sum_{i<j} \max\{0, \mu_{\alpha,j} - \mu_{\alpha,i} - 1\}.$$

If $i < j \leq k - \alpha$, then $\mu_{\alpha,j} - \mu_{\alpha,i} \leq 1$, so the pair (i, j) does not contribute to the sum above. Similarly, if $k - \alpha < i < j$, then $\mu_{\alpha,j} - \mu_{\alpha,i} \leq 1$, so again the pair (i, j) does not contribute to the sum above.

On the other hand, if $i \leq k - \alpha$ and $j > k - \alpha$, then the pair (i, j) does contribute to the sum. There are precisely $(k - \alpha)\alpha$ such pairs; each $\mu_{\alpha,i}$ with $i \leq k - \alpha$ appears in exactly α of these pairs, and each $\mu_{\alpha,j}$ with $j > k - \alpha$ appears in exactly $k - \alpha$ of these pairs. It follows that we may rewrite the sum above as

$$|\boldsymbol{\mu}| = (k - \alpha) \sum_{j=k-\alpha+1}^{k} \mu_j - \alpha \sum_{i=1}^{k-\alpha} \mu_i - (k - \alpha)\alpha$$

$$= (k - \alpha)(r + 1 - \alpha) + \alpha(g - d + r + k - \alpha) - (k - \alpha)\alpha$$

$$= \alpha(g - d + \alpha - 1) + (r + 1 - \alpha)k.$$

Subtracting both sides from g yields the result. □

Recall that the integers $g, r, d,$ and k are fixed. We will say that a splitting type is *maximal* if it is maximal with respect to the dominance order among all splitting types satisfying

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} \geq r + 1.$$

In the rest of this section, we show that the maximal splitting types are precisely the splitting types $\boldsymbol{\mu}_\alpha$ when either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$. We first prove the following reduction step.

LEMMA 2.7. *A maximal splitting type $\boldsymbol{\mu}$ satisfies*

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Proof. For the purposes of this argument, we define

$$h(\boldsymbol{\mu}) = \sum_{i=1}^{k} \max\{0, \mu_i + 1\}.$$

Let $\boldsymbol{\mu}$ be a splitting type satisfying

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and $h(\boldsymbol{\mu}) \geq r + 1$. We will show, by induction on $h(\boldsymbol{\mu})$, that there exists a splitting type $\boldsymbol{\lambda}$ such that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ and $h(\boldsymbol{\lambda}) = r + 1$.

Since $r \geq 0$, we see that $\mu_k \geq 0$, and since $h(\boldsymbol{\mu}) \geq r + 1 > d - g + 1$, we see that $\mu_1 < -1$. Therefore there exists an integer i such that $\mu_i > \mu_{i-1}$. Let j be the smallest such integer and j' the largest such integer. Since $\mu_1 < -1$ and $\mu_k \geq 0$, either $j < j'$, or $j = j'$ and $\mu_{j-1} < \mu_j - 1$. It follows that the vector $\boldsymbol{\mu}'$ obtained from $\boldsymbol{\mu}$ by adding 1 to μ_{j-1} and subtracting 1 from $\mu_{j'}$ is nondecreasing, and therefore a valid splitting type. Moreover, we have $\boldsymbol{\mu} < \boldsymbol{\mu}'$. Since $\mu_{j-1} < -1$ and $\mu_{j'} \geq 0$, we see that $h(\boldsymbol{\mu}') = h(\boldsymbol{\mu}) - 1$, and the result follows by induction. $\square$

We now show that every maximal splitting type is of the form $\boldsymbol{\mu}_\alpha$ for some α.

LEMMA 2.8. *Let $\boldsymbol{\mu}$ be a splitting type satisfying*

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Let α denote the number of nonnegative entries of $\boldsymbol{\mu}$. Then $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$.

Proof. By assumption we have

$$\sum_{i=k-\alpha+1}^{k} \mu_i = r + 1 - \alpha = \sum_{i=k-\alpha+1}^{k} \mu_{\alpha,i}.$$

It follows that

$$\sum_{i=1}^{k-\alpha} \mu_i = -(g - d + r) - (k - \alpha) = \sum_{i=1}^{k-\alpha} \mu_{\alpha,i}.$$

Because the entries of $\boldsymbol{\mu}$ are ordered from smallest to largest, for any $\ell \leq k - \alpha$, we see that

$$\sum_{i=1}^{\ell} \mu_i \leq \frac{\ell}{k-\alpha} \sum_{i=1}^{k-\alpha} \mu_i = \frac{-\ell(g-d+r)}{k-\alpha} - \ell.$$

Similarly, for any $\ell \leq \alpha$, we see that

$$\sum_{i=k-\alpha+1}^{k-\alpha+\ell} \mu_i \leq \frac{\ell}{\alpha} \sum_{i=k-\alpha+1}^{k} \mu_i = \frac{\ell(r+1)}{\alpha} - \ell.$$

Therefore by the definition of $\boldsymbol{\mu}_\alpha$ we have $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$. $\square$

COROLLARY 2.9. *If $\boldsymbol{\mu}$ is a maximal splitting type, then $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ for some integer α.*

Proof. Let $\boldsymbol{\mu}$ be a maximal splitting type. By Lemma 2.7 we see that

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Let α denote the number of nonnegative entries of $\boldsymbol{\mu}$. By Lemma 2.8 we have $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$, but since $\boldsymbol{\mu}$ is maximal, it follows that $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$. $\square$

We now show that if $\alpha < \min\{k - (g - d + r), r + 1\}$, then $\boldsymbol{\mu}_\alpha$ is not maximal.

LEMMA 2.10. *If $\alpha < \min\{k - (g - d + r), r + 1\}$, then $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}_{\alpha+1}$.*

Proof. Since $g - d + r < k - \alpha$, by definition we have $\mu_{\alpha,k-\alpha} = -1$. If $r + 1$ is not divisible by α, then consider the splitting type $\boldsymbol{\mu}$ obtained from $\boldsymbol{\mu}_\alpha$ by adding 1 to $\mu_{\alpha,k-\alpha}$ and subtracting 1 from $\mu_{\alpha,k-\beta+1}$. On the other hand, if $r + 1$ is divisible by α, then since $\alpha < r + 1$, we must have $\mu_{\alpha,k-\alpha+1} > 0$. In this case, consider the splitting type $\boldsymbol{\mu}$ obtained from $\boldsymbol{\mu}_\alpha$ by adding 1 to $\mu_{\alpha,k-\alpha}$ and subtracting 1 from $\mu_{\alpha,k-\alpha+1}$. In either case, we see that $\boldsymbol{\mu}$ is a splitting type with $\alpha + 1$ nonnegative entries satisfying $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}$. By Lemma 2.8 we have $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu} \leq \boldsymbol{\mu}_{\alpha+1}$. $\square$

Finally, we see that the remaining splitting types $\boldsymbol{\mu}_\alpha$ are maximal.

PROPOSITION 2.11. *The splitting type $\boldsymbol{\mu}$ is maximal if and only if $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ for some integer α satisfying either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$.*

Proof. By Corollary 2.9 every maximal splitting type is of the form $\boldsymbol{\mu}_\alpha$ for some integer α. By Lemma 2.10, if $\alpha < k - (g - d + r)$ and $\alpha \neq r + 1$, then $\boldsymbol{\mu}_\alpha$ is not maximal. It therefore suffices to show that if $\alpha \neq \gamma$ are both greater than or equal to $k - (g - d + r)$, then $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are incomparable.

Without loss of generality, assume that $\alpha < \gamma$. We write

$$r + 1 = q_\alpha \alpha + \beta_\alpha, \quad 0 \leq \beta_\alpha < \alpha$$
$$= q_\gamma \gamma + \beta_\gamma, \quad 0 \leq \beta_\gamma < \gamma.$$

Since $\alpha < \gamma$, we see that $q_\alpha \geq q_\gamma$. Moreover, since $\gamma \leq r + 1$, we see that both q_α and q_γ are positive. It follows that if $q_\alpha = q_\gamma$, then $\beta_\alpha > \beta_\gamma$. Thus, if j is the largest integer such that $\mu_{\alpha,j} \neq \mu_{\gamma,j}$, then $\mu_{\alpha,j} > \mu_{\gamma,j}$. If $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are comparable, then we see that $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}_\gamma$.

Since $k - \alpha \leq g - d + r$, we see by a similar argument that if j' is the smallest integer such that $\mu_{\alpha,j'} \neq \mu_{\gamma,j'}$, then $\mu_{\alpha,j'} > \mu_{\gamma,j'}$. It follows that if $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are comparable, then $\boldsymbol{\mu}_\alpha > \boldsymbol{\mu}_\gamma$. Combining these two observations, we see that $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are incomparable. $\square$

3. Divisor Theory of Chains of Loops

In this section, we survey the theory of special divisors on chains of loops, as discussed in [Pfl17a; Pfl17b; JR17]. We refer the reader to those papers for more
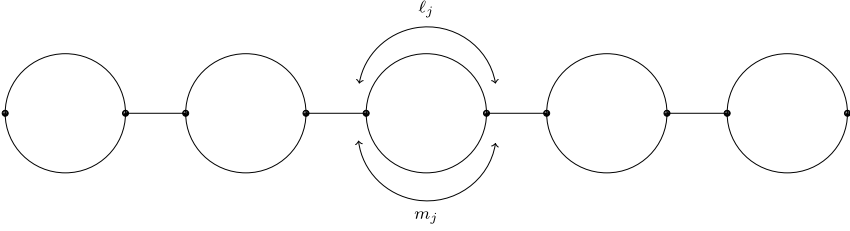
Figure 2   The chain of loops Γ.

details. For a more general overview of divisors on tropical curves, we refer the reader to [Bak08; BJ16]. For the uninitiated, we will not require most of the material of these papers; we will use only the classification of special divisors on chains of loops from [Pfl17a; Pfl17b].

3.1. Chains of Loops and Torsion Profiles

Let Γ be a chain of g loops with bridges, as pictured in Figure 2. Each of the g loops consists of two edges. We denote the lengths of the top and bottom edge of the jth loop by ℓ_j and m_j, respectively. The Brill–Noether theory of chains of loops is governed by the *torsion orders* of the loops.

DEFINITION 3.1 ([Pfl17b, Definition 1.9]). If $\ell_j + m_j$ is an irrational multiple of m_j, then the jth *torsion order* τ_j of Γ is 0. Otherwise, we define τ_j to be the minimum positive integer such that $\tau_j m_j$ is an integer multiple of $\ell_j + m_j$. The sequence $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_g)$ is called the *torsion profile* of Γ.

For the remainder of this paper, we assume that the torsion profile of Γ is given by

$$\tau_i := \begin{cases} 0 & \text{if } i < k \text{ or } i > g - k + 1, \\ k & \text{otherwise.} \end{cases}$$

This chain of loops with this torsion profile possesses a distinguished divisor class of rank 1 and degree k, given by $g_k^1 = kv_k$, where v_k is the lefthand vertex of the kth loop.

REMARK 3.2. Note that, unlike [Pfl17a, Definition 2.1], we do *not* require the first $k - 1$ loops or the last $k - 1$ loops to have torsion order k. This choice does not affect the gonality, or more generally the Brill–Noether theory, of this metric graph. A primary reason for this choice is that the space of such metric graphs has dimension equal to that of the Hurwitz space, namely $2g + 2k - 5$.

Pflueger [Pfl17b] classifies the special divisor classes on chains of loops. This classification generalizes that of special divisor classes on generic chains of loops in [CDPR12]. Specifically, Pflueger shows that $W_d^r(\Gamma)$ is a union of tori, where the tori are indexed by certain types of tableaux. Whereas Pflueger's analysis applies

to chains of loops with arbitrary torsion profiles, we record it only for the torsion profile above. For ease of notation, given a positive integer a, we write $[a]$ for the finite set $\{1, \ldots, a\}$.

DEFINITION 3.3 ([Pfl17a, Definition 2.5]). Let a and b be positive integers. Recall that a *tableau* on $[a] \times [b]$ with alphabet $[g]$ is a function $t : [a] \times [b] \to [g]$ satisfying

$$t(x, y) < t(x, y + 1) \quad \text{and} \quad t(x, y) < t(x + 1, y) \quad \text{for all } (x, y).$$

A tableau t is *standard* if t is injective. A tableau t is called a *k-uniform displacement tableau* if, whenever

$$t(x, y) = t(x', y'), \quad \text{we have} \quad x - y = x' - y' \pmod{k}.$$

It is standard to depict a tableau on $[a] \times [b]$ as a rectangle with a columns and b rows, where the box in position (x, y) is filled with the symbol $t(x, y)$. We draw our tableaux according to the English convention, so that the box $(1, 1)$ appears in the upper lefthand corner.

3.2. Coordinates on $\mathrm{Pic}(\Gamma)$

A nice feature of the chain of loops is that its Picard group has a natural system of coordinates. On the jth loop, let $\langle \xi \rangle_j$ denote the point located ξm_j units from the righthand vertex in the counterclockwise direction. Note that

$$\langle \xi \rangle_j = \langle \eta \rangle_j \quad \text{if and only if} \quad \xi = \eta \pmod{\tau_j}.$$

By the tropical Abel–Jacobi theorem [BN07] every divisor class D of degree d on Γ has a *unique* representative of the form

$$(d - g)\langle 0 \rangle_g + \sum_{j=1}^{g} \langle \xi_j(D) \rangle_j$$

for some real numbers $\xi_j(D)$. Because this expression is unique, the functions ξ_j form a system of coordinates on $\mathrm{Pic}^d(\Gamma)$. This representative of the divisor class D is known as the *break divisor* representative [MZ08; ABKS14].

DEFINITION 3.4 ([Pfl17b, Definition 3.5]). Given a degree d and a k-uniform displacement tableau t with alphabet $[g]$, we define the coordinate subtorus $\mathbb{T}(t)$ as follows:

$$\mathbb{T}(t) := \{D \in \mathrm{Pic}^d(\Gamma) | \xi_{t(x,y)}(D) = y - x \pmod{k}\}.$$

Note that the coordinate $\xi_j(D)$ of a divisor class D in $\mathbb{T}(t)$ is determined if and only if j is in the image of t. It follows that the codimension of $\mathbb{T}(t)$ in $\mathrm{Pic}^d(\Gamma)$ is the number of distinct symbols in t. The main combinatorial result of [Pfl17b] is a classification of special divisors on Γ.

THEOREM 3.5 ([Pfl17b, Theorem 1.4]). *For any positive integers r and d satisfying $r > d - g$, we have*

$$W_d^r(\Gamma) = \bigcup \mathbb{T}(t),$$

where the union is over k-uniform displacement tableaux on $[r + 1] \times [g - d + r]$ with alphabet $[g]$.

Notably, Pflueger does not consider the containment relations between the various tori $\mathbb{T}(t)$. These containment relations are the primary concern of Sections 5 and 6. We note the following, which will be explored in more detail in these later sections.

LEMMA 3.6. *Let t and t' be k-uniform displacement tableaux on $[a] \times [b]$. Then $\mathbb{T}(t) \subseteq \mathbb{T}(t')$ if and only if*

(1) *every symbol in t' is a symbol in t, and*
(2) *if $t(x, y) = t'(x', y')$, then $x - y = x' - y' \pmod{k}$.*

Under Pflueger's classification of special divisors, there is a natural interpretation of Serre duality. Given a tableau t on $[a] \times [b]$, define the *transpose tableau* to be the tableau t^T on $[b] \times [a]$ given by $t^T(x, y) = t(y, x)$.

LEMMA 3.7 ([Pfl17b, Remark 3.6]). *Let t be a k-uniform displacement tableau on $[r + 1] \times [g - d + r]$ with alphabet $[g]$, and let $D \in \mathbb{T}(t)$ be a divisor class. Then the Serre dual $K_\Gamma - D$ is contained in $\mathbb{T}(t^T)$.*

3.3. Scrollar Tableaux

In [JR17], Ranganathan and the second author consider a special type of k-uniform displacement tableaux, known as *scrollar tableaux*. Throughout this section, we fix positive integers a and b, and a positive integer $\alpha \leq \min\{a, k - 1\}$ satisfying either $\alpha \geq k - b$ or $\alpha = a$. As in Definition 2.5, we write

$$a = q\alpha + \beta, \quad 0 \leq \beta < \alpha$$

and

$$b = q'(k - \alpha) + \beta', \quad 0 \leq \beta' < k - \alpha.$$

DEFINITION 3.8. Let t be a tableau on $[a] \times [b]$. We define t to be *scrollar of type α* if it satisfies the following three conditions.

(1) $t(x, y) = t(x', y')$ if and only if there exists an integer ℓ such that both

$$x' - x = \ell\alpha \quad \text{and} \quad y' - y = \ell(\alpha - k).$$

(2) If $\alpha = a$, then $t(1, y) > t(a, y + a - k)$ for all $y > k - a$.
(3) If $\alpha = k - b$, then $t(x, 1) > t(x + b - k, b)$ for all $x > k - b$.

REMARK 3.9. When $k - b < \alpha < a$, Definition 3.8 agrees with [JR17, Definition 7.1], but in the edge cases the two definitions disagree. This is because when α is equal to a or $k - b$, every standard tableau satisfies [JR17, Definition 7.1] trivially. In Sections 5 and 6, however, we will see that $\mathbb{T}(t)$ is maximal only for

tableaux satisfying Definition 3.8. We note that when $\alpha < a$, condition (1) implies an inequality analogous to that of condition (2), because

$$t(1, y) = t(\alpha + 1, y + \alpha - k) > t(\alpha, y + \alpha - k).$$

Similarly, when $\alpha > k - b$, condition (1) implies an inequality analogous to that of condition (3).

For the reader interested in comparing the definitions in the two papers, we provide a brief dictionary. The integer α appearing here is the same as n in [JR17]. The integer β agrees with b in [JR17], and q is equal to $\lfloor \frac{a}{\alpha} \rfloor = \lfloor \frac{r+1}{n} \rfloor$.

EXAMPLE 3.10. A typical example of a scrollar tableau appears in Figure 3. Note that the boxes in the first α columns necessarily contain distinct symbols, as do the boxes in the last $k - \alpha$ rows. The symbols in the remaining boxes are obtained by repeatedly translating the symbols in this L-shaped region α boxes rightward and $k - \alpha$ boxes upward.

EXAMPLE 3.11. Figure 4 depicts three different 3-uniform displacement tableaux on $[3] \times [2]$. The first tableau t is scrollar of type 2. To see this, note that there is only one pair of boxes whose x coordinates differ by a multiple of 2 and whose y coordinates differ by the same multiple of -1, and these boxes contain the same symbol. The second tableau t' is scrollar of type 1, because it is standard, $t'(2, 1) > t'(1, 2)$, and $t'(3, 1) > t'(2, 2)$. The final tableau t^* is not scrollar of either type. Specifically, it is not scrollar of type 1 because $t^*(2, 1) < t^*(1, 2)$, and it is not scrollar of type 2 because $t^*(3, 1) \neq t^*(1, 2)$. By Lemma 3.6 we see that $\mathbb{T}(t^*) \subset \mathbb{T}(t)$.

The following observation from [JR17] is central to our argument.

Figure 3   A scrollar tableau of type 3, where $k = 5$.

Figure 4   Three different 3-uniform displacement tableaux. The first two are scrollar of different types, and the third is not scrollar.

PROPOSITION 3.12. *Let t be a scrollar tableau of type α on $[r+1] \times [g-d+r]$ with alphabet $[g]$. Then $g \geq |\boldsymbol{\mu}_\alpha|$ and*

$$\dim \mathbb{T}(t) = g - |\boldsymbol{\mu}_\alpha|.$$

Proof. By [JR17, Proposition 7.4] we have

$$\dim \mathbb{T}(t) = \rho(g, \alpha - 1, d) - (r + 1 - \alpha)k.$$

The result then follows from Lemma 2.6.   □

Proposition 3.12 suggests a connection between scrollar tableaux of type α and the splitting type $\boldsymbol{\mu}_\alpha$. This connection will be established in Proposition 4.1. The following lemma is a key to the proof of Proposition 4.1.

LEMMA 3.13 ([JR17, Corollary 7.3]). *Let t be a scrollar tableau of type α, and let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. Then*

(1) $\text{rk}(D - qg_k^1) = \beta - 1$, *and*
(2) $\text{rk}(D - (q+1)g_k^1) = -1$.

REMARK 3.14. In Lemma 3.13, when we say that the divisor class $D \in \mathbb{T}(t)$ is "sufficiently general", we mean that D lies in the complement of finitely many coordinate subtori of codimension at least 1 in $\mathbb{T}(t)$. In particular, the set of divisor classes in $\mathbb{T}(t)$ satisfying the conclusion of Lemma 3.13 is open and dense in $\mathbb{T}(t)$.

Much of [JR17] is devoted to a lifting result for divisor classes in $\mathbb{T}(t)$ when t is a scrollar tableau. Unfortunately, [JR17] does not establish this lifting result for all scrollar tableaux, but only for those that satisfy the following condition.

DEFINITION 3.15. We say that a tableau t *has no vertical steps* if

$$t(x, y+1) \neq t(x, y) + 1 \quad \text{for all } x, y.$$

We note that if $g \geq |\boldsymbol{\mu}_\alpha|$ and $\alpha > 1$, then there exists a scrollar tableau of type α with no vertical steps. For example, the transpose of the tableau defined in the proof of [Pfl17a, Lemma 3.5] has no vertical steps. Another example of such a tableau appears in Figure 5.

The following proposition is one of the main technical results of [JR17]. In this proposition and throughout Section 4, we let K be an algebraically closed, non-Archimedean-valued field of equicharacteristic zero.

1	2	3	7	8	9	13
4	5	6	10	11	12	16
7	8	9	13	14	15	19
10	11	12	16	17	18	20
13	14	15	19	21	22	23

Figure 5   A scrollar tableau of type 3, where $k = 5$, with no vertical steps.

PROPOSITION 3.16 ([JR17, Proposition 9.2]). *Let t be a scrollar tableau of type* α *with no vertical steps, and let* $D \in \mathbb{T}(t)$ *be a sufficiently general divisor class. Then there exist a curve C of genus g and gonality k over K with skeleton* Γ *and a divisor class* $\mathcal{D} \in W_d^r(C)$ *specializing to D.*

4. Connections Between Combinatorics and Algebraic Geometry

In this section, we demonstrate the connection between our combinatorial and geometric results. Specifically, we show that Theorem 1.4 implies Theorem 1.3. To begin, we establish the connection between scrollar tableaux of type α and the splitting types $\boldsymbol{\mu}_\alpha$.

PROPOSITION 4.1. *Let C be a curve of genus g and gonality k over K with skeleton* Γ. *Let t be a scrollar tableau of type* α, *let* $D \in \mathbb{T}(t)$ *be a sufficiently general divisor class, and let* $\mathcal{D} \in W_d^r(C)$ *be a divisor that specializes to D. Then* $\mathcal{D} \in W^{\boldsymbol{\mu}_\alpha}(C)$.

Proof. Let $\boldsymbol{\mu}$ denote the splitting type of $\pi_* \mathcal{O}(\mathcal{D})$. By Lemma 3.13 we have

$$\mathrm{rk}(D - q g_k^1) = \beta - 1,$$
$$\mathrm{rk}(D - (q+1)g_k^1) = -1.$$

By Baker's specialization lemma [Bak08] it follows that

$$h^0(\mathcal{D} - q g_k^1) \le \beta, \tag{1}$$
$$h^0(\mathcal{D} - (q+1)g_k^1) = 0. \tag{2}$$

Recall that if t^T denotes the transpose of t, then the Serre dual $K_\Gamma - D$ is contained in $\mathbb{T}(t^T)$. Note that t^T is also a scrollar tableau. Therefore by Lemma 3.13, since $K_\Gamma - D$ is sufficiently general, we see that

$$\mathrm{rk}(K_\Gamma - D - q' g_k^1) = \beta' - 1,$$
$$\mathrm{rk}(K_\Gamma - D - (q'+1)g_k^1) = -1.$$

By Baker's specialization lemma it follows that

$$h^0(K_C - \mathcal{D} - q' g_k^1) \le \beta', \tag{3}$$
$$h^0(K_C - \mathcal{D} - (q'+1)g_k^1) = 0. \tag{4}$$

By ($\star$), (2) implies that $\mu_k \le q$, and (1) implies that $\mu_{k-\beta} \le q-1$. It follows that

$$\mu_{k-\alpha+1} + \cdots + \mu_{k-\alpha+\ell} \le \mu_{\alpha,k-\alpha+1} + \cdots + \mu_{\alpha,k-\alpha+\ell} \quad \text{for all } \ell \le \alpha.$$

Similarly, (4) implies that $\mu_1 \ge -q'-2$, and (3) implies that $\mu_{\beta'+1} \ge -q'-1$. It follows that

$$\mu_1 + \cdots + \mu_\ell \ge \mu_{\alpha,1} + \cdots + \mu_{\alpha,\ell} \quad \text{for all } \ell \le k-\alpha.$$

Putting these together, we see that $\boldsymbol{\mu} \ge \boldsymbol{\mu}_\alpha$. By Proposition 2.11, however, $\boldsymbol{\mu}_\alpha$ is maximal, and hence $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$. $\qquad\square$

COROLLARY 4.2. *Let t be a scrollar tableau of type α with no vertical steps, and let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. Then there exist a curve C of genus g and gonality k over K with skeleton Γ and a divisor class $\mathcal{D} \in W^{\mu_\alpha}(C)$ specializing to D.*

Proof. By Proposition 3.16 there exists a curve C of genus g and gonality k over K with skeleton Γ, and a divisor class $\mathcal{D} \in W_d^r(C)$ specializing to D. By Proposition 4.1, the divisor class $\mathcal{D}$ is in $W^{\mu_\alpha}(C)$. $\square$

We now show that Theorem 1.4 implies Theorem 1.3. We do this in two steps. First, we obtain an upper bound on a particular component of $W_d^r(C)$.

PROPOSITION 4.3. *Let C and $\mathcal{D}$ be as in Corollary 4.2, and let $\mathcal{Y}$ be any irreducible component of $W_d^r(C)$ containing $\mathcal{D}$. Then*

$$\dim \mathcal{Y} \leq g - |\boldsymbol{\mu}_\alpha|.$$

Proof. By [Gub07, Theorem 6.9]

$$\dim \mathcal{Y} = \dim \operatorname{Trop} \mathcal{Y}.$$

By Baker's specialization lemma we see that $\operatorname{Trop} \mathcal{Y} \subseteq W_d^r(\Gamma)$. It follows that $\dim \mathcal{Y}$ cannot exceed the local dimension of $W_d^r(\Gamma)$ in a neighborhood of D. By Theorem 1.4, $\mathbb{T}(t)$ is maximal with respect to containment in $W_d^r(\Gamma)$, and since $D \in \mathbb{T}(t)$ is sufficiently general, the local dimension of $W_d^r(\Gamma)$ in a neighborhood of D is equal to that of $\mathbb{T}(t)$. Finally, by Proposition 3.12 we have

$$\dim \mathcal{Y} \leq \dim \mathbb{T}(t) = g - |\boldsymbol{\mu}_\alpha|.   \square$$

Proof that Theorem 1.4 implies Theorem 1.3. The case $k = 2$ is classical, so we assume that $k \geq 3$. Let $\alpha \leq \min\{r + 1, k - 1\}$ be a positive integer satisfying either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$. If $\alpha \geq k - (g - d + r)$, then applying Serre duality exchanges α with $k - \alpha$, so we may assume that $\alpha > 1$.

Since $|\boldsymbol{\mu}| \leq g$ and $\alpha > 1$, there exists a scrollar tableau t of type α with no vertical steps. Let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. By Corollary 4.2 there exist a curve C of genus g and gonality k over K with skeleton Γ and a divisor class $\mathcal{D} \in W^{\mu_\alpha}(C)$ specializing to D. If $\mathcal{Y}$ is an irreducible component of $W^{\mu_\alpha}(C)$ containing $\mathcal{D}$, then by Proposition 4.3 we have

$$\dim \mathcal{Y} \leq g - |\boldsymbol{\mu}_\alpha|.$$

It therefore suffices to prove the reverse inequality.

The rest of the proof is identical to that of [JR17, Theorem 9.3], which we reproduce here for completeness. Let $\mathcal{M}_g^k$ be the moduli space of curves of genus g that admit a degree k map to $\mathbb{P}^1$, let $\mathcal{C}_k$ be the universal curve, and let $\mathcal{W}^{\mu_\alpha}$ be the universal splitting-type locus over $\mathcal{M}_g^k$. Let $\widetilde{\mathcal{W}}^{\mu_\alpha}$ be the locus in the symmetric dth fiber power of $\mathcal{C}_k$ parameterizing divisors $\mathcal{D}$ such that $\pi_* \mathcal{O}(\mathcal{D})$ has splitting type $\boldsymbol{\mu}_\alpha$.

We work in the Berkovich analytic domain of k-gonal curves whose skeleton is a k-gonal chain of loops. By Corollary 4.2 the tropicalization of $\widetilde{\mathcal{W}}^{\boldsymbol{\mu}_\alpha,\mathrm{an}}$ has dimension at least

$$3g - 5 + 2k - |\boldsymbol{\mu}_\alpha| + r.$$

If $\pi_*\mathcal{O}(\mathcal{D}) \cong \mathcal{O}(\boldsymbol{\mu}_\alpha)$, then $\mathcal{D}$ has rank exactly r. It follows that $\mathcal{W}^{\boldsymbol{\mu}_\alpha}$ has dimension at least

$$3g - 5 + 2k - |\boldsymbol{\mu}_\alpha|.$$

By Corollary 4.2 there is an irreducible component of $\mathcal{W}^{\boldsymbol{\mu}_\alpha}$ whose tropicalization contains pairs of the form $(\Gamma, \mathcal{D})$ where Γ is a k-gonal chain of loops and $\mathcal{D} \in \mathbb{T}(t)$ is sufficiently general. The image of this component in $\mathcal{M}_g^{k,\mathrm{trop}}$ has dimension $2g - 5 + 2k$. It follows that this component dominates $\mathcal{M}_g^k$, and the fibers have dimension at least $g - |\boldsymbol{\mu}_\alpha|$.

Combining the two bounds, we see that there exists an irreducible component $\mathcal{Y}$ of $W^{\boldsymbol{\mu}_\alpha}(C)$, containing $\mathcal{D}$, of dimension $g - |\boldsymbol{\mu}_\alpha|$. If $\mathcal{Z}$ is a component of $W_d^r(C)$ containing $\mathcal{Y}$, then by Proposition 4.3 we see that

$$\dim \mathcal{Z} = \dim \mathcal{Y}.$$

It follows that $\mathcal{Z}$ is the closure of $\mathcal{Y}$. $\qquad\square$

5. Maximality of Scrollar Tableaux

Having established that Theorem 1.3 follows from our combinatorial results, it remains to prove the combinatorial results. The goal of this section is to prove the following:

THEOREM 5.1. *Let t be a scrollar tableau of type α on $[a] \times [b]$. Then $\mathbb{T}(t)$ is maximal with respect to containment.*

Before proving Theorem 5.1, we first make two simple observations. These will be useful because if $\mathbb{T}(t) \subseteq \mathbb{T}(t')$, then by Lemma 3.6, for every box (n, m) in $[a] \times [b]$, there exists a box (x, y) such that $t'(n, m) = t(x, y)$. Our argument will break into cases, depending on the location of (x, y) relative to that of (n, m).

LEMMA 5.2. *Let α be a positive integer and (n, m) any box in $[a] \times [b]$. For any box (x, y) in $[a] \times [b]$, there exists an integer ℓ such that one of the following holds*:

(1) $x \leq n - \ell\alpha$ *and* $y \leq m + \ell(k - \alpha)$,
(2) $x \geq n - \ell\alpha$ *and* $y \geq m + \ell(k - \alpha)$, *or*
(3) $n - (\ell + 1)\alpha < x < n - \ell\alpha$ *and* $m + \ell(k - \alpha) < y < m + (\ell + 1)(k - \alpha)$.

Proof. By the division algorithm, there exists an integer ℓ such that

$$n - (\ell + 1)\alpha < x \leq n - \ell\alpha.$$

If $y \leq m + \ell(k - \alpha)$, then case (1) holds. If $y \geq m + (\ell + 1)(k - \alpha)$, or if $x = n - \ell\alpha$ and $y \geq m + \ell(k - \alpha)$, then case (2) holds. Otherwise, $x \neq n - \ell\alpha$, and case (3) holds. $\qquad\square$
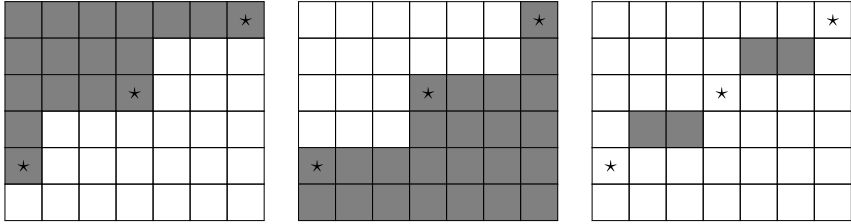
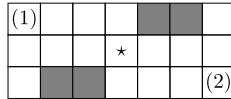**Figure 6**   The three regions described in Lemma 5.2.

Figure 7   When $\alpha = k - b$, the integer ℓ can be taken to be one of $-1, 0,$ or 1.

Lemma 5.2 is illustrated in Figure 6. Boxes of the form $(n - \ell\alpha, m + \ell(k - \alpha))$ are labeled with stars, and the three cases of Lemma 5.2 are depicted in gray. Note that every box is contained in one of the three gray regions.

REMARK 5.3. If α is equal to either a or $k - b$, then the integer ℓ in Lemma 5.2 can be taken to be one of $-1, 0,$ or 1, as illustrated in Figure 7. If $\ell = \pm 1$, then the box $(n - \ell\alpha, m + \ell(k - \alpha))$ is not contained in $[a] \times [b]$.

The following simple lemma is key to our argument.

LEMMA 5.4. *Let α be a positive integer, let (n, m) be any box in $[a] \times [b]$, and let (x, y) be a box satisfying condition (3) of Lemma 5.2. Then*

$$x - y \not\equiv n - m \pmod{k}.$$

Proof. Since

$$n - (\ell + 1)a < x < n - \ell a$$

and

$$m + \ell(k - a) < y < m + (\ell + 1)(k - a),$$

we have

$$(n - m) - (\ell + 1)k < x - y < (n - m) - \ell k.$$

Hence $x - y \not\equiv n - m \pmod{k}$. $\square$

We are now prepared to prove the main result of this section, the maximality of scrollar tableaux.

Proof of Theorem 5.1. Let t' be a k-uniform displacement tableau such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. We will show that $t = t'$. We first demonstrate, by induction, that $t'(n, m) \geq t(n, m)$ for all $(n, m) \in [a] \times [b]$. The base case $t'(1, 1) \geq t(1, 1)$ holds

because, by Lemma 3.6, $t'(1, 1)$ must be a symbol in t, and $t(1, 1)$ is the smallest symbol in t.

For our inductive hypothesis, suppose that $t'(x, y) \geq t(x, y)$ for all (x, y) such that $x \leq n$ and $y \leq m$, not both equal. We will show that $t'(n, m) \geq t(n, m)$. By Lemma 3.6 there exists $(x, y) \in [a] \times [b]$ such that $t'(n, m) = t(x, y)$. By Lemma 5.2, there exists an integer ℓ such that one of the following holds:

(1) $x \leq n - \ell\alpha$ and $y \leq m + \ell(k - \alpha)$,
(2) $x \geq n - \ell\alpha$ and $y \geq m + \ell(k - \alpha)$, or
(3) $n - (\ell + 1)\alpha < x < n - \ell\alpha$ and $m + \ell(k - \alpha) < y < m + (\ell + 1)(k - \alpha)$.

If (x, y) satisfies (3), then by Lemma 5.4, $x - y \not\equiv n - m \pmod{k}$, a contradiction to Lemma 3.6. Hence (x, y) must satisfy either (1) or (2).

There are now two cases to consider, the case where the box $(n - \ell\alpha, m + \ell(k - \alpha))$ is contained in $[a] \times [b]$, and the case where it is not. We first consider the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is contained in $[a] \times [b]$. Notice that if α is equal to a or $k - b$, then in this case, we must have $\ell = 0$. If (x, y) satisfies (2), then

$$t(x, y) \geq t(n - \ell\alpha, m + \ell(k - \alpha)) = t(n, m),$$

and hence $t'(n, m) \geq t(n, m)$, as desired. If (x, y) satisfies (1) and $(x, y) \neq (n - \ell\alpha, m + \ell(k - \alpha))$, then we have either

$$t'(n, m) = t(x, y) \leq t(n - \ell\alpha - 1, m + \ell(k - \alpha)) \quad \text{or}$$
$$t'(n, m) = t(x, y) \leq t(n - \ell\alpha, m + \ell(k - \alpha) - 1).$$

First, assume that $n, m > 1$. Since t is scrollar, we have

$$t(n - \ell\alpha - 1, m + \ell(k - \alpha)) = t(n - 1, m) \quad \text{and}$$
$$t(n - \ell\alpha, m + \ell(k - \alpha) - 1) = t(n, m - 1).$$

By our inductive hypothesis, however, we have $t(n - 1, m) \leq t'(n - 1, m)$ and $t(n, m - 1) \leq t'(n, m - 1)$. This guarantees that either $t'(n, m) \leq t'(n - 1, m)$ or $t'(n, m) \leq t'(n, m - 1)$, a contradiction. It follows that

$$t'(n, m) = t(x, y) = t(n - \ell\alpha, m + \ell(k - \alpha)) = t(n, m).$$

Now suppose that $m = 1$ and $n > 1$. The case where $n = 1$ will follow from a similar argument. Without loss of generality, let ℓ be the smallest integer such that (x, y) is above and to the right of $(n - \ell\alpha, m + \ell(k - \alpha))$. If $x < n - \ell\alpha$, then the conclusion follows from the argument above. On the other hand, if $x = n - \ell\alpha$, then since

$$m + (\ell - 1)(k - \alpha) < y < m + \ell(k - \alpha),$$

we see that $x - y \not\equiv n - m \pmod{k}$, a contradiction to Lemma 3.6.

We now turn to the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is not contained in $[a] \times [b]$. First, suppose that (x, y) satisfies (2). In this case, either $n - \ell\alpha \leq 0$ or $m + \ell(k - \alpha) \leq 0$, but not both. We will assume that $n - \ell\alpha \leq 0$; the other case follows by a similar argument. If ℓ is any integer satisfying $n - \ell\alpha \leq 0$, then (x, y) is below and to the right of $(1, m + \ell(k - \alpha))$. We may therefore assume
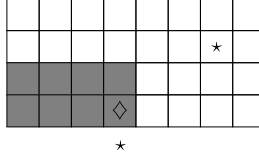
Figure 8   An illustration of the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is not contained in $[a] \times [b]$.

without loss of generality that ℓ is the minimal integer such that $n - \ell\alpha \leq 0$. If α is equal to a, then $\ell = 1$. Because t is scrollar, we observe that

$$t'(n, m) = t(x, y) \geq t(1, m + \ell(k - \alpha)) > t(\alpha, m + (\ell - 1)(k - \alpha))$$
$$\geq t(n - (\ell - 1)\alpha, m + (\ell - 1)(k - \alpha)) = t(n, m).$$

Now suppose that (x, y) satisfies (1). In this case, either $b < m + \ell(k - \alpha)$ or $a < n - \ell\alpha$, but not both. We will assume that $b < m + \ell(k - \alpha)$. The other case follows by a similar argument. Without loss of generality, assume that ℓ is the minimal integer such that $b < m + \ell(k - \alpha)$. As before, if $\alpha = k - b$, then $\ell = 1$. If $y \leq m + (\ell - 1)(k - \alpha)$, then by replacing ℓ with $\ell - 1$ we may reduce to the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is in $[a] \times [b]$. We may therefore assume that

$$m + (\ell - 1)(k - \alpha) < y \leq b < m + \ell(k - \alpha).$$

This situation is illustrated in Figure 8. The boxes $(n - \ell\alpha, m + \ell(k - \alpha))$ and $(n - (\ell - 1)\alpha, m + (\ell - 1)(k - \alpha))$ are labeled with stars, the box $(n - \ell\alpha, b)$ is labeled with a diamond, and the box (x, y) is located somewhere in the shaded region.

If $x = n - \ell\alpha$, then since

$$m + (\ell - 1)(k - \alpha) < y < m + \ell(k - \alpha),$$

we see that $x - y \not\equiv n - m \pmod{k}$, a contradiction to Lemma 3.6. We may therefore assume that $x < n - \ell\alpha$. Because t is scrollar, we have

$$t'(n, m) = t(x, y) \leq t(n - \ell\alpha - 1, b) < t(n - (\ell - 1)\alpha - 1, b + 1 - (k - \alpha))$$
$$\leq t(n - (\ell - 1)\alpha - 1, m + (\ell - 1)(k - \alpha)) = t(n - 1, m).$$

By induction, however, we have $t(n - 1, m) \leq t'(n - 1, m)$, and hence $t'(n, m) \leq t'(n - 1, m)$, a contradiction.

Thus, in every case, we see that $t'(n, m) \geq t(n, m)$. We now show that $t'(n, m) \leq t(n, m)$ for all $(n, m) \in [a] \times [b]$. Combining the two inequalities, we see that $t' = t$. Given a tableau t, define the "rotated" tableau t_R as follows:

$$t_R(x, y) = g + 1 - t(a + 1 - x, b + 1 - y)$$

(see Figure 9 for an example). Returning to our tableaux t and t', we see that by definition both t_R and t'_R are k-uniform displacement tableaux, the tableau t_R is scrollar, and $\mathbb{T}(t_R) \subseteq \mathbb{T}(t'_R)$. By the argument above, we see that $t'_R(n, m) \geq t_R(n, m)$ for all $(n, m) \in [a] \times [b]$, hence $t'(n, m) \leq t(n, m)$ for all $(n, m) \in [a] \times [b]$, and the conclusion follows. $\square$

1	2	4	5	10	11	12
3	7	8	9	13	16	18
5	10	11	12	15	17	20
9	13	16	18	19	22	23
12	15	17	20	21	24	26

26	24	21	20	17	15	12
23	22	19	18	16	13	9
20	17	15	12	11	10	5
18	16	13	9	8	7	3
12	11	10	5	4	2	1

1	3	6	7	10	12	15
4	5	8	9	11	14	18
7	10	12	15	16	17	22
9	11	14	18	19	20	24
15	16	17	22	23	25	26

Figure 9   To obtain the "rotation" of the tableau on the left, first rotate 180 degrees and then subtract each entry from $g + 1$.

6. Non-existence of Other Maximal Tableaux

In this section, we prove the following:

THEOREM 6.1. *Let t be a k-uniform displacement tableau on $[a] \times [b]$. Then there exists a scrollar tableau t' on $[a] \times [b]$ such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$.*

Together with Theorem 5.1, this establishes Theorem 1.4.

   To prove Theorem 6.1, we will describe an algorithm that, starting with t, produces a scrollar tableau t' by replacing certain symbols in t with other symbols in t. We first introduce a statistic on the boxes in a k-uniform displacement tableau.

DEFINITION 6.2. Let t be a k-uniform displacement tableau on $[a] \times [b]$. Given a box (x, y) such that $x + y \geq k$, we define a statistic $S_t(x, y)$ as follows. Consider the symbols appearing above (x, y) in column x and to the left of (x, y) in row y. Among these symbols, the $k - 1$ largest ones form a hook of width α and height $k - \alpha$. We define $S_t(x, y)$ to be α.

REMARK 6.3. Note that if $x + y < k$, then $S_t(x, y)$ is undefined. In this case the box (x, y) is left empty. Additionally, the statistic α cannot appear in any box (x, y) with $x < \alpha$ or $y < k - \alpha$. In particular, for any k-uniform displacement tableau t, we have $S_t(\alpha, k - \alpha) = \alpha$.

   Note also that S_t is well-defined. To see this, let i be the smallest positive integer such that $t(x - i, y) = t(x, y - j)$ for some positive integer j. By the definition of a k-uniform displacement tableau, $i + j$ must be a multiple of k. It follows that the hook from $(x - i, y)$ to $(x, y - j)$ contains at least $k - 1$ distinct symbols, all greater than $t(x - i, y)$.

EXAMPLE 6.4. Figure 10 depicts an example of a 5-uniform displacement tableau t on $[4] \times [4]$. The first figure is t, the second is S_t, and the last two depict example hooks of width 2 and 3, respectively.

Before proceeding further, we will first need the following property of the statistic S_t.

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

			4
		3	3
	2	3	2
1	2	3	3

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

Figure 10 A 5-uniform displacement tableau, its associated statistics, and some example hooks.

LEMMA 6.5. *Let t be a k-uniform displacement tableau. We have the following inequalities on statistics*:

$$S_t(x + 1, y) \leq S_t(x, y) + 1,$$
$$S_t(x, y - 1) \leq S_t(x, y) + 1,$$
$$S_t(x + 1, y - 1) \leq S_t(x, y) + 1.$$

Proof. Let H be the hook containing the $k - 1$ largest symbols appearing above (x, y) in column x and to the left of (x, y) in row y. By the definition of S_t, H contains the boxes

$$(x, y + 1 - k + S_t(x, y)) \quad \text{and} \quad (x + 1 - S_t(x, y), y),$$

but not the boxes

$$(x, y - k + S_t(x, y)) \quad \text{or} \quad (x - S_t(x, y), y).$$

It follows that

$$t(x - S_t(x, y), y) < t(x, y + 1 - k + S_t(x, y)) \quad \text{and}$$
$$t(x + 1 - S_t(x, y), y) > t(x, y - k + S_t(x, y)).$$

If $S_t(x + 1, y) > S_t(x, y) + 1$, then

$$t(x - S_t(x, y), y) \geq t(x + 2 - S_t(x + 1, y), y)$$
$$> t(x + 1, y - k + S_t(x + 1, y))$$
$$> t(x, y + 1 - k + S_t(x, y)),$$

a contradiction.

Similarly, if $S_t(x, y - 1) > S_t(x, y) + 1$, then

$$t(x + 1 - S_t(x, y), y) < t(x - S_t(x, y - 1), y - 1)$$
$$< t(x, y - k + S_t(x, y - 1))$$
$$\leq t(x, y - k + S_t(x, y)),$$

a contradiction.

Finally, if $S_t(x + 1, y - 1) > S_t(x, y) + 1$, then

$$t(x - S_t(x, y), y) > t(x + 2 - S_t(x + 1, y - 1), y - 1)$$
$$> t(x + 1, y - 1 - k + S_t(x + 1, y - 1))$$
$$> t(x, y - k + S_t(x, y)),$$

another contradiction. $\square$

Figure 11 Two admissible paths of type 3.

DEFINITION 6.6. Let t be a k-uniform displacement tableau on $[a] \times [b]$, and suppose that $a + b \geq k$. An *admissible path* P of type α in t is a sequence of boxes

$$P = (x_0, y_0), (x_1, y_1), \ldots, (x_{a+b-k}, y_{a+b-k})$$

satisfying the following conditions:

(1) $(x_0, y_0) = (\alpha, k - \alpha)$ and $(x_{a+b-k}, y_{a+b-k}) = (a, b)$.
(2) For all i, (x_i, y_i) is equal to either $(x_{i-1} + 1, y_{i-1})$ or $(x_{i-1}, y_{i-1} + 1)$.
(3) If $(x_i, y_i) = (x_{i-1} + 1, y_{i-1})$, then $S_t(x_i, y_i) \leq \alpha$.
(4) If $(x_i, y_i) = (x_{i-1}, y_{i-1} + 1)$, then $S_t(x_i, y_i) \geq \alpha$.

In other words, an admissible path is a sequence of pairwise adjacent boxes starting at $(\alpha, k - \alpha)$ and ending in the bottom right corner of the tableau. Every time the path moves right, the statistic in the new box must be at most α, and every time the path moves down, the statistic in the new box must be at least α.

EXAMPLE 6.7. Figure 11 depicts the statistics S_t for the tableau t from Example 6.4, together with two admissible paths of type 3 shaded. Note that the first path is admissible because the box labeled 2 is to the right of the previous box in the path.

Note that an admissible path of type a is completely vertical and that an admissible path of type $k - b$ is completely horizontal. If there is an admissible path of type a in a tableau t, then $S_t(a, y) = a$ for all $y \geq k - a$. It follows that $t(1, y + k - a) > t(a, y)$ for all $y > k - a$, so t is scrollar of type a. Similarly, if there is an admissible path of type $k - b$ in a tableau t, then t is scrollar of type $k - b$.

The first main goal of this section is to prove the existence of admissible paths. That is, given a k-uniform tableau t on $[a] \times [b]$, we show that there exist an integer α and a admissible path P of type α. Our argument will require the following lemma.

LEMMA 6.8. *Let t be a k-uniform displacement tableau. If P_1 and P_2 are two admissible paths in t of types α_1 and α_2, respectively, then $\alpha_1 = \alpha_2$.*

Proof. First, note that the last box in any admissible path is (a, b), so any two admissible paths intersect. Let (x, y) be the box in the intersection that minimizes $x + y$. Without loss of generality, assume that $\alpha_1 > \alpha_2$. Note that P_1 starts at $(\alpha_1, k - \alpha_1)$, which is above and to the right of $(\alpha_2, k - \alpha_2)$. Because (x, y) is

the first box at which the two paths cross, we see that P_1 must contain the box $(x, y - 1)$ and P_2 must contain the box $(x - 1, y)$. By the definition of admissible paths, we have

$$\alpha_1 \leq S_t(x, y) \leq \alpha_2,$$

contradicting our assumption that $\alpha_1 > \alpha_2$. □

We now prove that admissible paths exist.

PROPOSITION 6.9. *Let t be a k-uniform displacement tableau on* $[a] \times [b]$, *and suppose that* $a + b \geq k$. *Then there exists an admissible path in* t.

Proof. We proceed by induction on $a + b$. In the base case $b = k - a$ the admissible path consists of the single box (a, b).

If $a + b > k$, then by induction the tableau t_1 obtained by deleting the last row of t contains an admissible path P_1 of type α_1. Similarly, the tableau t_2 obtained by deleting the last column of t contains an admissible path P_2 of type α_2. We will show that either the path P_1' obtained by appending (a, b) to P_1 or the path P_2' obtained by appending (a, b) to P_2 is admissible. Note that P_1' is admissible if and only if $S_t(a, b) \geq \alpha_1$ and P_2' is admissible if and only if $S_t(a, b) \leq \alpha_2$.

If $S_t(a, b) < S_t(a, b - 1)$, then by Lemma 6.5 we have

$$S_t(a, b - 1) = S_t(a, b) + 1 \quad \text{and}$$
$$S_t(a - 1, b) \geq S_t(a, b - 1) - 1 = S_t(a, b).$$

It follows that either $S_t(a, b) \geq S_t(a, b - 1)$ or $S_t(a, b) \leq S_t(a - 1, b)$. We assume that $S_t(a, b) \geq S_t(a, b - 1)$; the case where $S_t(a, b) \leq S_t(a - 1, b)$ follows by a similar argument. If $S_t(a, b) \geq \alpha_1$, then P_1' is an admissible path of type α_1, and we are done. If P_1 contains the box $(a, b - 2)$, then $S_t(a, b) \geq S_t(a, b - 1) \geq \alpha_1$ by the definition of an admissible path. We may therefore assume that P_1 contains the box $(a - 1, b - 1)$, and $S_t(a, b) < \alpha_1$.

Now consider the path P_2. If the paths P_1 and P_2 intersect, let (x, y) be a box in the intersection, and let t_3 be the tableau obtained by restricting t to $[x] \times [y]$. The restrictions of P_1 and P_2 to t_3 are both admissible, and it follows from Lemma 6.8 that $\alpha_1 = \alpha_2$. Since $S_t(a, b) < \alpha_1$, we see that P_2' is an admissible path.

If P_1 and P_2 do not intersect, then P_1 lies entirely above and to the right of P_2, so $\alpha_1 > \alpha_2$. Let $(x, b - 1)$ be the leftmost box of P_1 in row $b - 1$. Because the two paths do not intersect, the boxes $(x - 1, b)$ and (x, b) must be contained in P_2. By the definition of admissible paths, we have $\alpha_1 \leq S_t(x, b - 1)$ and $\alpha_2 \geq S_t(x, b)$. By Lemma 6.5, however, we have

$$\alpha_1 \leq S_t(x, b - 1) \leq S_t(x, b) + 1 \leq \alpha_2 + 1.$$

It follows that $\alpha_1 = \alpha_2 + 1$. Since $S_t(a, b) < \alpha_1$, we see that $S_t(a, b) \leq \alpha_1 + 1 = \alpha_2$, and hence P_2' is an admissible path. □

Now that we know admissible paths exist, we can use them to construct a scrollar tableau from an arbitrary tableau.

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

1	2	3
4	6	7

1	2	3
4	6	7
5	8	11

1	2	3	5
4	6	7	10
5	8	11	13

1	2	3	5
4	6	7	10
5	8	11	13
10	14	15	16

Figure 12 Construction of a scrollar tableau from a given k-uniform displacement tableau and admissible path.

EXAMPLE 6.10. Before proving Theorem 6.1, we first illustrate the idea with an example. Figure 12 depicts an example of a 5-uniform displacement tableau t and an admissible path of type $\alpha = 3$ from Example 6.7. The proof of Theorem 6.1 provides us with an iterative procedure for constructing a scrollar tableau t' of type 3 such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. This procedure begins with the subtableau on $[\alpha] \times [k - \alpha] = [3] \times [2]$. It then follows the admissible path, extending the tableau one row or one column at a time. Every time we extend the tableau by a column, we replace each symbol in the new column with the symbol appearing α boxes to the left and $k - \alpha$ boxes below in the previous tableau. Similarly, every time we extend the tableau by a row, we replace each symbol in the new row with the symbol appearing α boxes to the right and $k - \alpha$ boxes above in the previous tableau. The definition of admissible paths guarantees that this construction yields a tableau.

Proof of Theorem 6.1. First, note that if $a + b \leq k$, then t is scrollar of type a for trivial reasons. We therefore assume that $a + b > k$. By Proposition 6.9 there exists an admissible path P in t of type α. We will prove, by induction on $a + b$, that there exists a scrollar tableau t' of type α on $[a] \times [b]$ such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. In addition, we will see that $t'(a - i, b) = t(a - i, b)$ for all $i < \alpha$ and $t'(a, b - j) = t(a, b - j)$ for all $j < k - \alpha$. We assume that P contains the box $(a - 1, b)$; the case where P contains the box $(a, b - 1)$ follows by a similar argument. By the definition of admissible paths this implies that $S_t(a, b) \leq \alpha$.

Let t_1 be the tableau obtained by deleting the last column from t. The restriction of P to t_1 is an admissible path of type α in t_1. By induction, therefore, there exists a scrollar tableau t_1' on $[a - 1] \times [b]$ such that $\mathbb{T}(t_1) \subseteq \mathbb{T}(t_1')$. Moreover, we have that $t_1'(a - 1 - i, b) = t_1(a - 1 - i, b)$ for all $i < \alpha$ and $t_1'(a - 1, b - j) = t_1(a - 1, b - j)$ for all $j < k - \alpha$. By Lemma 3.6 every symbol in t_1' is a symbol in t_1, and if $t_1(x, y) = t_1'(x', y')$, then $x - y \equiv x' - y' \pmod{k}$.

We now define a tableau t' on $[a] \times [b]$:

$$t'(x, y) = \begin{cases} t_1'(x, y) & \text{if } x < a, \\ t_1'(x - \alpha, y + k - \alpha) & \text{if } x = a \text{ and } y \leq b - k + \alpha, \\ t(x, y) & \text{if } x = a \text{ and } y > b - k + \alpha. \end{cases}$$

We first show that t' is a tableau. Let $(x, y) \in [a] \times [b]$. If $x < a$, then since t_1' is a tableau, we see that $t'(x, y) > t'(x - 1, y)$ and $t'(x, y) > t'(x, y - 1)$. If $y \leq b - k + \alpha$, then because t_1' is a tableau, we have $t'(a, y) > t'(a, y - 1)$, and

because t'_1 is scrollar of type α, we have

$$t'(a-1, y) < t'(a-\alpha, y+k-\alpha) = t'(a, y).$$

If $y > b - k + \alpha$, then since t is a tableau and $t'_1(a-1, y) = t(a-1, y)$, we have $t'(a-1, y) < t'(a, y)$. If $y > b + 1 - k + \alpha$, then since t is a tableau, we have $t'(a, y-1) < t'(a, y)$. Finally, since $S_t(a, b) \leq \alpha$, we have

$$t'(a, b-k+\alpha) = t(a-\alpha, b) < t(a, b+1-k+\alpha) = t'(a, b+1-k+\alpha).$$

To see that t' is scrollar, we show that if $b > k - \alpha$, then $t'(x, y) = t'(x + \alpha, y - k + \alpha)$ for all pairs (x, y). This is clear if $x + \alpha < a$, because t'_1 is scrollar of type α. On the other hand, if $x + \alpha = a$, then this holds by construction. If $\alpha = k - b$, then $t'(x, 1) > t'(x + b - k, b)$ for all $x < a$ because t'_1 is scrollar, and $t'(a, 1) > t'(a + b - k, b)$ because $S_t(a, b) \leq \alpha = k - b$.

Finally, we show that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. Note that the symbol $t'(x, y)$ is also a symbol in t_1 if and only if $x < a$ or $y \leq b - k + \alpha$. By construction every symbol in t_1 is also a symbol in t, and if

$$t'(x, y) = t_1(x', y') = t(x'', y''),$$

then

$$x - y = x' - y' = x'' - y'' \pmod{k}.$$

On the other hand, if $y > b - k + \alpha$, then the symbol $t'(a, y) = t(a, y)$ appears only in one box, and there is nothing to prove. By Lemma 3.6 it follows that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. □

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