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# BIRATIONAL CONTRACTIONS OF $\overline{M}_{3,1}$ AND $\overline{M}_{4,1}$

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ABSTRACT. We study the birational geometry of $\overline{M}_{3,1}$ and $\overline{M}_{4,1}$. In particular, we pose a pointed analogue of the Slope Conjecture and prove it in these low-genus cases. Using variation of GIT, we construct birational contractions of these spaces in which certain divisors of interest – the pointed Brill-Noether divisors – are contracted. As a consequence, we see that these pointed Brill-Noether divisors generate extremal rays of the effective cones for these spaces.

## 1. INTRODUCTION

The moduli spaces of curves are some of the most studied objects in algebraic geometry. In recent years, a great deal of progress has been made on understanding the birational geometry of these spaces. Examples include the work of Hassett and Hyeon on the minimal model program for $\overline{M}_g$ [HH09a], [HH09b] and the discovery by Farkas of previously unknown effective divisors on $\overline{M}_g$ [Far09]. Nevertheless, many fundamental questions remain open.

Many of these questions can be stated in terms of the cone of effective divisors $\overline{NE}^1(\overline{M}_g)$. Among the first to study this cone were Eisenbud, Harris and Mumford in a series of papers proving that $\overline{M}_g$ is of general type for $g \geq 24$ [HM82], [EH87]. A key element of these proofs is the computation of the class of certain divisors on $\overline{M}_g$. The original paper of Harris and Mumford focused on the $k$-gonal divisor in $\overline{M}_{2k-1}$, a specific case of the more general class of Brill-Noether divisors. In their argument, they use this calculation to show that the canonical class can be written as an effective sum of a Brill-Noether divisor, boundary divisors, and an ample divisor, and hence lies in the interior of $\overline{NE}^1(\overline{M}_g)$. The search for effective divisors with this property eventually led to the Harris-Morrison Slope Conjecture.

In their work, Harris and Eisenbud discovered that all of the Brill-Noether divisors lie on a single ray in $\overline{NE}^1(\overline{M}_g)$. One consequence of the Slope Conjecture would be that this ray is extremal. The Slope Conjecture has recently been proven false in [FP05] and subsequently in [Far09], but the statement is known to hold for certain small values of $g$. In several of these cases, the statement can be proved by use of the Contraction Theorem, which states that the set of exceptional divisors of a birational contraction $X \dashrightarrow Y$ spans a simplicial face of $\overline{NE}^1(X)$ (see [Rul01]). In other words, the Slope Conjecture has been shown to hold for small values of $g$ by constructing explicit birational models for the moduli space in which the Brill-Noether divisor is contracted. Moreover, these models arise naturally as geometric invariant theory quotients.

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The purpose of this paper is to carry out a pointed analogue of the discussion above in some low genus cases. In [Log03], Logan introduced the notion of **pointed Brill-Noether divisors**.

**Definition 1.** Let $Z = (a_0, \ldots, a_r)$ be an increasing sequence of nonnegative integers with $\alpha = \sum_{i=0}^{r}(a_i - i)$. Let $BN^r_{d,Z}$ be the closure of the locus of pointed curves $(p, C) \in M_{g,1}$ possessing a $g^r_d$ on $C$ with vanishing sequence $Z$ at $p$. When $g + 1 = (r + 1)(g - d + r) + \alpha$, this is a divisor in $\overline{M}_{g,1}$, called a **pointed Brill-Noether divisor**.

Logan's original motivation was to prove a pointed version of the Harris-Mumford general type result. In this setting, it is natural to consider an analogue of the Slope Conjecture:

**Question 1.** Is there an extremal ray of $\overline{NE}^1(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor?

We consider this question in certain low-genus cases. When $g = 2$, this question was answered in the affirmative by Rulla [Rul01]. He shows that the Weierstrass divisor $BN^1_{2,(0,2)}$ generates an extremal ray of $\overline{NE}^1(\overline{M}_{2,1})$ by explicitly constructing a birational contraction of $\overline{M}_{2,1}$. Our main result is an extension of this to higher genera:

**Theorem 1.1.** *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$. Similarly, there is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$.*

As a consequence, we identify an extremal ray of the effective cone.

**Corollary 1.2.** *For $g = 3, 4$, there is an extremal ray of $\overline{NE}^1(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor.*

The proof uses variation of GIT. In particular, we consider the following GIT problem: let $Y$ be a surface and fix a linear equivalence class $|D|$ of curves on $Y$. Now, let

$$X = \{(p, C) \in Y \times |D| \; |p \in C\}$$

be the universal family over this space of curves. In the case where $(Y, |D|)$ is $(\mathbb{P}^2, |\mathcal{O}(4)|)$ or $(\mathbb{P}^1 \times \mathbb{P}^1, |\mathcal{O}(3,3)|)$, the quotient of $X//\mathrm{Aut}(Y)$ is a birational model for $\overline{M}_{3,1}$ or $\overline{M}_{4,1}$, respectively. By varying the choice of linearization, we obtain a birational model in which the specified divisor is contracted.

The outline of the paper is as follows. In section 2 we provide some background on variation of GIT. In section 3, we develop a tool for studying GIT quotients of families of curves on surfaces. In particular, we construct a large class of divisors on these spaces that are invariant under the automorphism group of the surface, called Hessians. In sections 4 and 5 we then examine separately curves on $\mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$, yielding our result in the cases of $g = 3$ and 4.

We plan on discussing similar results for genus 5 and 6 in a later paper.

## 2. Variation of GIT

The birational contractions that we construct arise naturally as GIT quotients. This section contains a brief summary of results of Dolgachev-Hu [DH98] and Thaddeus [Tha96] on variation of GIT.

Given a group $G$ acting on a variety $X$, the GIT quotient $X//G$ is not unique; it depends on the choice of a $G$-ample line bundle. In particular, if $\mathcal{L} \in Pic^G(X)$, we have

$$X//_{\mathcal{L}}G = Proj \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})^G.$$

Following Dolgachev and Hu, we will call the set of all $G$-ample line bundles the **$G$-ample cone**. A study of how the quotient varies with the choice of the $G$-ample line bundle was carried out independently by Dolgachev-Hu [DH98] and Thaddeus [Tha96]. The following theorem is a summary of some of the results of those papers:

**Theorem 2.1** ([DH98], [Tha96]). *The $G$-ample cone is divided into a finite number of convex cones, called* **chambers**. *Two line bundles $\mathcal{L}$ and $\mathcal{L}'$ lie in the same chamber if $X^s(\mathcal{L}) = X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}') = X^s(\mathcal{L}')$. The chambers are bounded by a finite number of* **walls**. *A line bundle $\mathcal{L}$ lies on a wall if $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$. If $\mathcal{L}$ lies on a wall and $\mathcal{L}'$ lies is an adjacent chamber, then there is a morphism $X//_{\mathcal{L}'}G \to X//_{\mathcal{L}}G$. This map is an isomorphism over the stable locus.*

Both Thaddeus and Dolgachev-Hu examine the maps between quotients at a wall in the $G$-ample cone. Specifically, let $\mathcal{L}_+$, $\mathcal{L}_-$ be $G$-ample line bundles in adjacent chambers of the $G$-ample cone, and define $\mathcal{L}(t) = \mathcal{L}_+^t \otimes \mathcal{L}_-^{1-t}$. Suppose that the line between them crosses a wall precisely at $\mathcal{L}(t_0)$. Following Thaddeus, define

$$X^{\pm} = X^{ss}(\mathcal{L}_{t_0}) \backslash X^{ss}(\mathcal{L}_{\mp}),$$
$$X^0 = X^{ss}(\mathcal{L}_{t_0}) \backslash (X^{ss}(\mathcal{L}_+) \cup X^{ss}(\mathcal{L}_-)).$$

**Theorem 2.2** ([Tha96]). *Let $x \in X^0$ be a smooth point of $X$. Suppose that $G \cdot x$ is closed in $X^{ss}(\mathcal{L}_{t_0})$ and that $G_x \cong \mathbb{C}^*$. Then the natural map $X//_{\mathcal{L}_\pm}G \to X//_{\mathcal{L}_{t_0}}G$ is an isomorphism outside of $X^{\pm}//_{\mathcal{L}_\pm}G$. Over a neighborhood of $x$ in $X^0//_{\mathcal{L}_{t_0}}G$, $X^{\pm}//_{\mathcal{L}_\pm}G$ are fibrations whose fibers are weighted projective spaces.*

In order to determine whether a point is (semi)stable, we will make frequent use of Mumford's numerical criterion. Given a $G$-ample line bundle $\mathcal{L}$ and a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$, it is standard to choose coordinates so that $\lambda$ acts diagonally on $H^0(X, \mathcal{L})^*$. In other words, it is given by $\mathrm{diag}(t^{a_1}, t^{a_2}, \dots, t^{a_n})$. We will refer to the $a_i$'s as the weights of the $\mathbb{C}^*$-action. For a point $x \in X$, Mumford defines

$$\mu_\lambda(x) = \min(a_i | x_i \neq 0).$$

Then $x$ is stable (semistable) if and only if $\mu_\lambda(x) < 0$ (resp. $\mu_\lambda(x) \leq 0$) for every nontrivial 1-parameter subgroup $\lambda$ of $G$ (see Theorem 2.1 in [MFK94]).

## 3. Hessians

Here we set up the GIT problem that appears in sections 4 and 5. We also identify a collection of $G$-invariant divisors that will be useful for analyzing this problem.

Let $Y$ be a smooth projective surface over $\mathbb{C}$, $\mathcal{L}'$ an effective line bundle on $Y$, and $Z = \mathbb{P}H^0(Y, \mathcal{L}')$. Let

$$X = \{(p, C) \in Y \times Z | p \in C\}.$$

We denote the various maps as in the following diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \times Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & Y \\ Z \arrow[r, "id"] & Z. \end{tikzcd}$$

If $\mathcal{L}'$ is base-point free, then $X$ is a projective space bundle over $Y$, so it is smooth and $PicX \cong PicY \times \mathbb{Z}$. We will later study the GIT quotients of $X$ by the natural action of $\mathrm{Aut}(Y)$.

If $C$ is a curve on $Y$ and $\mathcal{L}$ is another line bundle on $Y$, then for every point $p \in C$ there are $n+1 = h^0(C, \mathcal{L}|_C)$ different orders of vanishing of sections $s \in H^0(C, \mathcal{L}|_C)$.

**Definition 2.** When written in increasing order,

$$a_0^{\mathcal{L}}(p) < \cdots < a_n^{\mathcal{L}}(p),$$

the orders of vanishing are called the **vanishing sequence** of $\mathcal{L}$ at $p$. The **weight** of $\mathcal{L}$ at $p$ is defined to be $w^{\mathcal{L}}(p) = \sum_{i=0}^{n}(a_i^{\mathcal{L}}(p) - i)$. A point is said to be an **$\mathcal{L}$-flex** if the weight of $\mathcal{L}$ at the point is nonzero.

In other words, $p$ is an $\mathcal{L}$-flex if the vanishing sequence of $\mathcal{L}$ at $p$ is anything other than $0 < 1 < \cdots < n$.

**Definition 3.** The **divisor of $\mathcal{L}$-flexes** is $\sum_{p\in C} w^{\mathcal{L}}(p)p$. It corresponds to a section $W_{\mathcal{L}}$ of a certain line bundle called the **Wronskian** of $\mathcal{L}$. We say that a curve $H$ on $Y$ is an **$\mathcal{L}$-Hessian** if the restriction of $H$ to $C$ is precisely the divisor of $\mathcal{L}$-flexes.

Returning to our family of curves $f : X \to Z$ above, suppose that $\mathcal{L}$ is a line bundle on $Y$ such that the pushforward $f_*(\pi_1 \circ i)^*\mathcal{L}$ is locally free of rank $n+1$. We define a relative $\mathcal{L}$-Hessian to be a divisor $H \subseteq X$ whose restriction to each fiber is the divisor of $f_*(\pi_1 \circ i)^*\mathcal{L}$-flexes. Relative $\mathcal{L}$-Hessians were studied by Cukierman [Cuk97], who shows:

**Proposition 3.1** ([Cuk97]). *The class of the relative $\mathcal{L}$-Hessian is*

$$(n+1)c_1(\pi_1 \circ i)^*\mathcal{L} + \binom{n+1}{2} c_1\Omega^1_{X/Z} - c_1 f^* f_*(\pi_1 \circ i)^*\mathcal{L}.$$

In our particular case, we can determine this class more explicitly.

**Corollary 3.2.** *For $X, Y$, and $Z$ as above, the class of the relative $\mathcal{L}$-Hessian is*

$$(n+1)c_1(\pi_1 \circ i)^*\mathcal{L} + \binom{n+1}{2}(c_1\pi_1^*\Omega^1_Y|_X + c_1(\pi_1 \circ i)^*\mathcal{L}' + c_1 f^*\mathcal{O}_Z(1))$$
$$-h^0(Y, \mathcal{L} \otimes \mathcal{L}'^*)(c_1 f^*\mathcal{O}_Z(1)).$$

*Proof.* We follow the proof in [Cuk97]. If $I$ is the ideal sheaf of $X$ in $Z \times Y$, then we have the exact sequence

$$0 \to I/I^2 \to \pi_1^*\Omega^1_Y|_X \to \Omega^1_{X/Z} \to 0,$$

so we have

$$c_1\Omega^1_{X/Z} = c_1\pi_1^*\Omega^1_Y|_X - c_1 I/I^2.$$

Also, $X$ is the scheme of zeros of a section of the line bundle $E = (\pi_1 \circ i)^*\mathcal{L}' \otimes f^*\mathcal{O}_Z(1)$ on $Y \times Z$. Note that $I/I^2 \cong E^* \otimes \mathcal{O}_X = E^*|_X$. It follows that

$$\begin{aligned} c_1\Omega^1_{X/Z} &= c_1(\pi_1 \circ i)^*\Omega^1_Y|_X + c_1 E \\ &= c_1(\pi_1 \circ i)^*\Omega^1_Y|_X + c_1(\pi_1 \circ i)^*\mathcal{L}' + c_1 f^*\mathcal{O}_Z(1). \end{aligned}$$

Now, consider the exact sequence on $Y \times Z$

$$0 \to \pi_1^* L \otimes E^* \to \pi_1^* L \to \pi_1^* L|_X \to 0.$$

From the projection formula, we see that

$$\pi_{2*}(\pi_1^*\mathcal{L} \otimes E^*) = H^0(Y, \mathcal{L} \otimes \mathcal{L}'^*) \otimes \mathcal{O}_Z(-1)$$

and $R^1\pi_{2*}(\pi_1^* L \otimes E^*) = 0$. This gives us the exact sequence on $Z$

$$0 \to \pi_{2*}(\pi_1^* L \otimes E^*) \to \pi_{2*}\pi_1^* L \to \pi_{2*}(\pi_1^* L|_X) \to 0.$$

Since the middle term is a trivial bundle, the result follows from Proposition 3.1. $\square$

For the remainder of this section, we identify specific examples that will appear in the arguments to follow.

In section 4 we consider the case that $Y = \mathbb{P}^2$ and $\mathcal{L}' = \mathcal{O}_Y(d)$ for some $d \geq 3$. By the above, we see that for every $m$ and $d$, a relative $\mathcal{O}_Y(m)$-Hessian $H_m$ exists. Since $c_1\pi_1^*\Omega^1_Y|_X = \mathcal{O}_X(-3, 0)$, if $m < d$, $H_m$ is cut out by a $G$-invariant section $W_m$ of

$$\mathcal{O}_X((n+1)m + \binom{n+1}{2}(d-3), \binom{n+1}{2}),$$

where $n + 1 = h^0(Y, \mathcal{L}) = \binom{m+2}{2}$.

In particular, $H_1$ is cut out by a section $W_1 \in H^0(\mathcal{O}_X(3(d-2), 3))$. $W_1$ vanishes at $(p, C)$ if $C$ is smooth at $p$ and the tangent line to $C$ at $p$ intersects $C$ with multiplicity at least 3, or if $p$ is a singular point of $C$. Similarly, $H_2$ is defined by a section of $W_2 \in H^0(\mathcal{O}_X(15d - 33, 15))$. $W_2$ vanishes at $(p, C)$ if $C$ is smooth at $p$ and the osculating conic to $C$ at $p$ intersects $C$ with multiplicity at least 6, or if $p$ is a singular point of $C$.

It is known that $H_2 = H_1 \cup H_2'$ is reducible (see Proposition 6.6 in [CF91]). Indeed, if a line meets $C$ with multiplicity 3 at $p$, then the double line meets $C$ with multiplicity 6 at $p$. The points of $H_2' \cap C$ are classically known as the **sextatic points** of $C$, and $H_2'$ is cut out by a $G$-invariant section $W_2'$ of $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$. A simple calculation shows that $H_2' \cap C$ also contains those points of $C$ where $w^{\mathcal{O}_C(1)}(p) > 1$. These include singular points and points where the tangent line to $C$ is a **hyperflex** (a line that intersects $C$ at $p$ with multiplicity $\geq 4$).

Similarly, in section 5 we consider the case that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{L}' = \mathcal{O}_Y(d, d)$. Note that, for every $(m_1, m_2, d)$ with $m_i < d$, a relative $\mathcal{O}_Y(m_1, m_2)$-Hessian $H'_{m_1,m_2}$ exists. In this case, our formulas show that the rank of $f_*(\pi_1 \circ i)^*\mathcal{O}_Y(m_1, m_2)$ is

$$n + 1 = h^0(\mathcal{O}_Y(m_1, m_2)) = (m_1 + 1)(m_2 + 1).$$

Also, since $c_1\pi_1^*\Omega_Y^1|_X = \mathcal{O}_X(-2,-2,0)$, we see that $H'_{m_1,m_2}$ is cut out by a section $W'_{m_1,m_2} \in H^0(\mathcal{O}_X(a_1,a_2,b))$ for

$$a_i = (n+1)m_i + \binom{n+1}{2}(d-2),$$
$$b = \binom{n+1}{2}.$$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ has a natural involution, we know that $W'_{m_1,m_2}$ cannot be $G$-invariant if $m_1 \neq m_2$. Notice, however, that $W'_{m_1,m_2} \otimes W'_{m_2,m_1}$ is a $G$-invariant section of $\mathcal{O}_X(a,a,b)$ for

$$n+1 = (m_1+1)(m_2+1),$$
$$a = (n+1)(m_1+m_2) + 2\binom{n+1}{2}(d-2),$$
$$b = 2\binom{n+1}{2}.$$

We will use $W_{m_1,m_2}$ to denote the $G$-invariant section described here, and $H_{m_1,m_2}$ to denote its zero locus.

In particular, $W_{0,1} \in H^0(\mathcal{O}_X(2(d-1),2(d-1),2))$. It vanishes at a point $(p,C)$ if $C$ intersects one of the two lines through $p$ with multiplicity at least 2 (or, equivalently, if the osculating $(1,1)$ curve is a pair of lines). Similarly, $W_{1,1} \in H^0(\mathcal{O}_X(2(3d-4),2(3d-4),6))$. It vanishes at a point $(p,C)$ if there is a curve of bidegree $(1,1)$ that intersects $C$ with multiplicity 4 or more at $p$.

## 4. Contraction of $\overline{M}_{3,1}$

In this section, we prove our main result in the genus 3 case:

**Theorem 4.1.** *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$.*

In order to construct a birational model for $\overline{M}_{3,1}$, we consider GIT quotients of the universal family over the space of plane quartics. The image of the Weierstrass divisor in this model is precisely the Hessian $H_1$, and we exhibit a GIT quotient in which this locus is contracted. For most of this section we will consider, more generally, plane curves of any degree $d \geq 3$.

Specifically, following the setup of the previous section, we let

$$X = \{(p,C) \in \mathbb{P}^2 \times |\mathcal{O}(d)| \; |p \in C\}.$$

Then $\pi_2 : X \to |\mathcal{O}(d)|$ is the family of all plane curves of degree $d$. Our goal is to study the GIT quotients of $X$ by the action of $G = PSL(3,\mathbb{C})$. By the above, we know that $PicX \cong \mathbb{Z} \times \mathbb{Z}$, so the quotient $X//_{\mathcal{L}}G$ depends on a single parameter $t$ which we call the **slope** of $\mathcal{L}$.

**Definition 4.** We say a line bundle $\mathcal{L}$ has **slope** $t$ if $\mathcal{L} = \pi_1^*\mathcal{O}(a) \otimes \pi_2^*\mathcal{O}(b)$ with $t = \frac{a}{b}$. We write $X^s(t)$ and $X^{ss}(t)$ for the sets of stable and semistable points, and $X//_tG$ for the corresponding GIT quotient.

Here we describe the numerical criterion for points in $X$. Let $p = (x_0, x_1, x_2)$ and

$$C = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k.$$

Then a basis for $H^0(\mathcal{O}_X(a,b))$ consists of monomials of the form

$$\prod_{\alpha=1}^{a} x_{l_\alpha} \prod_{\beta=1}^{b} a_{i_\beta, j_\beta, k_\beta}.$$

The one-parameter subgroup with weights $(r_0, r_1, r_2)$ acts on the monomial above with weight

$$\sum_{\alpha=1}^{a} r_{l_\alpha} - \sum_{\beta=1}^{b} (i_\beta r_0 + j_\beta r_1 + k_\beta r_2).$$

In our case, we will only be interested in maximizing or minimizing this weight, so it suffices to consider monomials of the form $x_l^a a_{i,j,k}^b$. In this case, the one-parameter subgroup acts with weight $ar_l - b(ir_0 + jr_1 + kr_2)$, which is proportional to

$$\mu_\lambda(x_l, a_{i,j,k}) := tr_l - (ir_0 + jr_1 + kr_2).$$

The $G$-ample cone of $X$ has two edges, one of which occurs when $t = 0$. In the case where $d = 4$, we obtain the well-known moduli space of plane quartics. Descriptions of $X^s(0)$ and $X^{ss}(0)$ appear in [MFK94], and the quotient $X//_0 G$ plays an important role in the birational geometry of $\overline{M}_3$. For example, Hyeon and Lee show that this quotient is a log canonical model for $\overline{M}_3$ [HL10], and the space also appears in work on moduli of $K3$ surfaces [Art09] and cubic threefolds [CML09].

We will see that, when $t$ is large, stability conditions reflect the inflectionary behavior of linear series at the marked point. Thus, as $t$ increases, the curve is allowed to have more complicated singularities, but vanishing sequences at the marked point become more well-behaved.

Our first result is to identify the other edge of the $G$-ample cone. It is determined by the Wronksian $W_1$.

**Proposition 4.2.** *An edge of the G-ample cone occurs at* $t = d - 2$.

*Proof.* It suffices to show that $X^{ss}(d-2) \neq X^s(d-2) = \emptyset$. It is clear that $X^{ss}(d-2) \neq \emptyset$, since $W_1$ is a $G$-invariant section of $\mathcal{O}_X(3(d-2), 3)$.

To show that $X^s(d-2) = \emptyset$, we invoke the numerical criterion. Let $(p, C) \in X$. By change of coordinates, we may assume that $p = (0,0,1)$ and the tangent line to $C$ at $p$ is $x_0 = 0$. So in the coordinates described above, we have $a_{0,0,d} = a_{0,1,d-1} = 0$.

Now consider the 1-parameter subgroup with weights $(-1, 0, 1)$. We have

$$\mu_\lambda(x_2, a_{i,j,k}) = d - 2 + i - k,$$

which is negative whenever $i - k - 2 < -d = -i - j - k$, or $2i + j < 2$. This only occurs when both $i = 0$ and $j < 2$, in other words, when either $a_{0,0,d}$ or $a_{0,1,d-1}$ is nonzero. By assumption, however, this is not the case, so $(p, C) \notin X^s(d-2)$. Since $(p, C)$ was arbitrary, it follows that $X^s(d-2) = \emptyset$. □

Next, we identify the adjacent chamber in the $G$-ample cone. It lies between the slopes corresponding to $W_1$ and $W_2'$. In what follows, we let $S$ denote the set of all pointed curves $(p, C)$ admitting the following description: $C$ consists of a smooth conic together with $d-2$ copies of the tangent line through a point $q \neq p$ on $C$. Notice that $S \subset H_2'$.

**Proposition 4.3.** *For any $t \in (d-\frac{9}{4}, d-2)$, $X^s(t) = X^{ss}(t) = X \backslash (H_1 \cup S)$.*

*Proof.* We first show that $X^{ss}(t) \subseteq X\backslash H_1$. Suppose that $(p, C) \in H_1$. As before, by change of coordinates, we may assume that $p = (0,0,1)$ and the tangent line to $C$ at $p$ is $x_0 = 0$. Since $(p, C) \in H_1$, either $p$ is a singular point of $C$ or this tangent line intersects $C$ at $p$ with multiplicity at least 3. Thus we have $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ (if $p$ is singular) or $a_{0,2,d-2} = 0$ (if $p$ is a flex).

We first examine the case where $p$ is a flex. In this case, consider the 1-parameter subgroup with weights $(-5,1,4)$. Then

$$\mu_\lambda(x_2, a_{i,j,k}) = 4t + 5i - j - 4k > 4d - 9 + 5i - j - 4k = 9i + 3j - 9,$$

which is nonnegative when $3i + j \geq 3$. Since, by assumption, $C$ has no nonzero terms with both $i = 0$ and $j < 3$, we see that $(p, C) \notin X^{ss}(t)$.

Next we look at the case where $p$ is a singular point. Consider the 1-parameter subgroup with weights $(-1,-1,2)$. Then we have

$$\mu_\lambda(x_2, a_{i,j,k}) = 2t + i + j - 2k > 2d - \frac{9}{2} + i + j - 2k = 3i + 3j - \frac{9}{2},$$

which is nonnegative when $i + j \geq \frac{3}{2}$. By assumption, $C$ has no nonzero terms where one of $i, j$ is 0 and the other is at most 1, so $(p, C) \notin X^{ss}(t)$. It follows that $X^{ss}(t) \subseteq X\backslash H_1$.

Next we show that $X^{ss}(t) \subseteq X\backslash S$. Suppose that $(p, C) \in S$. Without loss of generality, we may assume that $C$ is of the form

$$C = x_0^{d-2}(a_{d,0,0}x_0^2 + a_{d-1,1,0}x_0x_1 + a_{d-2,2,0}x_1^2 + a_{d-1,0,1}x_0x_2).$$

Now, consider the 1-parameter subgroup with weights $(-1,0,1)$. Then

$$\mu_\lambda(x_l, a_{i,j,k}) \geq -t + i - k > 2 - d + i - k,$$

which is nonnegative when $i - k \geq d-2$. It follows that $(p, C) \notin X^{ss}(t)$.

Now we show that $X\backslash(H_1 \cup S) \subseteq X^s(t)$. Suppose that $(p, C) \notin X^s(t)$. Then there is a nontrivial 1-parameter subgroup that acts on $(p, C)$ with nonnegative weight. By change of basis, we may assume that this subgroup acts with weights $(r_0, r_1, r_2)$, with $r_0 \leq r_1 \leq r_2$. Since this is a nontrivial subgroup of $PSL(3,\mathbb{C})$, we know that $r_0 < 0 < r_2$ and $r_0 + r_1 + r_2 = 0$. We then have

$$\mu_\lambda(x_l, a_{i,j,k}) = tr_l - (r_0 i + r_1 j + r_2 k) \geq 0.$$

We divide this into three cases, depending on $p$.

*Case* 1 *– $p = (0,0,1)$:* In this case, $r_l = r_2$. If $r_1 \geq 0$, then $tr_2 < (d-2)r_2 \leq 2r_1 + (d-2)r_2$. On the other hand, if $r_1 < 0$, then $tr_2 < (d-2)r_2 < r_0 + (d-1)r_2$. Since the subgroup acts with nonnegative weight, it follows that $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ or $a_{0,2,d-2} = 0$. Hence, $(p, C) \in H_1$.

*Case* 2 *– $p$ lies on the line $x_0 = 0$, but not on the line $x_1 = 0$:* In this case, $r_l = r_1$. If $r_1 > 0$, then since $r_1 \leq r_2$, we have $tr_1 < dr_1 \leq r_1 j + r_2(d-j)$, so we see that $a_{0,0,d} = a_{0,1,d-1} = \cdots = a_{0,d,0} = 0$. This means that $p$ lies on a linear component of $C$, and therefore $(p, C) \in H_1$.

On the other hand, if $r_1 \leq 0$, then since $r_2 \geq -2r_1$, we see that $tr_1 \leq (d-3)r_1 \leq (d-1)r_1 + r_2 \leq r_1 j + (d-j)r_2 + r_2$ for $j \leq d-1$. Note furthermore that if $r_1 < 0$, then the first of these inequalities is strict, whereas if $r_1 = 0$, the second inequality is strict. It follows that $a_{0,0,d} = a_{0,1,d-1} = \cdots = a_{0,d-1,1} = 0$. This means that either $p$ lies on a linear component of $C$ or the only point of $C$ lying on the line $x_0 = 0$ also lies on the line $x_1 = 0$. Again, we see that $(p, C) \in H_1$.

*Case 3 – $p$ does not lie on the line $x_0 = 0$:* In this case, $r_l = r_0$. Since $r_0 < 0$ and $r_0 \leq r_1 \leq r_2$, we see that $tr_0 < (d-3)r_0 = (d-2)r_0 + r_1 + r_2 < r_0 i + r_1 j + r_2 k$ for $i \leq d-2, k \neq 0$. Now, if $r_0 \geq 4r_1$, then we have $tr_0 < (d-\frac{9}{4})r_0 = (d-\frac{5}{4})r_0 + r_1 + r_2 \leq (d-1)r_0 + r_2$. It follows that $C$ is of the form

$$C = \sum_{i+j=d} a_{i,j,0} x_0^i x_1^j.$$

In other words, $C$ is a union of $d$ lines. In this case, the tangent line to every point of $C$ is a component of $C$ itself, so $(p, C) \in H_1$.

On the other hand, if $r_0 < 4r_1$, then $tr_0 < (d - \frac{9}{4})r_0 = (d-3)r_0 + \frac{3}{4}r_0 < (d-3)r_0 + 3r_1$. It follows that $C$ is of the form

$$C = x_0^{d-2}(a_{d,0,0}x_0^2 + a_{d-1,1,0}x_0x_1 + a_{d-2,2,0}x_1^2 + a_{d-1,0,1}x_0x_2);$$

hence $C \in S$. □

We now consider the wall in the $G$-ample cone determined by $W_2'$.

**Proposition 4.4.** *A wall of the G-ample cone occurs at $t = d - \frac{9}{4}$. More specifically, $X^{ss}(t) = X \backslash ((H_1 \cap H_2') \cup S)$, and $X^s(t) \subseteq X \backslash (H_1 \cup S)$.*

*Proof.* First, notice that if $(p, C) \notin H_2'$, then $(p, C) \in X^{ss}(t)$, since $W_2'$ is a $G$-invariant section of $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$ that does not vanish at $(p, C)$. Moreover, by general variation of GIT we know that, when passing from a chamber to a wall, we have

$$X^{ss}(t+\epsilon) \subseteq X^{ss}(t),$$
$$X^s(t) \subseteq X^s(t+\epsilon).$$

Thus, $X^s(t) \subseteq X \backslash (H_1 \cup S)$ and $X \backslash ((H_1 \cap H_2') \cup S) \subseteq X^{ss}(t)$.

Now, suppose that $(p, C) \in S$. Using the same argument as above with the same 1-parameter subgroup, we see that $(p, C) \notin X^{ss}(t)$.

Next, suppose that $(p, C) \in H_1$. If $p$ is a singular point of $C$, then we see that $(p, C) \notin X^{ss}(t)$ by the same argument as before, using the subgroup with weights $(-1, -1, 2)$.

The only other possibility is that $p$ is a flex. In this case, we again consider the 1-parameter subgroup with weights $(-5, 1, 4)$. As before, we have

$$\mu_\lambda(x_2, a_{i,j,k}) = 4d - 9 + 5i - j - 4k = 9i + 3j - 9,$$

which is nonnegative when $3i + j \geq 3$. As before, we see that $(p, C) \notin X^s(t)$.

Notice furthermore that if $(p, C) \in H_1 \cap H_2'$, then either $a_{0,3,d-3} = 0$ or $a_{1,0,d-1} = 0$. Now consider the 1-parameter subgroup with weights $(-5-\epsilon, 1+\epsilon, 4)$. For $\epsilon > 0$, we see that any curve with $a_{0,3,d-3} = 0$ is unstable. Conversely, if $\epsilon < 0$, we see that any curve with $a_{1,0,d-1} = 0$ is unstable. From our observations above, we may therefore conclude that $X^{ss}(t) \subseteq X \backslash ((H_1 \cap H_2') \cup S)$. □

We are left to consider the behavior of our quotient at the wall crossing defined by $t_0 = d - \frac{9}{4}$. As in Theorem 2.2, we let

$$X^{\pm} = X^{ss}(t_0)\backslash X^{ss}(t_0 \mp \epsilon),$$
$$X^0 = X^{ss}(t_0)\backslash(X^{ss}(t_0+\epsilon) \cup X^{ss}(t_0-\epsilon)).$$

Our first task is to determine $X^-$ and $X^0$ in this situation.

**Proposition 4.5.** *With the setup above, $X^- = H_1\backslash H_2'$. $X^0$ is the set of all pointed curves $(p, C)$ consisting of a cuspidal cubic plus $d-3$ copies of the projectivized tangent cone at the cusp. The point $p$ is the unique smooth flex point of the cuspidal cubic.*

*Proof.* We have already seen that $X^{ss}(t_0) = X\backslash((H_1 \cap H_2') \cup S)$ and $X^{ss}(t_0+\epsilon) = X\backslash(H_1 \cup S)$. Thus, $X^- = H_1\backslash H_2'$.

To prove the statement about $X^0$, let $(p, C) \in X^0$. Notice that, since $X^0 \subseteq X^-$, $p$ is a smooth point of $C$ and the tangent line to $C$ at $p$ intersects $C$ with multiplicity exactly 3. Since $(p, C) \notin X^{ss}(t_0 - \epsilon)$, there must be a nontrivial 1-parameter subgroup that acts on $(p, C)$ with strictly positive weight. Again we assume that this subgroup acts with weights $(r_0, r_1, r_2)$, with $r_0 \leq r_1 \leq r_2$. As before, we know that $r_0 < 0 < r_2$ and $r_0 + r_1 + r_2 = 0$. Again we have

$$\mu_\lambda(x_l, a_{i,j,k}) = tr_l - (r_0 i + r_1 j + r_2 k) > 0.$$

We divide this into three cases, depending on $p$.

*Case* 1 – $p = (0,0,1)$*:* In this case, $r_l = r_2$. Now, if $tr_2 \geq r_0 + (d-1)r_2$, then $(d - \frac{9}{4})r_2 > r_0 + (d-1)r_2$, so $r_1 > \frac{1}{4}r_2$. This means that $tr_2 < (d - \frac{9}{4})r_2 < 3r_1 + (d-3)r_2$. It follows that $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ or $a_{0,2,d-2} = a_{0,3,d-3} = 0$. But we know that $p$ is a smooth point of $C$ and the tangent line to $C$ at $p$ intersects $C$ with multiplicity exactly 3, so neither of these is a possibility.

*Case* 2 – *$p$ lies on the line $x_0 = 0$, but not on the line $x_1 = 0$:* Using the same argument as before, we see that $p$ lies on a linear component of $C$, which is impossible.

*Case* 3 – *$p$ does not lie on the line $x_0 = 0$:* In this case, $r_l = r_0$. Again, since $r_0 < 0$ and $r_1 < r_0 < r_2$, we see that $tr_0 < (d-3)r_0 = (d-2)r_0 + r_1 + r_2 < r_0 i + r_1 j + r_2 k$ for $i \leq d-2, k \neq 0$. Notice that, if $tr_0 < (d-1)r_0 + r_2$, then as before we see that $C$ is the union of $d$ lines, which is impossible.

We therefore see that $(d - \frac{12}{5})r_0 > tr_0 \geq (d-1)r_0 + r_2$. But then $\frac{7}{5}r_0 < -r_2 = r_0 + r_1$, so $r_0 < \frac{5}{2}r_1$. It follows that $tr_0 < (d - \frac{12}{5})r_0 < (d-4)r_0 + 4r_1 \leq r_0 i + r_1 j$ for $j \geq 4$.

We see that $C$ is of the form

$$C = x_0^{d-3}(a_{d,0,0}x_0^3 + a_{d-1,1,0}x_0^2 x_1 + a_{d-2,2,0}x_0 x_1^2 + a_{d-3,3,0}x_1^3 + a_{d-1,0,1}x_0^2 x_2).$$

Thus, $C$ consists of a cuspidal cubic together with $d-3$ copies of the projectivized tangent cone to the cusp. The point $p$ is the unique flex point of the cuspidal cubic.

It is clear that this $(p, C) \in X^-$, since the tangent line to $C$ at $p$ intersects $C$ with multiplicity exactly 3. To see that $(p, C) \notin X^{ss}(t_0 - \epsilon)$, consider again the 1-parameter subgroup with weights $(5, -1, -4)$. The characterization of $X^0$ above then follows from the fact that all cuspidal plane cubics are projectively equivalent. □

**Corollary 4.6.** *The map $X//_{t_0-\epsilon}G \to X//_{t_0}G$ contracts the locus $H_1 \backslash H_2'$ to a point. Outside of this locus, the map is an isomorphism.*

*Proof.* Let $(p, C) \in X^0$. Since all cuspidal plane cubics are projectively equivalent, $G \cdot (p, C) = X^0$, so $G \cdot (p, C)$ is closed in $X^{ss}(t_0)$ and $X^0//G$ is a point. An automorphism of $\mathbb{P}^1$ extends to $(p, C)$ if and only if it fixes the point $p$ and the cusp, and thus the stabilizer of $(p, C)$ is isomorphic to $\mathbb{C}^*$. The conclusion follows from Theorem 2.2. $\square$

We are particularly interested in the case where $d = 4$, because in this case $X//_{t_0-\epsilon}G$ is a birational model for $\overline{M}_{3,1}$. In particular, we have the following:

**Proposition 4.7.** *There is a birational contraction $\beta : \overline{M}_{3,1} \dashrightarrow X//_{t_0-\epsilon}G$.*

*Proof.* It suffices to exhibit a morphism $\beta^{-1} : V \to \overline{M}_{3,1}$, where $V \subseteq X//_{t_0-\epsilon}G$ is open with complement of codimension $\geq 2$ and $\beta^{-1}$ is an isomorphism onto its image. To see this, let $U \subseteq X^{ss}(t_0 - \epsilon)$ be the set of all moduli stable pointed curves $(p, C) \in X^{ss}(t_0 - \epsilon)$. Notice that $U$ is invariant under the action of the group and its complement is strictly contained in the discriminant locus $\Delta$, which is an irreducible $G$-invariant hypersurface in $X$. Note furthermore that there are stable points contained in both $X \backslash \Delta$ and $\Delta \cap U$. Thus, the containments $(X \backslash U)//_{t_0-\epsilon}G \subset \Delta//_{t_0-\epsilon}G$ and $\Delta//_{t_0-\epsilon}G \subset X//_{t_0-\epsilon}G$ are strict. It follows that the complement of $U//G$ in the quotient has codimension $\geq 2$.

By the universal property of the moduli space, since $U$ is a family of moduli stable curves, it admits a unique map $U \to \overline{M}_{3,1}$. Since $U$ is contained in the semistable locus and this map is $G$-equivariant, it factors uniquely through a map $U//_{t_0-\epsilon}G \to \overline{M}_{3,1}$. Since every degree 4 plane curve is canonical, two such curves are isomorphic if and only if they differ by an automorphism of $\mathbb{P}^2$. It follows that this map is an isomorphism onto its image. $\square$

**Theorem 4.8.** *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$. Furthermore, the divisors $BN^1_{3,(0,3)}$, $BN^1_2$, $\Delta_1$ and $\Delta_2$ span a simplicial face of $\overline{NE}^1(\overline{M}_{3,1})$.*

*Proof.* The composition $\overline{M}_{3,1} \dashrightarrow X//_{t_0-\epsilon}G \to X//_{t_0}G$ is a birational contraction. By the above, the Weierstrass divisor is contracted by this map. It therefore suffices to show that the isomorphism $\beta^{-1}$ constructed in the preceding theorem does not contain in its image the generic point of $BN^1_2$ or $\Delta_i$ for $i \geq 1$. For $BN^1_2$ this is automatic, since every smooth curve in $X$ is canonically embedded and hence non-hyperelliptic. For $\Delta_i$ this follows directly from the fact that $\Delta \cap U$ is an irreducible divisor in $U$ whose generic point is an irreducible nodal curve. $\square$

## 5. Contraction of $\overline{M}_{4,1}$

We now turn to the case of genus 4 curves. Our main result will be the following:

**Theorem 5.1.** *There is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$.*

In a similar way to the previous section, we will construct a birational model for $\overline{M}_{4,1}$ by considering GIT quotients of the universal family over the space of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Here, the Hessian $H_{0,1}$ is again the image of a pointed Brill-Noether divisor. As above, our goal is to find a GIT quotient in which this locus is contracted. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and

$$X = \{(p,C) \in Y \times |\mathcal{O}(d,d)| \; |p \in C\}.$$

Then $\pi_2 : X \to |\mathcal{O}(d,d)|$ is the family of all curves of bidegree $(d,d)$. Our goal, as before, is to study the GIT quotients of $X$ by the action of $G = PSO(4,\mathbb{C})$. By the above, we know that $PicX \cong \mathbb{Z}^3$, but we are only interested in those line bundles of the form $\mathcal{O}_X(a,a,b)$. We can therefore define the slope of a line bundle $\mathcal{L} \in PicX$ as above.

**Definition 5.** We say a line bundle $\mathcal{L}$ has **slope** $t$ if $\mathcal{L} = \pi_1^*\mathcal{O}(a,a) \otimes \pi_2^*\mathcal{O}(b)$ with $t = \frac{a}{b}$. We write $X^s(t)$ and $X^{ss}(t)$ for the sets of stable and semistable points, and $X//_tG$ for the corresponding GIT quotient.

Here we describe the numerical criterion for points in $X$. Let $p = (x_0, x_1 : y_0, y_1)$ and

$$C = \sum_{0 \le i,j \le d} a_{i,j} x_0^i x_1^{d-i} y_0^j y_1^{d-j}.$$

Then a basis for $H^0(\mathcal{O}_X(a,a,b))$ consists of monomials of the form

$$\prod_{\alpha_0=1}^{a} x_{l_{\alpha_0}} y_{m_{\alpha_1}} \prod_{\beta=1}^{b} a_{i_\beta, j_\beta}.$$

The one-parameter subgroup with weights $(-r_0, r_0, -r_1, r_1)$ acts on the monomial above with weight

$$\sum_{\beta=1}^{b} (r_0(i_\beta - (d - i_\beta)) + r_1(j_\beta - (d - j_\beta))) - \sum_{\alpha_0=1}^{a} ((-1)^{l_{\alpha_0}} r_0 + (-1)^{m_{\alpha_1}} r_1).$$

In our case, we will only be interested in maximizing or minimizing this weight, so it suffices to consider monomials of the form $x_l^a y_m^a a_{i,j}^b$. In this case, the one-parameter subgroup acts with weight $b(r_0(2i-d) + r_1(2j-d)) - a((-1)^l r_0 + (-1)^m r_1)$, which is proportional to

$$\mu_\lambda(x_l, y_m, a_{i,j}) := r_0(2i-d) + r_1(2j-d) - t((-1)^l r_0 + (-1)^m r_1).$$

As in the previous section, when $t = 0$, we obtain a moduli space of curves of bidegree $(d,d)$. In particular, the case $d = 3$ is notable for being a birational model for $\overline{M}_4$. We will see that as $t$ increases, stable curves are allowed to have more complicated singularities, but the vanishing sequences of linear series at the marked point become more well-controlled. We begin by identifying an edge of the $G$-ample cone corresponding to the Wronskian $W_{0,1}$.

**Proposition 5.2.** *An edge of the G-ample cone occurs at* $t = d - 1$.

*Proof.* It suffices to show that $X^{ss}(d-1) \neq X^s(d-1) = \emptyset$. It is clear that $X^{ss}(d-1) \neq \emptyset$, since $W_{0,1}$ is a $G$-invariant section of $\mathcal{O}_X(2(d-1), 2(d-1), 2)$.

To show that $X^s(d-1) = \emptyset$, we invoke the numerical criterion. Let $(p,C) \in X$. By change of coordinates, we may assume that $p = (0,1 : 0,1)$. So, in the coordinates described above, we have $a_{0,0} = 0$.

Now consider the 1-parameter subgroup with weights $(-1,1,-1,1)$. We have

$$\mu_\lambda(x_1, y_1, a_{i,j}) = 2(d-1) + (2i-d) + (2j-d),$$

which is negative whenever $(2i-d)+(2j-d) < -2(d-1)$, or $i+j < 1$. This only occurs when $i=j=0$, in other words, when $a_{0,0}$ is nonzero. By assumption, however, this is not the case, so $(p,C) \notin X^s(d-1)$. Since $(p,C)$ was arbitrary, it follows that $X^s(d-1) = \emptyset$. □

As above, we identify the adjacent chamber in the $G$-ample cone. It lies between the slopes corresponding to the Wronskians $W_{0,1}$ and $W_{1,1}$. In what follows, we let $S$ denote the set of all pointed curves $(p,C)$ admitting the following description: $C$ consists of a smooth curve of bidegree $(1,1)$ together with $d-1$ copies of the two lines through a point $q \neq p$ on $C$. Notice that $S \subset H_{1,1}$.

**Proposition 5.3.** *For any $t \in (d-\frac{4}{3}, d-1)$, $X^s(t) = X^{ss}(t) = X \backslash (H_{0,1} \cup S)$.*

*Proof.* We first show that $X^{ss}(t) \subseteq X\backslash H_{0,1}$. Suppose that $(p,C) \in H_{0,1}$. As before, by change of coordinates, we may assume that $p = (0,1:0,1)$. Since $(p,C) \in H_{0,1}$, $C$ intersects one of the two lines through $p$ with multiplicity at least 2. Without loss of generality, we may assume this line to be $x_0 = 0$. Thus, if we write $C$ as above, then $a_{0,0} = a_{0,1} = 0$. Now, consider the 1-parameter subgroup with weights $(-2,2,-1,1)$. Then

$$\begin{aligned}\mu_\lambda(x_1, y_1, a_{i,j}) &= 3t + 2(2i-d) + (2j-d) > 3d - 4 + 2(2i-d) + (2j-d)\\ &= 2(2i+j-2),\end{aligned}$$

which is nonnegative when $2i+j \geq 2$. Since, by assumption, $C$ has no nonzero terms with both $i=0$ and $j \leq 1$, we see that $(p,C) \notin X^{ss}(t)$.

Next we show that $X^{ss}(t) \subseteq X\backslash S$. Suppose that $(p,C) \in S$. Without loss of generality, we may assume that $C$ is of the form

$$C = x_0^{d-1} y_0^{d-1} (a_{d,d} x_0 y_0 + a_{d-1,d} x_1 y_0 + a_{d,d-1} x_0 y_1).$$

Now, consider the 1-parameter subgroup with weights $(1,-1,1,-1)$. Then

$$\begin{aligned}\mu_\lambda(x_l, y_m, a_{i,j}) &\geq -2t - (2i-d) - (2j-d) > -2d + 2 - (2i-d) - (2j-d)\\ &= -2((d-i) + (d-j) - 1),\end{aligned}$$

which is nonnegative when $(d-i)+(d-j) \leq 1$. It follows that $(p,C) \notin X^{ss}(t)$.

Now we show that $X\backslash(H_{0,1} \cup S) \subseteq X^s(t)$. Suppose that $(p,C) \notin X^s(t)$. Then there is a nontrivial 1-parameter subgroup that acts on $(p,C)$ with nonnegative weight. By change of basis, we may assume that this subgroup acts with weights $(-r_0, r_0, -r_1, r_1)$, with $0 \leq r_0 \leq r_1$ and $r_1 > 0$. We then have

$$\mu_\lambda(x_l, y_m, a_{i,j}) = r_0(2i-d) + r_1(2j-d) - t((-1)^l r_0 + (-1)^m r_1) \geq 0.$$

We divide this into four cases, depending on $p$.

*Case* 1 – $p = (0,1:0,1)$*:* In this case, $l = m = 1$. We have $t(-r_0 - r_1) > (d-1)(-r_0 - r_1) \geq -(d-2)r_0 - dr_1$. It follows that $a_{0,0} = a_{1,0} = 0$, so $(p,C) \in H_{0,1}$.

*Case* 2 – *$p$ lies on the line $y_0 = 0$, but not on the line $x_0 = 0$:* In this case, $l = 1$ and $m = 0$. Here, $t(-r_0 + r_1) \geq (d-2)(-r_0 + r_1) \geq -dr_0 + kr_1$ for all $k \leq d-2$. Note further that if $r_0 \neq r_1$, then the first inequality is strict, whereas if $r_0 = r_1$, then the second inequality is strict. We therefore see that $a_{0,k} = 0$ for all $k \leq d-2$. If $a_{0,d} \neq 0$, then every point of $C$ that lies on the line $x_0 = 0$ also lies on the line $y_0 = 0$, a contradiction. We therefore see that $a_{0,d} = 0$ as well, but this means that $p$ lies on a linear component of $C$, and therefore $(p,C) \in H_{0,1}$.

*Case* 3 – *$p$ lies on the line $x_0 = 0$, but not on the line $y_0 = 0$:* In this case, $l = 0$ and $m = 1$. Note that $t(r_0 - r_1) \geq d(r_0 - r_1) \geq dr_0 - kr_1$ for all $k < d$. Again, if $r_0 \neq r_1$, then the first inequality is strict, whereas if $r_0 = r_1$, then the second inequality is strict. It follows that $a_{k,0} = 0$ for all $k < d$, which means that either $y_0 = 0$ is a linear component of $C$ or every point of $C$ that lies on the line $y_0 = 0$ also lies on the line $y_0 = 0$. Thus $(p, C) \in H_{0,1}$.

*Case 4 – $p$ does not lie on either of the lines $x_0 = 0$ or $y_0 = 0$:* In this case, $l = m = 0$. Now note that $t(r_0 + r_1) > (d-2)(r_0 + r_1)$, so $a_{k_0,k_1} = 0$ if $k_0$ and $k_1$ are both less than $d$. Furthermore, since $r_0 \leq r_1$, $(d-2)(r_0+r_1) \geq dr_0 + (d-4)r_1$, so $a_{d,k} = 0$ for $k \leq d-2$. Now, if $(d - \frac{4}{3})(r_0 + r_1) \leq (d-4)r_0 + dr_1$, then $2r_0 \leq r_1$, so $t(r_0 + r_1) > (d - \frac{4}{3})(r_0 + r_1) \geq dr_0 + (d-2)r_1$. It follows that either $a_{d,d-1} = 0$, in which case $C$ is a union of $2d$ lines and hence $(p, C) \in H_{0,1}$, or $a_{k,d} = 0$ for all $k \leq d-2$, in which case $C \in S$. □

We now consider the wall in the $G$-ample cone determined by the Wronskian $W_{1,1}$.

**Proposition 5.4.** *A wall of the $G$-ample cone occurs at $t = d - \frac{4}{3}$. More specifically, $X^{ss}(t) = X \backslash ((H_{0,1} \cap H_{1,1}) \cup S)$, and $X^s(t) \subseteq X \backslash (H_{0,1} \cup S)$.*

*Proof.* First, notice that if $(p, C) \notin H_{1,1}$, then $(p, C) \in X^{ss}(t)$, since $W_{1,1}$ is a $G$-invariant section of $\mathcal{O}_X(6(d - \frac{4}{3}), 6(d - \frac{4}{3}), 6)$ that does not vanish at $(p, C)$. Moreover, by general variation of GIT we know that, when passing from a chamber to a wall, we have

$$X^{ss}(t + \epsilon) \subseteq X^{ss}(t),$$
$$X^s(t) \subseteq X^s(t + \epsilon).$$

Thus, $X^s(t) \subseteq X \backslash (H_{0,1} \cup S)$ and $X \backslash ((H_{0,1} \cap H_{1,1}) \cup S) = X^{ss}(t)$.

Now, suppose that $(p, C) \in S$. Using the same argument as before with the same 1-parameter subgroup, we see that $(p, C) \notin X^{ss}(t)$.

Next, suppose that $(p, C) \in H_{0,1}$. In this case, we again consider the 1-parameter subgroup with weights $(-2, 2, -1, 1)$. As before, we have

$$\mu_\lambda(x_1, y_1, a_{i,j}) = 3d - 4 + 2(2i - d) + (2j - d) = 2(2i + j - 2),$$

which is nonnegative when $2i + j \geq 2$. Since, by assumption, $C$ has no nonzero terms with both $i = 0$ and $j \leq 1$, we see that $(p, C) \notin X^s(t)$.

Notice furthermore that if $(p, C) \in H_{0,1} \cap H_{1,1}$, this means that the osculating $(1, 1)$ curve to $C$ at $p$ is the pair of lines through that point, and this curve intersects $C$ with multiplicity at least 4. This means that either $a_{0,1} = 0$ or $a_{2,0} = 0$, which implies that the expression $2i + j - 2$ above is zero for at most one term, and strictly positive for all of the others. Now consider the 1-parameter subgroup with weights $(-2 - \epsilon, 2 + \epsilon, -1, 1)$. For $\epsilon > 0$, we see that any curve with $a_{0,1} = 0$ is unstable. Conversely, if $\epsilon < 0$, we see that any curve with $a_{2,0} = 0$ is unstable. It follows that $(p, C) \notin X^{ss}(t)$, and thus $X^{ss}(t) = X \backslash ((H_{0,1} \cap H_{1,1}) \cup S)$. □

Again, we want to use Theorem 2.2 to study the wall crossing at $t_0 = d - \frac{4}{3}$. Again, we let

$$X^{\pm} = X^{ss}(t_0) \backslash X^{ss}(t_0 \mp \epsilon),$$
$$X^0 = X^{ss}(t_0) \backslash (X^{ss}(t_0 + \epsilon) \cup X^{ss}(t_0 - \epsilon))$$

and determine $X^-$ and $X^0$.

**Proposition 5.5.** *With the setup above, $X^- = H_{0,1} \backslash H_{1,1}$. $X^0$ is the set of all pointed curves $(p, C)$ admitting the following description: $C$ consists of a smooth curve of bidegree $(1,2)$ (or $(2,1)$), together with $d-1$ copies of the tangent line to this curve through a point that has a tangent line, and $d-2$ copies of the other line through this same point. The marked point $p$ is the unique other point on the smooth $(1,2)$ curve that has a tangent line.*

*Proof.* We have already seen that $X^{ss}(t_0) = X \backslash ((H_{0,1} \cap H_{1,1}) \cup S)$ and $X^{ss}(t_0+\epsilon) = X \backslash (H_{0,1} \cup S)$. Thus, $X^- = H_{0,1} \backslash H_{1,1}$.

To prove the statement about $X^0$, let $(p, C) \in X^0$. Notice that, since $X^0 \subseteq X^-$, exactly one of the two lines through $p$ intersects $C$ with multiplicity exactly 2. Since $(p, C) \notin X^{ss}(t_0 - \epsilon)$, there must be a nontrivial 1-parameter subgroup that acts on $(p, C)$ with strictly positive weight. Again we assume that this subgroup acts with weights $(-r_0, r_0, -r_1, r_1)$, with $0 \leq r_0 \leq r_1$ and $r_1 > 0$. Again we have

$$\mu_\lambda(x_l, y_m, a_{i,j}) = r_0(2i - d) + r_1(2j - d) - t((-1)^l r_0 + (-1)^m r_1) > 0.$$

We divide this into four cases, depending on $p$.

*Case* 1 *- $p = (0, 1 : 0, 1)$:* In this case, $l = m = 1$. Again we have $t(-r_0 - r_1) > (d-1)(-r_0 - r_1) \geq -(d-2)r_0 - dr_1$. Now, if $t(-r_0 - r_1) \leq -dr_0 - (d-2)r_1$, then $(d - \frac{4}{3})(-r_0 - r_1) < -dr_0 - (d-2)r_1$, so $r_1 > 2r_0$. This means that $t(-r_0 - r_1) < (d - \frac{4}{3})(-r_0 - r_1) < -(d-4)r_0 - dr_1$. It follows that $a_{0,0} = a_{1,0} = 0$, and either $a_{0,1} = 0$ or $a_{2,0} = 0$. But we know that exactly one of the two lines through $p$ intersects $C$ with multiplicity exactly 2, so neither of these is a possibility.

*Case* 2 *- $p$ lies on the line $y_0 = 0$, but not on the line $x_0 = 0$:* Following the same argument as above we see that either $p$ lies on a linear component of $C$, or every point of $C$ that lies on the line $x_0 = 0$ also lies on the line $y_0 = 0$. It follows that $(p, C) \notin X^-$, a contradiction.

*Case* 3 *- $p$ lies on the line $x_0 = 0$, but not on the line $y_0 = 0$:* Again, following the same argument as above we see that $p$ lies on a linear component of $C$. This implies that $(p, C) \notin X^-$, which is impossible.

*Case* 4 *- $p$ does not lie on either of the lines $x_0 = 0$ or $y_0 = 0$:* In this case, $l = m = 0$. As above, we see that $a_{k_0,k_1} = 0$ if $k_0$ and $k_1$ are both less than $d$, and $a_{d,k} = 0$ for $k < d-1$. Now, if $(d - \frac{3}{2})(r_0 + r_1) \leq (d-6)r_0 + dr_1$, then $3r_0 \leq r_1$, so $t(r_0 + r_1) > (d - \frac{3}{2})(r_0 + r_1) \geq dr_0 + (d-2)r_0$. It follows that either $a_{d,d-1} = 0$, in which case $C$ is a union of $2d$ lines, which is impossible, or $a_{k,d} = 0$ for all $k < d-2$. We therefore see that $C$ is of the form

$$C = x_0^{d-2} y_0^{d-1} (a_{d,d} x_0^2 y_0 + a_{d,d-1} x_0^2 y_1 + a_{d-1,d} x_0 x_1 y_0 + a_{d-2,d} x_1^2 y_0).$$

Thus, $C$ consists of three components. One is a curve of bidegree $(2,1)$. The other two components consist of multiple lines through one of the points on this curve that has a tangent line. The point $p$ is forced to be the unique other such point.

It is clear that this $(p, C) \in X^-$, since by definition, one of the lines through $p$ intersects $C$ with multiplicity greater than 1, and it is impossible for it to intersect a smooth curve of bidegree $(2,1)$ with higher multiplicity than 2, or for the other line through $p$ to intersect the curve with multiplicity at all. To see that $(p, C) \notin X^{ss}(t_0 - \epsilon)$, consider the 1-parameter subgroup with weights $(-1, 1, -2, 2)$.

Finally, notice that all such curves are in the same orbit of the action of $G$, so $X^0$ must be the set of *all* such curves. To see this, note that if we fix the two points that have tangent lines to be $(1, 0 : 1, 0)$ and $(0, 1 : 0, 1)$, then the curve is determined uniquely by the third point of intersection of the curve with the diagonal. Since $PSL(2, \mathbb{C})$ acts 3-transitively on points of $\mathbb{P}^1$, we obtain the desired result. $\square$

**Corollary 5.6.** *The map $X//_{t_0-\epsilon}G \to X//_{t_0}G$ contracts the locus $H_{0,1} \backslash H_{1,1}$ to a point. Outside of this locus, the map is an isomorphism.*

*Proof.* Let $C = x_1^{d-2}y_1^{d-1}(x_0^2y_1 + x_1^2y_0)$, and $p = (0,1:0,1)$. Then $(p, C) \in X^0$. As we have seen, $X^0$ is the orbit of $(p, C)$, so $G \cdot (p, C)$ is closed in $X^{ss}(t_0)$ and $X^0//_{t_0}G$ is a point. Notice that the stabilizer of $(p, C)$ must fix $p = (0,1:0,1)$, and the other ramification point, which is $(1,0:1,0)$. Thus, the stabilizer of $(p, C)$ must consist solely of pairs of diagonal matrices. A quick check shows that the stabilizer of $(p, C)$ is the one-parameter subgroup with weights $(-1, 1, -2, 2)$, which is isomorphic to $\mathbb{C}^*$. Again, the conclusion follows from Theorem 2.2. $\square$

Our main interest is the case where $d = 3$. As above, this is because in this case $X//_{t_0-\epsilon}G$ is a birational model for $\overline{M}_{4,1}$. In particular, we have the following:

**Proposition 5.7.** *There is a birational contraction $\beta : \overline{M}_{4,1} \dashrightarrow X//_{t_0-\epsilon}G$.*

*Proof.* As above, it suffices to exhibit a morphism $\beta^{-1} : V \to \overline{M}_{4,1}$, where $V \subseteq X//_{t_0-\epsilon}G$ is open with complement of codimension $\geq 2$ and $\beta^{-1}$ is an isomorphism onto its image. Again, we let $U \subseteq X^{ss}(t_0 - \epsilon)$ be the set of all moduli stable pointed curves $(p, C) \in X^{ss}(t_0 - \epsilon)$. The proof in this case is exactly like that in the case of $\mathbb{P}^2$, as the discriminant locus $\Delta \subseteq X$ is again an irreducible $G$-invariant hypersurface.

By the universal property of the moduli space, since $U$ is a family of moduli stable curves, it admits a unique map $U \to \overline{M}_{4,1}$. Since $U$ is contained in the semistable locus and this map is $G$-equivariant, it factors uniquely through a map $U//_{t_0-\epsilon}G \to \overline{M}_{4,1}$. Since every curve of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is canonical, two such curves are isomorphic if and only if they differ by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. It follows that this map is an isomorphism onto its image. $\square$

**Theorem 5.8.** *There is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$. Moreover, if $P$ is the Petri divisor, then the divisors $BN^1_{3,(0,2)}$, $P$, $\Delta_1$, $\Delta_2$, and $\Delta_3$ span a simplicial face of $\overline{NE}^1(\overline{M}_{4,1})$.*

*Proof.* The composition $\overline{M}_{4,1} \dashrightarrow X//_{t_0-\epsilon}G \to X//_{t_0}G$ is a birational contraction. By the above, the given pointed Brill-Noether divisor is contracted by this map. It therefore suffices to show that the isomorphism $\beta^{-1}$ constructed in the preceding theorem does not contain in its image the generic point of $P$ or $\Delta_i$ for $i \geq 1$. Every smooth curve in $X$ is Gieseker-Petri general, since its canonical embedding lies on a smooth quadric, so the generic point of $P$ is not contained in the image of $\beta^{-1}$. For $\Delta_i$ this again follows directly from the fact that $\Delta \cap U$ is an irreducible divisor in $U$ whose generic point is an irreducible nodal curve. $\square$

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