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CONTROLLING FORMAL FIBERS OF PRINCIPAL PRIME IDEALS

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ABSTRACT. Let $(T, \mathfrak{m})$ be a complete local (Noetherian) ring, S_0 the prime subring of T and $p \neq 0$ a regular and prime element of T. Given a finite set of incomparable prime ideals $C = \{Q_1, \dots, Q_n\}$ of T such that either $Q_i \cap S_0 = (0)$ for all i or $Q_i \cap S_0 = pS_0$ for all i, we provide necessary and sufficient conditions for T to be the completion of a local domain A such that $p \in A$ and the formal fiber of pA is semi-local with maximal ideals the elements of C. We also show that in a special case the domain A we construct is excellent.

1. Introduction. Much research has been devoted to understanding the relationship between a local (Noetherian) ring and its completion. In these efforts, an important question to address has been the following: given a complete local ring T with maximal ideal $\mathfrak{m}$, when is it the completion of a ring A with certain properties? One major result of this type comes from Lech. Specifically, in [**5**], Lech shows that a complete local (Noetherian) ring T is the completion of a local (Noetherian) domain if and only if the following conditions hold.

(1) The prime ring of T is a domain that acts on T without torsion;

(2) Unless equal to (0), the maximal ideal of T does not belong to (0) as an associated prime ideal.

To understand the relationship between a local ring and its completion it is often useful to consider the formal fibers of the ring. If A is a local (Noetherian) ring with maximal ideal $\mathfrak{m}$ and P a prime ideal of A, we define the *formal fiber* of A at P to be $\operatorname{Spec}(\widehat{A} \otimes_A k(P))$ where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A and $k(P) = A_P/PA_P$. It is important to note that there is a one-to-one correspondence between the formal fiber of A at P and the inverse image of P under the map $\operatorname{Spec}\widehat{A} \to \operatorname{Spec} A$. If A is an integral domain, we call the formal fiber

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of A at the zero ideal the *generic formal fiber* of A. If $Q \in \operatorname{Spec} \widehat{A}$ and $Q \cap A = P$ we will say that Q is in the formal fiber of A at P. If the ring $\widehat{A} \otimes_A k(P)$ is semi-local with maximal ideals $Q_1 \otimes_A k(P)$, $Q_2 \otimes_A k(P), \dots, Q_n \otimes_A k(P)$, we will say that the formal fiber of A at P is semi-local with maximal ideals $Q_1, Q_2, \dots, Q_n$.

In [**1**], it is shown that semi-local generic formal fibers are more common than one might think. In particular, the authors present necessary and sufficient conditions on a complete local ring T and a finite set of incomparable nonmaximal prime ideals C of T for the existence of a local domain A which completes to T and has semi-local generic formal fiber with maximal ideals the elements of C (the set C is the set of maximal elements of $\mathcal{G}$ in the following theorem):

Theorem 1.1 (Charters and Loepp [**1**]). *Let $(T, \mathfrak{m})$ be a complete local ring and $\mathcal{G} \subseteq \operatorname{Spec} T$ such that $\mathcal{G}$ is nonempty and the number of maximal elements of $\mathcal{G}$ is finite. Then there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly $\mathcal{G}$ if and only if T is a field and $\mathcal{G} = \{(0)\}$ or the following conditions hold:*

(1) *$\mathfrak{m} \notin \mathcal{G}$, and $\mathcal{G}$ contains all the associated prime ideals of T.*

(2) *If $Q \in \mathcal{G}$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in \mathcal{G}$.*

(3) *If $Q \in \mathcal{G}$, then $Q \cap$ prime subring of $T = (0)$.*

Note that since the ring A in this result is a local domain, it must be the case that Lech's conditions are implied by those of Charters and Loepp. This is not difficult to show.

In this paper we present the following generalizations of Theorem 1.1. In Section 2, we prove a slightly stronger version of Theorem 1.1. In Section 3, instead of constructing our integral domain A to have a specified semi-local *generic* formal fiber, we construct it so that it contains a *height one* principal prime ideal with specified semi-local formal fiber. Specifically, let T be a complete local ring with maximal ideal $\mathfrak{m}$, prime subring S_0 and p a nonzero regular prime element of T. Let $C = \{Q_1, Q_2, \dots, Q_n\}$ be a finite set of incomparable prime ideals of T such that either $S_0 \cap Q_i = (0)$ for every i or $S_0 \cap Q_i = pS_0$ for every i. We show that there exists a local domain A such that $\widehat{A} = T$,

$p \in A$, and the formal fiber of pA is semi-local with maximal ideals the elements of C if and only if the following conditions are satisfied.

(1) $p \in Q_i$ for every i.

(2) If $\dim T = 1$, then $C = \{\mathfrak{m}\}$.

(3) If $\dim T > 1$, then $\mathfrak{m} \notin C$.

We then show that under certain conditions the ring A we construct is excellent.

1.1 Notation. We begin with some remarks about notation and conventions used in the paper. All rings we consider will be commutative with identity. A *quasi-local ring* is a ring with exactly one maximal ideal. If T is a quasi-local ring with maximal ideal $\mathfrak{m}$, we will denote it $(T, \mathfrak{m})$. If a quasi-local ring is Noetherian, then we will call it a *local ring*. The ($\mathfrak{m}$-adic) completion of a local ring $(T, \mathfrak{m})$ will be denoted $\widehat{T}$.

2. Generic formal fibers.

Let T be a complete local ring and C a finite set of incomparable prime ideals of T. In [**1**], the authors provide necessary and sufficient conditions for T to be the completion of a local domain A such that the generic formal fiber of A is semi-local with maximal elements the elements of C. In this section we prove a stronger version of their theorem.

The construction in [**1**] works only when C is finite solely because of its dependence on the following lemma:

Lemma 2.1 (Charters and Loepp [**1**]). *Let $(T, \mathfrak{m})$ be a complete local ring such that $\dim T \geq 1$, C a finite set of nonmaximal prime ideals such that no ideal in C is contained in any other ideal of C. Let D be a subset of T such that $|D| < |T|$, and let I be an ideal of T such that $I \not\subseteq P$ for all $P \in C$. Then $I \not\subseteq \cup\{r + P \mid r \in D, P \in C\}$.*

The following lemma can also be found in [**1**]. The authors of that paper have recently informed us that there is a problem in part of their proof, and so we provide an alternate proof here.

Lemma 2.2. *Let (T, M) be a complete local ring of dimension at least one. Let P be a nonmaximal prime ideal of T. Then, $|T/P| = |T| \geq c$ where c denotes the cardinality of the real numbers.*

Proof. Let $|T/M| = d$. We will show that $|T| = d^{\aleph_0}$ and $|T/P| = d^{\aleph_0}$. Since $c = 2^{\aleph_0}$, the result will follow. We start by showing $|T| \leq d^{\aleph_0}$. Define a map $f : T \to \prod_{i=1}^{\infty} T/M^i$ by $f(t) = (t+M, t+M^2, t+M^3, \dots)$. Suppose $f(t) = f(s)$. Then $t - s \in M^n$ for all $n \in \mathbf{N}$. By Krull's intersection theorem, $\cap_{n=1}^{\infty} M^n = (0)$. It follows that $t = s$ and so f is injective. This gives us that $|T| \leq |T/M|^{\aleph_0} = d^{\aleph_0}$.

We now show that if T is a domain, $|T| \geq d^{\aleph_0}$. Let $0 \neq y \in M$. Let X be a full set of coset representatives for T/M. Now we define a map $g : \prod_{i=1}^{\infty} X \to T$ by $g(a_1, a_2, \dots) = a_1 y + a_2 y^2 + a_3 y^3 + \cdots$. We claim that g is injective. Suppose not. Then there exists $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ in $\prod_{i=1}^{\infty} X$ such that $g(a_1, a_2, \dots) = g(b_1, b_2, \dots)$ but $(a_1, a_2, \dots) \neq (b_1, b_2, \dots)$. Let n be the smallest integer such that $a_n \neq b_n$. Note that we have $a_n + M \neq b_n + M$ since X was a full set of coset representatives. Now we have

$$a_n y^n + a_{n+1} y^{n+1} + \cdots = b_n y^n + b_{n+1} y^{n+1} + \cdots$$

and so $y^n((a_n - b_n) + (a_{n+1} - b_{n+1})y + \cdots) = 0$. Since T is a domain, we have $(a_n - b_n) + (a_{n+1} - b_{n+1})y + \cdots = 0$. But this implies that $a_n - b_n \in M$, a contradiction. It follows that g is injective and so $|T| \geq d^{\aleph_0}$. We have shown that if (T, M) is a complete local domain of dimension at least one, then $|T| = |T/M|^{\aleph_0} = d^{\aleph_0}$.

In the general case, note that T/P is a complete local domain of dimension at least one so using the above fact we have that $|T/P| = |(T/P)/(M/P)|^{\aleph_0} = |T/M|^{\aleph_0} = d^{\aleph_0}$. This also gives that $|T| \geq |T/P| = d^{\aleph_0}$. So we have $|T| = d^{\aleph_0}$ as desired. $\square$

Armed with Lemma 2.2, we can now prove a variation of Lemma 2.1.

Lemma 2.3. *Let $(T, \mathfrak{m})$ be a complete local ring such that $\dim T \geq 1$. Let C be a subset of $\operatorname{Spec} T$ such that $|C| < |T|$ and suppose that there exists a finite set of nonmaximal incomparable prime ideals $\{H_1, H_2, \dots, H_n\}$ of T satisfying the property that if $P \in C$, then $P \subseteq H_i$ for some $i = 1, 2, \dots, n$. Let D be a subset of T such that*

$|D| < |T|$, and let I be an ideal of T such that $I \not\subseteq \cup_{P \in C} P$. Then $I \not\subseteq \cup\{r + P \mid r \in D, P \in C\}$.

Proof. Choose an element $t \in I - \cup_{P \in C} P$. Now let $C_i = \{P \in C \mid P \subseteq H_i\}$ and consider any $P \in C_i$. If $r + P \in (t + P)(T/P)$ for some $r \in D$, then $r + P = (t + P)(s + P)$ for some $s \in T$. For each pair (r, P) with $r \in D$ and $P \in C_i$ that satisfies $r + P \in (t + P)(T/P)$, select a coset representative s from the coset $s + P$. Call this set of representatives S_i. Note that since $|C| < |T|$ and $|D| < |T|$, we have $|S_i| < |T/H_i| = |T|$ by Lemma 2.2.

We first prove the lemma if $n = 1$. In this case, we can find an $x \in T$ such that $x + H_1 \neq s + H_1$ for all $s \in S_1$. Now, suppose $tx \in r + P$ for some $r \in D$ and $P \in C = C_1$. Then $tx + P = r + P = (t + P)(s + P)$ for some $s \in S_1$. But $(t + P)(x + P) = (t + P)(s + P)$ implies that $x + P = s + P$, a contradiction, since if $x + P = s + P$, then clearly $x + H_1 = s + H_1$. Thus, $tx \notin \cup\{r + P \mid r \in D, P \in C\}$.

If $n > 1$, then we have that $(H_i + \cap_{j=1, j\neq i}^{n} H_j)/H_i$ is not the zero ideal of T/H_i since $H_1, H_2, \ldots, H_n$ are incomparable. It follows that

$$\left|\frac{T}{H_i}\right| = \left|\frac{H_i + \cap_{j=1, j\neq i}^{n} H_j}{H_i}\right|.$$

So, for every $i = 1, 2, \ldots, n$, there exists an $x_i \in \cap_{j=1, j\neq i}^{n} H_j$ such that $x_i + H_i \neq s + H_i$ for every $s \in S_i$. We claim that $t(x_1 + x_2 + \cdots + x_n) \notin \cup\{r + P \mid r \in D, P \in C\}$. Suppose on the contrary that $t(x_1 + x_2 + \cdots + x_n) \in r + P$ for some $r \in D$ and $P \in C$. Then $P \subseteq H_i$ for some $i = 1, 2, \ldots, n$. Now, $r + P = (t + P)(s + P)$ for some $s \in S_i$ and so we have that $(x_1 + x_2 + \cdots + x_n) + P = s + P$. But this implies that $(x_1 + x_2 + \cdots + x_n) + H_i = s + H_i$, which by the choice of the x's implies that $x_i + H_i = s + H_i$, a contradiction. It follows that $t(x_1 + x_2 + \cdots + x_n) \notin \cup\{r + P \mid r \in D, P \in C\}$, and so the lemma holds. □

We conclude this section with our variation of Theorem 1.1:

Theorem 2.4. *Let $(T, \mathfrak{m})$ be a complete local ring and $\mathcal{G} \subseteq \operatorname{Spec} T$ such that $\mathcal{G}$ is nonempty and such that there exists a finite set of non-maximal incomparable prime ideals $\{H_1, H_2, \ldots, H_n\}$ of T satisfying* *the property that if $P \in \mathcal{G}$, then $P \subseteq H_i$ for some $i = 1, 2, \ldots, n$. Let C be the set of maximal elements of $\mathcal{G}$ and suppose $|C| < |T|$. Then there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly $\mathcal{G}$ if and only if T is a field and $\mathcal{G} = \{(0)\}$ or the following conditions hold:*

(1) *$\mathfrak{m} \notin \mathcal{G}$, and $\mathcal{G}$ contains all the associated prime ideals of T.*

(2) *If $Q \in \mathcal{G}$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in \mathcal{G}$.*

(3) *If $Q \in \mathcal{G}$, then $Q \cap$ prime subring of $T = (0)$.*

(4) *If I is an ideal of T with $I \not\subseteq P$ for all $P \in C$, then $I \not\subseteq \cup_{P \in C} P$.*

Proof. Note that condition (4) is an added condition to the original theorem and always holds if C is finite or infinitely countable. Condition (4) is sufficient because it allows us to invoke Lemma 2.3. To show that this condition is necessary, suppose that there does exist a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly $\mathcal{G}$. Suppose it were the case that there exists an ideal I such that $I \not\subseteq P$ for all $P \in C$ but with $I \subseteq \cup_{P \in C} P$. Then $I \cap A = (0)$. Let $\operatorname{Ass}(T/I) = \{P_1, P_2, \ldots, P_m\}$, and suppose $P_i \cap A \neq (0)$ for every $i = 1, 2, \ldots, m$. Let $x_i \in P_i \cap A$ with $x_i \neq 0$. Then $x = \prod_{i=1}^{m} x_i \in P_1 \cap P_2 \cap \cdots \cap P_m \cap A$ and, as A is an integral domain, we have $x \neq 0$. Now, $\sqrt{I} = P_1 \cap \cdots \cap P_m$, so we have $x \in \sqrt{I} \cap A$ and it follows that $x^l \in I \cap A$ for some $l \geq 1$. But, $x^l \neq 0$ contradicting that $I \cap A = (0)$. Hence, $P_i \cap A = (0)$ for some $i = 1, 2, \ldots, m$. It follows that P_i is in the generic formal fiber of A and so $P_i \subseteq P$ for some $P \in C$. Hence, $I \subseteq P_i \subseteq P$, a contradiction. Thus, (4) is a necessary condition. Now, substituting Lemma 2.3 where in [**1**] the authors used Lemma 2.1, the result follows and thus the details will not be given here. $\square$

3. Formal fibers of domains at height one primes. In this section we address the following question: Given a complete local ring $(T, \mathfrak{m})$, a nonzero regular prime element p of T and a finite set of incomparable prime ideals $C = \{Q_1, \ldots, Q_n\}$ of T, when does there exist a local domain A such that $p \in A$, the completion of A is T, and the formal fiber of pA is semi-local with maximal elements the elements of C? In other words, we want $Q_i \cap A = pA$ for every

$i \in \{1, 2, \dots, n\}$ and, if $P \in \operatorname{Spec} T$ with $P \cap A = pA$, then $P \subseteq Q_i$ for some $i \in \{1, 2, \dots, n\}$. We have examined this problem for two cases: $Q_i \cap$ prime subring of $T = (0)$ for all i and the case when $Q_i \cap$ prime subring of T is generated by p for all i. Specifically, let $(T, \mathfrak{m})$ be a complete local ring, S_0 the prime subring of T and p a nonzero regular prime element of T. Suppose $C = \{Q_1, \dots, Q_n\}$ is a finite set of incomparable prime ideals of T such that either $Q_i \cap S_0 = (0)$ for all i or $Q_i \cap S_0 = pS_0$ for all i. We show that there exists a local domain A such that $\widehat{A} = T$, $p \in A$, and the formal fiber of pA is semi-local with maximal ideals the elements of C if and only if the following conditions are satisfied.

(1) $p \in Q_i$ for every i.

(2) If $\dim T = 1$, then $C = \{\mathfrak{m}\}$.

(3) If $\dim T > 1$, then $\mathfrak{m} \notin C$.

This result is Theorem 3.13 and is the main result of this section. The techniques we use are similar to those used in [**1, 3, 6**]. We begin with some useful definitions.

Definition 3.1. Let $(T, \mathfrak{m})$ be a complete local ring, and let p be a nonzero regular element of T. Suppose $C = \{Q_1, Q_2, \dots, Q_n\}$ is a finite set of prime ideals of T all containing p. Suppose that $(R, R \cap \mathfrak{m})$ is a quasi-local subring of T with the following properties:

(1) $|R| < |T|$.

(2) If P is an associated prime ideal of T, then $R \cap P = (0)$.

(3) If P is a prime ideal of T with $P \subseteq Q_i$ for some i and $p \notin P$, then $R \cap P = (0)$.

Then we call R a *p-including subring* of T, and we will denote it by *pin-subring.*

The properties of $p$in-subrings will be essential in the proof of our major result in this section. To show the existence of our local domain A, we construct a chain of intermediate $p$in-subrings and then let A be the union of these subrings. Ideally, for each Q_i, we would like $Q_i \cap S = pS$ for each of these intermediate subrings S. The following three lemmas show that, given a $p$in-subring R, we can find a larger $p$in-subring S with the property that $pT \cap S = pS$. We will see in Corollary 3.6 that this property implies $Q_i \cap S = pS$ for every $i = 1, 2, \dots, n$ when pT is a prime ideal of T.

For the rest of the section, we use the following conventions: Let $(T, \mathfrak{m})$ be a complete local ring with $\dim T \geq 1$ and $C = \{Q_1, \dots, Q_n\}$ a finite set of incomparable nonmaximal prime ideals of T all containing p. Thus, when we say a $p$in-subring of T, we shall mean a $p$in-subring with respect to the set C. We will also assume for the rest of the section that p is not a zerodivisor in T.

Lemma 3.2. *Given $(R, R \cap \mathfrak{m})$ a $p$in-subring of $(T, \mathfrak{m})$ with $C = \{Q_1, Q_2, \dots, Q_n\}$ a finite set of prime ideals and $c \in pT \cap R$, there exists a $p$in-subring S of T such that $R \subseteq S \subseteq T$ and $c \in pS$. Moreover, $|S| \leq \sup(\aleph_0, |R|)$.*

Proof. Since $c \in pT \cap R$, $c = pu$ for some element u in T. We claim that $S = R[u]_{(R[u] \cap \mathfrak{m})}$ is the desired subring. First note that $c \in pS$ and $|S| < |T|$. Let $P \in \operatorname{Ass} T$, and let $f \in P \cap R[u]$. Then $f = r_n u^n + \cdots + r_1 u + r_0 \in P$ and $p^n f = r_n c^n + \cdots + r_1 p^{n-1} c + r_0 p^n \in P \cap R = (0)$. Since p is not a zerodivisor, $f = 0$ and we have that $P \cap S = 0$. Now suppose that $P \in \operatorname{Spec} T$ with $P \subseteq Q_i$ for some $i, p \notin P$, and let $f \in P \cap S$. Then, by a similar argument, $f = r_n u^n + \cdots + r_1 u + r_0 \in P$ and $p^n f = r_n c^n + \cdots + r_1 p^{n-1} c + r_0 p^n \in P \cap R = (0)$. So $f = 0$ and we have that $P \cap S = 0$. It follows that S is a $p$in-subring of T. The fact that S satisfies the cardinality condition is clear. $\square$

Definition 3.3. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.

Lemma 3.4. *Given $(R, R \cap \mathfrak{m})$ a $p$in-subring of $(T, \mathfrak{m})$ with C a finite set of prime ideals, there exists a $p$in-subring S of T such that $R \subseteq S \subseteq T$ and $pT \cap R \subseteq pS$. Moreover, $|S| \leq \sup(\aleph_0, |R|)$.*

Proof. Let $\Omega = pT \cap R$. We know that $|\Omega| \leq |R|$. Well order Ω, and let 0 denote its first element. If Ω is infinite, we will well order it so that there is no maximal element in the following way. Let $|\Omega| = \kappa$. Then κ is an infinite cardinal. (For a discussion of this see, for example, [**2**, subsection 3.6].) By [**2**, Theorem 3.6.1], κ is a limit ordinal. It follows that κ is a well-ordered set with no maximal element (see the discussion on page 24 in [**2**]). Also note that this set has the same cardinality as Ω. So, we use this set to well order Ω. Define $R_0 = R$ and suppose $\alpha \in \Omega$. If $\gamma(\alpha) < \alpha$, then construct R_α from $R_{\gamma(\alpha)}$ using Lemma 3.2 with $c = \gamma(\alpha)$. Otherwise, $\gamma(\alpha) = \alpha$ so define $R_\alpha = \cup_{\beta<\alpha} R_\beta$. We show by transfinite induction that, for every $\alpha \in \Omega$, R_α is a $p$in-subring of T and that $|R_\alpha| \leq \sup(\aleph_0, |R|)$. Since R_0 satisfies these conditions, the base case holds. Suppose $\alpha \in \Omega$ and that for every $\beta < \alpha$ we have that R_β is a $p$in-subring of T and $|R_\beta| \leq \sup(\aleph_0, |R|)$. If $\gamma(\alpha) < \alpha$, then by the way we defined R_α, it is a $p$in-subring. Moreover, $|R_\alpha| \leq \sup(\aleph_0, |R_{\gamma(\alpha)}|) \leq \sup(\aleph_0, |R|)$. On the other hand, suppose $\gamma(\alpha) = \alpha$. Then R_α clearly satisfies the last two conditions of being a $p$in-subring. Now, $|R_\alpha| \leq \sum_{\beta<\alpha} |R_\beta| \leq |R| \sup_{\beta<\alpha} |R_\beta| \leq |R| \sup(\aleph_0, |R|) = \sup(\aleph_0, |R|)$. It follows that R_α is a $p$in-subring of T. If Ω is finite, let d denote its maximal element. In this case, define S to be the $p$in-subring obtained from Lemma 3.2 using $R = R_d$ and $c = d$. If Ω is infinite let $S = \cup_{\alpha\in\Omega} R_\alpha$. Then we have that $|S| \leq \sum_{\alpha\in\Omega} |R_\alpha| \leq |R| \cdot \sup(\aleph_0, |R|) = \sup(\aleph_0, |R|)$. It is now clear that S is a $p$in-subring of T. Additionally, if Ω is infinite and $c \in pT \cap R_0$, then $c = \gamma(\alpha)$ for some α in Ω with $\gamma(\alpha) < \alpha$. Thus $c \in pR_\alpha \subseteq pS$, so $pT \cap R_0 \subseteq pS$. In the case when Ω is finite, this condition is easy to verify. □

Lemma 3.5. *Given $(R, R \cap \mathfrak{m})$, a $p$in-subring of $(T, \mathfrak{m})$ with C a finite set of prime ideals, there exists a $p$in-subring S of T such that $R \subseteq S \subseteq T$ and $pT \cap S = pS$. Moreover, $|S| \leq \sup(\aleph_0, |R|)$.*

Proof. Let $R_0 = R$. We now define R_i recursively. For R_{i-1}, we use Lemma 3.4 to find a $p$in-subring R_i with $pT \cap R_{i-1} \subseteq pR_i$ and $|R_i| \leq \sup(\aleph_0, |R|)$. Let $S = \cup_{i=1}^{\infty} R_i$. It is easy to show that S is a $p$in-subring and that it satisfies the cardinality condition. Suppose $c \in pT \cap S$. Then there exists an $n \in \mathbf{N}$ such that $c \in pT \cap R_n \subseteq pR_{n+1} \subseteq pS$. Thus, $pT \cap S \subseteq pS$, so $pT \cap S = pS$. □

Corollary 3.6 will be essential to the proofs of Lemmas 3.9 and 3.10.

Corollary 3.6. *Let $(R, R \cap \mathfrak{m})$ be a pin-subring of $(T, \mathfrak{m})$ and $C = \{Q_1, Q_2, \dots, Q_n\}$ a finite set of prime ideals of T where R has the property that $pT \cap R = pR$ and pT is a prime ideal of T. Then we have $Q_i \cap R = pR$ for all $i = 1, 2, \dots, n$.*

Proof. Fix i and suppose $f \in Q_i \cap R$. We know there exists a height 1 prime ideal J in Q_i which contains fT. If $p \notin J$, then $J \cap R = (0)$ since R is a $p$in-subring, so $f = 0 \in pR$. Otherwise $p \in J$, so J is an associated prime of pT and $J = pT$. Thus, $J \cap R = pT \cap R = pR$, so $f \in pR$ and $Q_i \cap R \subseteq pR$, giving us that $Q_i \cap R = pR$. □

The following proposition establishes the fact that the conditions in the hypotheses of later lemmas imply that T is an integral domain. This fact is useful because then any subring of T is also an integral domain.

Proposition 3.7. *Let T be a complete local ring and let p be a nonzero element of T that is not a zerodivisor. If pT is a prime ideal of T, then T is an integral domain.*

Proof. Suppose pT is a prime ideal of T. Since p is not a zerodivisor, $\operatorname{ht} pT = 1$. So we have a principal prime ideal with height $\neq 0$, so by [**7**, Theorem 15.33], T is an integral domain. □

The following is Proposition 1 from [**4**]. It helps us to ensure that the final ring we create completes to T.

Proposition 3.8 (Heitmann [**4**]). *If $(R, \mathfrak{m} \cap R)$ is a quasi-local subring of a complete local ring $(T, \mathfrak{m})$, the map $R \to T/\mathfrak{m}^2$ is onto and $IT \cap R$ for every finitely generated ideal I of R, then R is Noetherian and the natural homomorphism $\widehat{R} \to T$ is an isomorphism.*

In light of this proposition, we will construct A so that the map $A \to T/\mathfrak{m}^2$ is onto. Lemma 3.9 allows us to adjoin an element of a coset of T/J to a $p$in-subring R where $J \in \operatorname{Spec} T$ such that $J \not\subseteq Q_i$ for every i to get a new $p$in-subring. Using this lemma with $J = \mathfrak{m}^2$ will eventually give us that A satisfies this property. We prove this lemma for an arbitrary ideal J because it is also used to show that the formal fiber of the ideal pA is as desired and that in a special case A is excellent.

Lemma 3.9. *Let $(T, \mathfrak{m})$ be a complete local ring and $C = \{Q_1, Q_2, \dots, Q_n\}$ a finite set of incomparable nonmaximal prime ideals of T. Let R be a pin-subring of T such that $pT \cap R = pR$ where pT is a prime ideal of T, and $u + J \in T/J$ where J is an ideal of T with $J \not\subseteq Q_i$ for all i. Then there exists a pin-subring S of T such that $R \subseteq S \subseteq T$, $u + T$ is in the image of the map $S \to T/J$ and S has the property that $pT \cap S = pS$. Moreover, $|S| = \sup(\aleph_0, |R|)$ and if $u \in J$ then $S \cap J \neq (0)$.*

Proof. Fix i and suppose P is a prime ideal contained in Q_i with $p \notin P$. Suppose that $(u + t) + P$ is algebraic over $R/R \cap P \cong R$ for some $t \in T$. Then there exists a polynomial $r_n(u + t)^n + \dots + r_1(u + t) + r_0 \in P$ where at least one $r_i \neq 0$. But $P \subset Q_i$, so $r_n(u+t)^n + \dots + r_1(u+t) + r_0 \in Q_i$. If any $r_j \notin R \cap Q_i$, then $(u+t) + Q_i$ is algebraic over $R/(R \cap Q_i)$. Suppose then that $r_j \in R \cap Q_i$ for all j. Since $R \cap Q_i = pR$ by Corollary 3.6, $r_j = ps_j$ for some $s_j \in R$. So $r_n(u+t)^n + \dots + r_1(u+t) + r_0 = p[s_n(u+t)^n + \dots + s_1(u+t) + s_0] \in P$. P is prime, and $p \notin P$, so $s_n(u + t)^n + \dots + s_1(u + t) + s_0 \in P \subset Q_i$. Again, if any $s_j \notin R \cap Q_i$, then $(u + t) + Q_i$ is algebraic over $R/R \cap Q_i$. Otherwise, we repeat the process. If, for every j, $r_j \in p^n R$ for all n, then $r_j \in \cap_{n=1}^{\infty}(pR)^n \subseteq \cap_{n=1}^{\infty}(pT)^n = (0)$ by the Krull intersection theorem. However, this implies that $(u + t) + P$ is not algebraic over R, a contradiction. Thus $(u + t) + Q_i$ is algebraic over $R/R \cap Q_i$. The contrapositive of this result says that if $(u + t) + Q_i$ is transcendental over $R/(R \cap Q_i)$, then $(u + t) + P$ is transcendental over R.

Now let $D_{(Q_i)}$ be a full set of coset representatives of the cosets $t + Q_i$ with $t \in T$ that make $(u + t) + Q_i$ algebraic over $R/R \cap Q_i$. Let $D = \cup_{i=1}^{n} D_{(Q_i)}$. Since $|R| < |T|$ and $|D_{(Q_i)}| < |T|$ we have that $|D| < |T|$.

We can now employ Lemma 2.1 with $I = J$ to find an $x \in J$ such that $x \notin \cup\{r + P \mid r \in D, P = Q_i \text{ for some } i\}$ since the set $\{Q_1, \dots, Q_n\}$ is finite. We claim that $S' = R[u + x]_{(R[u+x] \cap \mathfrak{m})}$ is a pin-subring. Since $u + x$ is transcendental over R we have $|S'| = \sup(\aleph_0, |R|) < |T|$. Now suppose that $f \in R[u+x] \cap P$, where P is a prime ideal of T such that $P \subseteq Q_i$ for some i and $p \notin P$. Then $f = r_n(u+x)^n + \cdots + r_1(u+x) + r_0 \in P$, where $r_i \in R$. But we chose x such that $(u+x)+Q_i$ is transcendental over $R/R \cap Q_i$. Thus $(u+x)+P$ is transcendental over $R/R \cap P$. Therefore, $r_i \in R \cap P = (0)$ for all i, and it follows that $f = 0$. Observe that since T is a domain, S' is a domain. Thus S' is a $p$in-subring. Note further that if $u \in J$, then $u+x \in J$. Since $(u+x)+P$ is transcendental over R, we have $u+x \neq 0$. It follows that $S' \cap J \neq (0)$. Now employ Lemma 3.5 to find a $p$in-subring S where $S' \subseteq S$ and $pT \cap S = pS$. Moreover we get that $|S| = \sup(\aleph_0, |R|)$. $\square$

The following lemma will help us ensure that $IT \cap A = I$ for every finitely generated ideal I of A. This is one of the conditions from Proposition 3.8 needed to show that $\widehat{A} = T$.

Lemma 3.10. *Let R be a $p$in-subring of $(T, \mathfrak{m})$ and $C = \{Q_1, Q_2, \dots, Q_n\}$ a finite set of incomparable nonmaximal prime ideals of T with the property that $pT \cap R = pR$ where pT is a prime ideal of T. Also let I be a finitely generated ideal of R with $c \in IT \cap R$. Then there exists a $p$in-subring S such that $R \subseteq S \subseteq T$, $c \in IS$ and $pT \cap S = pS$. Moreover, $|S| \leq \sup(\aleph_0, |R|)$.*

Proof. We induct on the number of generators of I. Suppose $I = aR$. If $a = 0$, then $c = 0$ so $S = R$ is the desired $p$in-subring. If $a \neq 0$, then $c = au$ for some $u \in T$. We claim that $S = R[u]_{(R[u] \cap \mathfrak{m})}$ is the desired subring. First note that $|S| \leq \sup(\aleph_0, |R|) < |T|$. Suppose $P \subseteq Q_i$ for some i and $p \notin P$ and $f \in R[u] \cap P$. Then $f = r_n u^n + \cdots + r_1 u + r_0 \in P$, and $a^n f = r_n c^n + \cdots + r_1 c a^{n-1} + r_0 a^n \in P \cap R = (0)$. We know that a is not a zerodivisor since T is a domain. Thus $f = 0$ and we have that S is a $p$in-subring of T. Now use Lemma 3.5 to get the desired $p$in-subring. We have proven our base case, when the number of generators of I is one.

Now let I be an ideal of R that is generated by $m > 1$ elements, and assume that the lemma is true for ideals with $m-1$ generators. Let $I = (y_1, \dots, y_m)R$. Suppose first that $y_j \notin pR$ for some j. Without loss of generality, reorder the generators of I so that this element is y_2. We have that $c = y_1 t_1 + y_2 t_2 + \cdots + y_m t_m$ for some $t_i \in T$. We can add 0 to get that $c = y_1t_1 + y_1y_2t - y_1y_2t + y_2t_2 + \cdots + y_mt_m = y_1(t_1 + y_2t) + y_2(t_2 - y_1t) + y_3t_3 + \cdots + y_mt_m$ for any $t \in T$. We will choose our t shortly. Now let $x_1 = t_1 + y_2t$ and $x_2 = t_2 - y_1t$. If $(t_1 + y_2t) + Q_i = (t_1 + y_2t') + Q_i$ for any i, then we have that $y_2(t - t') \in Q_i$. However, $y_2 \in R$ and $R \cap Q_i = pR$. Since $y_2 \notin pR$, $y_2 \notin Q_i$ and since Q_i is prime, we now have that $(t - t') \in Q_i$, which means $t + Q_i = t' + Q_i$. The contrapositive of this result says that if $t + Q_i \neq t' + Q_i$, then $(t_1 + y_2t) + Q_i \neq (t_1 + y_2t') + Q_i$. Let $D_{(Q_i)}$ be a full set of coset representatives of the cosets $t + Q_i$ that make $x_1 + Q_i$ algebraic over $R/R \cap Q_i$. Let $D = \cup_{i=1}^n D_{(Q_i)}$. Note that $|D| < |T|$, so we can use Lemma 2.1 where $I = T$ to find an element $t \in T$ such that $t \notin \cup\{r + P \mid r \in D, P = Q_i \text{ for } 1 \leq i \leq n\}$. Thus we have that $x_1 + Q_i$ is transcendental over $R/R \cap Q_i$ for all i. It's clear that $R' = R[x_1]_{(R[x_1] \cap \mathfrak{m})}$ is a $p$in-subring of T and $|R'| = \sup(\aleph_0, |R|)$. Let $J = (y_2, \ldots, y_m)R'$ and $c^* = c - y_1x_1$. Then $c^* \in JT \cap R'$ and so we use the induction assumption to find a $p$in-subring S of T such that $R' \subseteq S \subseteq T$ and $c^* \in JS$. Moreover, we have $|S| \leq \sup(\aleph_0, |R'|) = \sup(\aleph_0, |R|)$. Now, $c^* = y_2s_2 + \cdots + y_ms_m$ for some $s_i \in S$, so $c = y_1x_1 + y_2s_2 + \cdots + y_ms_m \in IS$, and S is our desired $p$in-subring.

Now suppose that $y_j \in p^kR$ for all j, where k is the largest such integer and $k \geq 1$. Since $c = y_1t_1 + \cdots + y_mt_m$, we now have that $c/p^k = (y_1/p^k)t_1 + \cdots + (y_m/p^k)t_m$. Let $I' = (y_1/p^k, \ldots, y_m/p^k)$. We know that $y_j \in pR$ for all j, so $c \in pT \cap R$, which means $c \in pR$ and $c/p \in R$. Similarly, we have that $y_j \in p^2R$, so $y_j/p \in pR$ and $c/p \in pT \cap R$, which means $c/p \in pR$ and $c/p^2 \in R$. We can repeat this process until we get that since $y_j \in p^kR$ for all j, $c/p^k \in R$. Thus, $c/p^k \in I'T \cap R$. I' is a finitely generated ideal in R where at least one of the generators is not a multiple of p, so we can use the induction in the previous paragraph to get a $p$in-subring S satisfying the cardinality condition and such that $c/p^k \in I'S = (y_1/p^k, \ldots, y_m/p^k)S$, so we have $c \in (y_1, \ldots, y_m)S = IS$. □

Lemma 3.11 allows us to create a subring S of T that satisfies many of the conditions we want to be true for our final ring A.

Lemma 3.11. *Let $(T, \mathfrak{m})$ be a complete local ring and $p \neq 0$ a regular and prime element of T. Let $C = \{Q_1, Q_2, \ldots, Q_n\}$ be a set* *of incomparable nonmaximal prime ideals such that $p \in \cap_{i=1}^{n} Q_i$. Also, let J be an ideal of T with $J \not\subseteq Q_i$ for all i, and let $u + J \in T/J$. Suppose R is a pin-subring such that $pT \cap R = pR$. Then there exists a pin-subring S of T such that*

(1) $R \subseteq S \subseteq T$.

(2) *If* $u \in J$, *then* $S \cap J \neq (0)$.

(3) $u + J$ *is in the image of the map* $S \to T/J$.

(4) *For every finitely generated ideal* I *of* S, *we have* $IT \cap S = I$.

(5) $|S| = \sup(\aleph_0, |R|)$.

Proof. We first apply Lemma 3.9 to find an infinite $p$in-subring R' of T such that $pT \cap R' = pR'$, $R \subseteq R' \subseteq T$, $u + J$ is in the image of the map $R \to T/J$, and if $u \in J$ then $R' \cap J \neq (0)$. Moreover, we have that $|R'| = \sup(\aleph_0, |R|)$. We will construct the desired S such that $R' \subseteq S \subseteq T$ which will ensure that the first three conditions of the lemma hold true. Now let $\Omega = \{(I, c) \mid I \text{ is a finitely generated ideal of } R' \text{ and } c \in IT \cap R'\}$. Letting $I = R'$, we can see that $|\Omega| \geq |R'|$. Since R' is infinite, the number of finitely generated ideals of R' is $|R'|$, and therefore $|R'| \geq |\Omega|$, giving us the equality $|R'| = |\Omega|$. Moreover, as R' is a $p$in-subring of T, we have $|\Omega| = |R'| < |T|$. Well order Ω so that it does not have a maximal element (just as in the proof of Lemma 3.4), and let 0 denote its first element. We will now inductively define a family of $p$in-subrings of T, one for each element of Ω. Let $R_0 = R'$, and let $\alpha \in \Omega$. Assume that R_β has been defined for all $\beta < \alpha$. If $\gamma(\alpha) < \alpha$ and $\gamma(\alpha) = (I, c)$, then define R_α to be the $p$in-subring obtained from Lemma 3.10. In this manner, R_α will have the properties that $R_{\gamma(\alpha)} \subseteq R_\alpha \subseteq T$, and $c \in IR_\alpha$. Moreover, $|R'| \leq \sup(\aleph_0, |R_{\gamma(\alpha)}|)$. If $\gamma(\alpha) = \alpha$, define $R_\alpha = \cup_{\beta<\alpha} R_\beta$. Note that in both cases, R_α is a $p$in-subring of T such that $pT \cap R_\alpha = pR_\alpha$. Also, by transfinite induction, $|R_\alpha| = \sup(\aleph_0, |R|)$ for every $\alpha \in \Omega$. Now let $R_1 = \cup_{\alpha \in \Omega} R_\alpha$. Then $|R_1| = \sup(\aleph_0, |R|) < |T|$. Moreover, as $R_\alpha \cap P = (0)$ for every P such that $P \subseteq Q_i$ where $p \notin P$ and every $\alpha \in \Omega$, we have $R_1 \cap P = (0)$ for all such P as well. It follows that R_1 is a $p$in-subring. Note also that, because $pT \cap R_\alpha = pR_\alpha$ for each $\alpha \in \Omega$, we have $pT \cap R_1 = pR_1$. Furthermore, notice that if I is a finitely generated ideal of R_0 and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$ with $\gamma(\alpha) < \alpha$. It follows from the construction that $c \in IR_\alpha \subseteq IR_1$. Thus $IT \cap R_0 \subseteq IR_1$ for every finitely generated ideal I of R_0.

Following this same pattern, build a $p$in-subring R_2 of T such that $pT \cap R_2 = pR_2$, $R_1 \subseteq R_2 \subseteq T$ and $IT \cap R_1 \subseteq IR_2$ for every finitely generated ideal I of R_1. Continue to form a chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of $p$in-subrings of T such that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal I of R_n. Also, $|R_n| = \sup(\aleph_0, |R|)$.

We now claim that $S = \cup_{i=1}^{\infty} R_i$ is the desired $p$in-subring. To see this, first note that S is indeed a $p$in-subring and that $R \subseteq S \subseteq T$. Now set $I = (y_1, y_2, \ldots, y_k)S$, and let $c \in IT \cap S$. Then there exists an $N \in \mathbf{N}$ such that $c, y_1, \ldots, y_k \in R_N$. Thus $c \in (y_1, \ldots, y_k)T \cap R_N \subseteq (y_1, \ldots, y_k)R_{N+1} \subseteq IS$. From this it follows that $IT \cap S = I$, so the fourth condition of the Lemma holds. Note that the fourth condition implies $pT \cap S = pS$ by setting $I = pS$. The cardinality condition is easy to check. $\square$

In Lemma 3.12 we construct a domain A that has the desired completion and so that the formal fiber of pA is semi-local.

Lemma 3.12. *Let $(T, \mathfrak{m})$ be a complete local ring, S_0 the prime subring of T and $p \neq 0$ a regular and prime element of T. Let $C = \{Q_1, Q_2, \ldots, Q_n\}$ be a set of incomparable nonmaximal prime ideals all containing p and such that either $Q_i \cap S_0 = (0)$ for every i or $Q_i \cap S_0 = pS_0$ for every i. Then there exists a local domain A such that*

(1) $p \in A$.

(2) $\widehat{A} = T$.

(3) *The formal fiber of pA is semi-local with maximal ideals the elements of C.*

(4) *If J is an ideal of T satisfying $J \not\subseteq Q_i$ for every i, then the map $A \to T/J$ is onto.*

Proof. Let $\Omega = \{u + J \in T/J \mid J$ is an ideal of T with $J \not\subseteq Q_i$ for all $i\}$. We claim that $|\Omega| \leq |T|$. Since T is infinite and Noetherian, $|\{J$ is an ideal of T with $J \not\subseteq Q_i$ for all $i\}| \leq |T|$. Now, if J is an ideal of T, then $|T/J| \leq |T|$. It follows that $|\Omega| \leq |T|$. Well order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. Define R'_0 to be $S_0[p]$, and let R_0 simply denote R'_0 localized at $R'_0 \cap \mathfrak{m}$. We claim that R_0 is a $p$in-subring. First note that if $Q_i \cap S_0 = pS_0$ for every i, then $R'_0 = S_0$. In this case it is easy to see that the localization R_0 is a $p$in-subring of T. On the other hand, suppose $Q_i \cap S_0 = (0)$ for every i. Then we have that the first two conditions of the definition of $p$in-subring are easily satisfied. Let P be a prime ideal of T with $P \subset Q_i$ for some i and $p \notin P$. Suppose $f \in S_0[p] \cap P$. Then $f = s_m p^m + \cdots + s_1 p + s_0 \in P \subset Q_i$ for some $s_j \in S_0$. But $p \in Q_i$ so $s_0 \in Q_i \cap S_0 = (0)$. It follows that $f = p(s_m p^{m-1} + \cdots + s_1) \in P$. Since $p \notin P$, we have $s_m p^{m-1} + \cdots + s_1 \in P$. Repeat this process to show that $s_j = 0$ for all $j = 0, 1, 2, \ldots, m$. It follows that $f = 0$ and so $R_0 \cap P = (0)$. Hence R_0 is a countable $p$in-subring of T.

Now recursively define a family of $p$in-subrings as follows, starting with R_0. Let $\lambda \in \Omega$ and assume that R_β has already been defined for all $\beta < \lambda$. Then $\gamma(\lambda) = u + J$ for some ideal J of T with $J \not\subseteq Q_i$ for all i. If $\gamma(\lambda) < \lambda$, use Lemma 3.11 to obtain a $p$in-subring R_λ such that $R_{\gamma(\lambda)} \subseteq R_\lambda \subseteq T$, $u + J$ is in the image of the map $R_\lambda \to T/J$ and $IT \cap R_\lambda = I$ for every finitely generated ideal I of R_λ. Moreover, this gives us $|R_\lambda| = \sup(\aleph_0, |R_{\gamma(\lambda)}|)$, $R_\lambda \cap J \neq (0)$ if $u \in J$ and $pT \cap R_\lambda = pR_\lambda$. If $\gamma(\lambda) = \lambda$, define $R_\lambda = \cup_{\beta<\lambda} R_\beta$. We claim R_λ is a $p$in-subring for all $\lambda \in \Omega$.

We first show that by transfinite induction $|R_\lambda| \leq \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \lambda\}|)$ for all $\lambda \in \Omega$. The base case is trivial. So assume $\lambda \in \Omega$ and that for all $\beta < \lambda$ we have $|R_\beta| \leq \sup(\aleph_0, |R_0|, |\{\kappa \in \Omega \mid \kappa < \beta\}|)$. If $\gamma(\lambda) < \lambda$, then

$$\begin{aligned} |R_\lambda| &= \sup(\aleph_0, |R_{\gamma(\lambda)}|) \\ &\leq \sup(\aleph_0, \sup(\aleph_0, |R_0|, |\{\kappa \in \Omega \mid |\kappa < \gamma(\lambda)\}|)) \\ &\leq \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \lambda\}|). \end{aligned}$$

On the other hand, if $\gamma(\lambda) = \lambda$, then

$$\begin{aligned} |R_\lambda| &\leq \sum_{\beta<\lambda} |R_\beta| \\ &\leq |\{\beta \in \Omega \mid \beta < \lambda\}| \sup_{\beta<\lambda} |R_\beta| \end{aligned}$$

$$\leq |\{\beta \in \Omega \mid \beta < \lambda\}| \sup_{\beta<\lambda}(\sup(\aleph_0, |R_0|, |\{\kappa \in \Omega \mid \kappa < \beta\}|))$$
$$\leq |\{\beta \in \Omega \mid \beta < \lambda\}| \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \lambda\}|)$$
$$= \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \lambda\}|).$$

It follows that $|R_\lambda| \leq \sup(\aleph_0, |R_0|, |\{\beta \in \Omega \mid \beta < \lambda\}|)$ for all $\lambda \in \Omega$. By the way we well ordered Ω, this implies that $|R_\lambda| < |T|$ for every $\lambda \in \Omega$. The other conditions for $p$in-subring are easy to check. It follows that R_λ is a $p$in-subring for every $\lambda \in \Omega$.

We claim that $A = \cup_{\lambda\in\Omega} R_\lambda$ is the desired domain. First note that by construction condition (4) of the lemma is satisfied. We now show that the completion of A is T. To do this, we use Proposition 3.8. Note that as each prime ideal Q_i is nonmaximal in T, we have that $\mathfrak{m}^2$ is not contained in any Q_i. Thus, by the construction, the map $A \to T/\mathfrak{m}^2$ is onto. Now let I be a finitely generated ideal of A with $I = (y_1, \dots, y_k)$. Let $c \in IT \cap A$. Then $(c, y_1, \dots, y_k) \subseteq R_\lambda$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. Again, by the construction, $(y_1, \dots, y_k)T \cap R_\lambda = (y_1, \dots, y_k)R_\lambda$. As $c \in (y_1, \dots, y_k)T \cap R_\lambda$, we have that $c \in (y_1, \dots, y_k)R_\lambda \subseteq I$. Hence $IT \cap A = I$ as desired, and it follows from Proposition 3.8 that A is Noetherian and its completion is T.

Now we show that the formal fiber of pA has the desired properties. As each R_λ is a $p$in-subring, we have $R_\lambda \cap P = (0)$ for each prime ideal $P \subseteq Q_i$ such that $p \notin P$. Therefore, $A \cap P = (0)$ for every such P as well. Now let $a \in Q_i \cap A$ for some i. Then there exists a height one prime ideal I of T such that $a \in I$ and $I \subseteq Q_i$. If $p \notin I$, then $I \cap A = (0)$, and thus $a = 0$. Otherwise, $p \in I$, but pT is the only height one prime containing p. Thus, $I = pT$, and $a \in pT \cap A = pA$. It follows that $Q_i \cap A \subseteq pA$. Because $p \in Q_i \cap A$, we have $Q_i \cap A = pA$.

Now, let J be an ideal of T with $J \not\subseteq Q_i$ for all i. By the Prime Avoidance theorem, we know that $J \not\subseteq \cup_{i=1}^n Q_i$, so there is an $a \in J$ such that $a \notin Q_i$ for all i. From this, we know that there exists a height one prime $I \subseteq J \subseteq T$ such that $a \in I$. Note that $0 + I \in \Omega$. Therefore, $\gamma(\lambda) = 0 + I$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. By construction, $R_\lambda \cap I \neq (0)$. It follows that $I \cap A \neq (0)$. Also, because $a \in I$ and $a \notin Q_i$ for all i, we have $I \not\subseteq Q_i$, so $I \neq pT$. Note that pT is the only height one prime ideal of T that contains p, so $p \notin I$. Thus $I \cap A \neq pA$. Also, because the zero ideal is the only prime ideal in A contained in pA and $I \cap A \neq (0)$, it follows that $I \cap A \not\subseteq pA$. There is therefore an element $x \in A \cap I$ with $x \notin pA$. So, $x \notin pT \cap A = pA$. Now, $x \in I \cap A \subseteq J \cap A$. It follows that $J \cap A \neq pA$. Hence, the formal fiber of pA is semi-local with maximal ideals the elements of C. □

Theorem 3.13 is our main result. The previous work in this section has been devoted to showing sufficiency of the conditions below, so the majority of Theorem 3.13 demonstrates their necessity.

Theorem 3.13. *Let $(T, \mathfrak{m})$ be a complete local ring, S_0 the prime subring of T and $p \neq 0$ a regular and prime element of T. Let $C = \{Q_1, Q_2, \dots, Q_n\}$ be a finite set of incomparable prime ideals of T such that either $S_0 \cap Q_i = (0)$ for every i or $S_0 \cap Q_i = pS_0$ for every i. Then there exists a local domain A such that $\widehat{A} = T$, $p \in A$, and the formal fiber of pA is semi-local with maximal ideals the elements of C if and only if the following conditions are satisfied.*

(1) *$p \in Q_i$ for every i.*

(2) *If $\dim T = 1$, then $C = \{\mathfrak{m}\}$.*

(3) *If $\dim T > 1$, then $\mathfrak{m} \notin C$.*

Proof. We will first show that the conditions are necessary. The first condition is clearly necessary. By Lemma 3.7, T is an integral domain, so (0) is the only height zero prime ideal of T. Also, because $pT \neq (0)$ is a prime ideal of T, $\dim T \neq 0$. Suppose that $\dim T > 1$, and that there exists a local domain A such that $\widehat{A} = T$ and the formal fiber of pA is semi-local with maximal ideals the elements of C. Suppose that $\mathfrak{m} \in C$. Then from our assumptions $\mathfrak{m} \cap A = pA$. Because pA is generated by one element, it must have height less than or equal to one, and because $pA \neq (0)$, it cannot be height zero, so pA is height one. Thus, $\mathfrak{m} \cap A$ is height one, but the height of $\mathfrak{m} \cap A$ is equal to the height of $\mathfrak{m}$, so $\dim T = 1$. This is a contradiction, so $\mathfrak{m} \notin C$.

Otherwise, suppose that $\dim T = 1$. Because pT is generated by one element, it must have height less than or equal to one, and because $pT \neq (0)$, it cannot be height zero, so pT is height one. Thus $\mathfrak{m} = pT$. Because $p \in Q_i$ for every i, and $p \neq 0$, no Q_i can be height zero. It follows that $C = \{pT\} = \{\mathfrak{m}\}$.

Now we will show the sufficiency of the conditions. If $\dim T = 1$, setting $A = T$ produces the desired result. Otherwise, $\dim T > 1$. Now, use Lemma 3.12 to construct the desired domain A. $\square$

We note here that the generic formal fiber of A is easy to describe. In fact, it is the set $\{P \in \operatorname{Spec} T \mid P \subseteq Q_i \text{ for some } i \text{ and } p \notin P\}$. We also note that $\{P \in \operatorname{Spec} T \mid P \cap A = pA\} = \{P \in \operatorname{Spec} T \mid P \subseteq Q_i \text{ for some } i \text{ and } p \in P\}$.

Example 3.14. Let $T = \widehat{\mathbf{Z}_{(7)}}[[x, y]]$, $p = 7$ and $C = \{(x, 7), (y, 7)\}$. Is there a local domain A such that $7 \in A$, $\widehat{A} = T$ and the formal fiber of $7A$ is semi-local with maximal ideals the elements of C? T is local, with maximal ideal $\mathfrak{m} = (x, y, 7)$, and C is finite. Also note that the prime subring of T is $\mathbf{Z}$ and we have $(x, 7) \cap \mathbf{Z} = 7\mathbf{Z}$ and $(y, 7) \cap \mathbf{Z} = 7\mathbf{Z}$. Thus we may use Theorem 3.13. Certainly $\mathfrak{m} \notin C$. There therefore exists a domain A such that $\widehat{A} = T$ and the formal fiber of $7A$ is semi-local with maximal ideals the elements of C.

We now state the local version of Theorem 3.13.

Corollary 3.15. *Let $(T, \mathfrak{m})$ be a complete local ring, S_0 the prime subring of T and $p \neq 0$ a regular and prime element of T. Let Q be a prime ideal of T such that either $S_0 \cap Q = (0)$ or $S_0 \cap Q = pS_0$. Then there exists a local domain A such that $\widehat{A} = T$, $p \in A$, and the formal fiber of pA is local with maximal ideal Q if and only if $p \in Q$ and either $Q \neq \mathfrak{m}$ or T is dimension one and $Q = \mathfrak{m}$.*

In this paper, we have covered the cases when $Q_i \cap S_0 = (0)$ for every i and $Q_i \cap S_0 = pS_0$ for every i. We note here that we do not know if similar kinds of results can be obtained in other cases. For example, we would like to prove similar theorems when p is not a prime element of T or when the condition that $Q_i \cap S_0 = (0)$ for every i or $Q_i \cap S_0 = pS_0$ for every i is not satisfied.

We now show that in a special case we can construct the ring A to be excellent.

Lemma 3.16. *Let $(T, \mathfrak{m})$ be a complete local ring containing the rationals and $p \neq 0$ a regular and prime element of T. Let $C = \{Q_1, Q_2, \dots, Q_n\}$ be a set of incomparable nonmaximal prime ideals all containing p such that T_{Q_i} and $(T/pT)_{Q_i}$ are regular local rings for every i. Then there exists an excellent local domain A such that*

(1) $p \in A$.

(2) $\widehat{A} = T$.

(3) *The formal fiber of pA is semi-local with maximal ideals the elements of C.*

Proof. First note that since T contains the rationals, we have that $Q_i \cap S_0 = (0)$ for every i. Use Lemma 3.12 to construct the domain A. Then we must only show that A is excellent. Note that T is a domain and so A is formally equidimensional. It follows that A is universally catenary. We have left to show that the formal fibers of A are geometrically regular.

First let P be a prime ideal of A with $P \neq (0)$ and $P \neq pA$. Define $k(P)$ to be the field A_P/PA_P. Now if $PT \subseteq Q_i$ for some i then $P = PT \cap A \subset Q_i \cap A = pA$, a contradiction. So we have that $PT \not\subseteq Q_i$ for every i. By condition (4) of Lemma 3.12 we have that the map $A \to T/PT$ is onto. Hence, $A/P \cong T/PT$. Now,

$$T \otimes_A k(P) \cong (T/PT)_{\overline{A-P}} \cong (A/P)_{\overline{A-P}} \cong A_P/PA_P = k(P),$$

a field. Also note that if L is a finite field extension of $k(P)$, then we have that

$$T \otimes_A L \cong T \otimes_A k(P) \otimes_{k(P)} L \cong k(P) \otimes_{k(P)} L \cong L,$$

also a field. It follows that the fiber over P is geometrically regular.

Now we show that the fiber over (0) is geometrically regular. By construction, the maximal ideals of $T \otimes_A k((0))$ are the maximal elements of the set $X = \{P \in \operatorname{Spec} T \mid P \subseteq Q_i \text{ for some } i \text{ and } p \notin P\}$. Let J be a maximal element of X. Then $T \otimes_A k((0))$ localized at J is isomorphic to T_J. But since T_{Q_i} is a regular local ring for every i, we have that T_J is a regular local ring. Now, since T contains the rationals, $k((0))$ is a field of characteristic zero. It follows that the formal fiber of A at (0) is geometrically regular.

Now let $P = pA$. We have by construction that $T \otimes_A k(pA)$ is a semi-local ring with maximal ideals the elements of C. Since $T \otimes_A k(pA)$ is isomorphic to $(T/pT)_{\overline{A-pA}}$ we have that the ring $T \otimes_A k(pA)$ localized at Q_i for some i is isomorphic to $(T/pT)_{Q_i}$ which is a regular local ring by assumption. Since T contains the rationals, $k(pA)$ is a field of characteristic zero, and it follows that the formal fiber of pA is geometrically regular. Therefore, A is excellent as desired. $\square$

Example 3.17. Let $T = \mathbf{C}[[x, y, z]]$, $Q_1 = (x, y)$, $Q_2 = (x, z)$ and $p = x$. Then the conditions of Lemma 3.16 are satisfied so we know there is an excellent local domain A such that $x \in A$, $\widehat{A} = T$ and the formal fiber of xA is semi-local with maximal ideals Q_1 and Q_2.

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