Chip Firing and Algebraic Curves

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The chip firing game is played with poker chips on the vertices of a graph. Though seemingly simple, this game has deep connections to various fields of mathematics. In this article, we discuss one of these connections, to the theory of algebraic curves. We will see that the chip firing is a combinatorial analogue of *Brill-Noether theory*, the study of divisors on algebraic curves. This observation allows us to prove theorems in algebraic geometry using graph theory, and vice versa.

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1. Basics of Algebraic Curves

Algebraic geometers are interested in the solutions to systems of polynomial equations, which are called *varieties*. Among the simplest varieties are the one-dimensional ones, which are called *algebraic curves*. To 19th century algebraic geometers, a curve was simply a one-dimensional subset of some ambient projective space, determined by polynomial equations. In the 20th century, however, much of mathematics underwent a fundamental shift, with objects defined in terms of their abstract properties, without reference to an ambient space. To a modern algebraic geometer, a given curve is not equipped with a specific embedding. Instead, we think of the curve as admitting many different maps to projective spaces, and an interesting question is to describe *all* maps from the given curve to projective space. The study of such maps is known as *Brill-Noether theory*. Readers interested in learning more about the geometry of algebraic curves are encouraged to read the standard text [ACGH85].

Two important invariants of such a map are its *degree* and its *rank*. Given a curve C and a map $\varphi : C \to \mathbb{P}^r$ to projective space, the degree of φ is the number of points in the preimage of a general hyperplane. The *rank* of φ is the dimension of the projective space spanned by the image $\varphi(C)$. For example, Figure 1 depicts an embedding of a curve in $\mathbb{P}^2$. Because a general line meets the curve in four points,[1] the degree of this embedding is 4, and because the curve spans the whole plane (that is, it is not contained in a line), the rank of this embedding is 2.

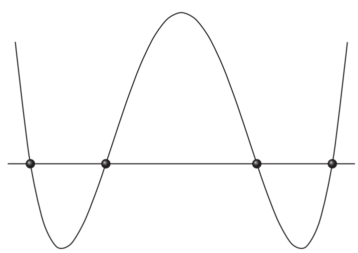

Figure 1. An embedding of degree 4 and rank 2.

Given a curve C, it is natural to ask the following question.

Question 1.1. Does C possess a map of degree d and rank r? If so, how many?

It is standard to recast this discussion in the equivalent language of divisors. A *divisor* on a curve C is a formal sum of points of C. The *degree* of a divisor $D = \sum_{p\in C} D(p) \cdot p$ is the integer $\deg D = \sum_{p\in C} D(p)$. A divisor D is called *effective* if its coefficients $D(p)$ are nonnegative for all $p \in C$. The connection between maps and divisors comes from hyperplane sections, like the linear section depicted in Figure 1. If $\varphi : C \to \mathbb{P}^r$ is a map of degree d, then the preimage of a hyperplane (with appropriate multiplicities) is an effective divisor of degree d. We say that two effective divisors D_1 and D_2 are *linearly equivalent* if there exists a map from C to a projective space such that both D_1 and D_2 are preimages of hyperplanes. Equivalently, D_1 and D_2 are linearly equivalent if there is a rational function on C with zeros equal to D_1 and poles equal to D_2. In this way, we see that equivalence classes of divisors on a curve C correspond in a natural way to maps from C to projective space.

The set of hyperplanes in $\mathbb{P}^r$ forms an r-dimensional space, and each hyperplane corresponds to an effective divisor in a single linear equivalence class. Thus, one can define the rank of a divisor D to be the dimension of its *complete linear series*, which is the set of effective divisors linearly equivalent to D. We can reinterpret the rank in the following way. If $\varphi(C)$ spans a projective space of dimension r, then there is a hyperplane passing through any r points of $\varphi(C)$. In other words, if D is the preimage of a general hyperplane, E is the sum of the r points, and E' is the sum of the other $d - r$ points of C mapping into the hyperplane, then D is linearly equivalent to $E + E'$. In particular, $D - E$ is linearly equivalent to an effective divisor. Thus, a divisor D has rank at least r if, for every effective divisor E of degree r, $D - E$ is equivalent to an effective divisor. If D is not equivalent to an effective divisor, we say that it has rank -1. The set of divisor classes on C is called the *Picard group* of C, denoted $\mathrm{Pic}(C)$, and the set of divisor classes of degree d is denoted $\mathrm{Pic}^d(C)$. We define

$$W^r_d(C) \coloneqq \{[D] \in \mathrm{Pic}^d(C) \mid \mathrm{rank}(D) \geq r\}.$$

The *Brill-Noether varieties* $W^r_d(C)$ are central to the study of algebraic curves. For example, Question 1.1 above can be rephrased as follows.

Question 1.2. For which values of r and d is $W^r_d(C)$ nonempty? In these cases, how "big" is it?

An early and important development in this theory was the Riemann-Roch Theorem. To understand the statement, we first need to define a few more terms. Any meromorphic 1-form on C determines a divisor class. This is called the *canonical class*, and typically denoted K_C. The *genus* of C is one more than the rank of its canonical class K_C.

Riemann-Roch Theorem. *Let C be a curve of genus g, and let D be a divisor on C. Then*

$$\mathrm{rank}(D) - \mathrm{rank}(K_C - D) = \deg D - g + 1.$$

The Riemann-Roch Theorem has many important consequences. The degree of the canonical divisor K_C is $2g-2$. Since every effective divisor has nonnegative degree, a divisor of negative degree must have rank -1. By Riemann-Roch, it follows that every divisor of degree $d > 2g - 2$ has rank $d - g$. On a fixed curve C, the rank of a divisor is therefore completely determined by its degree for all but finitely many possible degrees. Using Riemann-Roch, it is a straightforward exercise to compute the rank of every divisor on a curve of genus zero, one, or two. The first interesting case occurs in genus three—some curves of genus three or more possess a divisor of degree 2 and rank 1, and some do not. Those that do are are known as *hyperelliptic curves*. More generally, the *gonality* of a curve C is the minimum degree of a divisor of positive rank on C. Hyperelliptic curves are precisely the curves of gonality 2. The gonality of a curve is an important invariant that determines many interesting things about the geometry of the curve.

[1] *The astute reader will note that many lines do not appear to meet the curve at all, because Figure 1 depicts only the points with real coordinates. This is among the reasons that algebraic geometers prefer to work over algebraically closed fields, though we tend to draw pictures like Figure 1 anyway.*

2. The Chip Firing Game

The discussion above has an interesting combinatorial analogue. While the chip firing game has been studied for decades, the perspective of it as combinatorial Brill-Noether theory is relatively new. Throughout, we choose our terminology and notation to reflect the analogy with algebraic curves. All graphs are assumed to be finite, connected, and loopless, though possibly with multiedges. For more information on the chip firing game, we recommend the textbooks [CP18, Kli19].

Definition 2.1. A *divisor* D on a graph G is a formal $\mathbb{Z}$-linear combination of vertices of G,

$$D = \sum_{v \in V(G)} D(v) \cdot v$$

with $D(v) \in \mathbb{Z}$.

For example, Figure 2 depicts a divisor on the wedge of two triangles. In this and other examples, a vertex with coefficient zero is pictured as an unlabeled vertex.

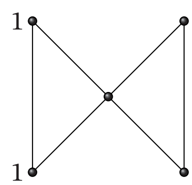

Figure 2. A divisor on a graph.

Divisors on graphs were studied in combinatorics, computer science, and dynamics long before they attracted the interest of algebraic geometers. In these disciplines it is more common to refer to divisors on graphs as *chip configurations* or *abelian sandpiles*. One can think of a divisor as a stack of poker chips on each vertex of the graph, hence the term "chip configuration." From this perspective, the vertices with negative coefficients are "in debt."

In the chip firing game, there is only one move. We choose a vertex to "fire," which results in that vertex giving a chip to each of its neighbors. More concretely, we have the following definition.

Definition 2.2. The *chip-firing move* at a vertex v takes a divisor D to D', where

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v, \\ D(w) + \text{\# of edges between } w \text{ and } v & \text{if } w \neq v, \end{cases}$$

where $\mathrm{val}(v)$ denotes the valence of the vertex v.

In our example, if we fire the top left vertex, we get the divisor pictured in Figure 3.

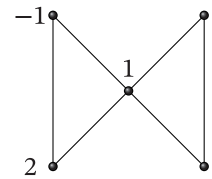

Figure 3. The result of firing a vertex.

Two divisors D, D' are *linearly equivalent* if D' can be obtained from D by a sequence of chip-firing moves. In analogy with divisors on curves, a divisor $D = \sum_{v \in V(G)} D(v) \cdot v$ is called *effective* if $D(v) \geq 0$ for all $v \in V(G)$, and the *degree* of D is the integer $D = \sum_{v \in V(G)} D(v)$. Note that the degree of a divisor is invariant under chip firing. The set of divisor classes on G is called the *Picard group* of G, denoted $\mathrm{Pic}(G)$, and the set of divisor classes of degree d is denoted $\mathrm{Pic}^d(G)$. The set $\mathrm{Pic}^d(G)$ is finite. Indeed, it is a consequence of Kirchoff's Matrix Tree Theorem that $|\mathrm{Pic}^d(G)|$ is equal to the number of spanning trees in G.

Given a divisor D on a graph, how should we define its rank? Its complete linear series is a finite set, so we cannot define the rank of D to be the dimension of this linear series. Our alternate definition of rank is more appropriate to the combinatorial setting. We say that a divisor D has *rank* at least r if, for every effective divisor E of degree r, $D - E$ is equivalent to an effective divisor. We may think of this as a sort of game—starting with a chip configuration D on a graph G, our opponent is allowed to "steal" r chips, and then our job is to perform a sequence of chip-firing moves to eliminate any possible debt. If we can always win the game no matter how our opponent plays, then the divisor D has rank at least r. As an example, we encourage the reader to check that the divisor pictured in Figure 2 has rank 1.

In analogy with Question 1.1, given a graph G, we may ask the following.

Question 2.3. Does G possess a divisor of degree d and rank r? If so, how many?

As an approach to Question 2.3, we can modify the game described above to search for low-degree divisors of positive rank. We are given a number of chips to place on the graph, and after we place them, our opponent is allowed to steal a chip, creating a debt of 1 at some vertex of the graph. If we can eliminate the debt via a sequence of chip-firing moves, no matter how our opponent plays, then we have found a divisor of positive rank. In analogy with algebraic curves, the minimum number of chips required to win this game is called the *gonality* of the graph. The wedge of two triangles pictured in Figure 2 has gonality 2—the pictured divisor has the smallest degree among all divisors of positive rank. In fact, as we shall see shortly, every graph with first Betti number 2 has gonality 2. In general, however, computing the gonality of a graph is NP-hard [GSvdW20].

There has been significant work on the gonality of graphs. There is a complete classification of hyperelliptic graphs [Cha13] and a partial classification of graphs of gonality 3 [ADM$^+$19]. The gonality of a graph is bounded below by its treewidth [vDdBG20], an invariant that has received a lot of attention in the combinatorics literature.

The *genus* of a graph G is its first Betti number, $g = |E(G)| - |V(G)| + 1$. The *canonical divisor* of a graph G is the divisor

$$K_G \coloneqq \sum_{v \in V(G)} (\mathrm{val}(v) - 2) \cdot v.$$

A simple double-counting argument shows that the degree of K_G is $2g - 2$. Amazingly, the Riemann-Roch Theorem also holds in this setting.

Riemann-Roch Theorem for graphs ([BN07]). *Let G be a graph of genus g, and let D be a divisor on G. Then*

$$\mathrm{rank}(D) - \mathrm{rank}(K_G - D) = \deg D - g + 1.$$

Neither the Riemann-Roch Theorem for curves nor the Riemann-Roch Theorem for graphs is known to imply the other. Nevertheless, in this setting the Riemann-Roch Theorem has all the same consequences as it does for divisors on curves. For example, Riemann-Roch implies that the rank of the canonical divisor K_G is $g - 1$. Given a chip configuration D with some vertices in debt, we may ask whether it is possible to perform a sequence of chip-firing moves to eliminate the debt. This is equivalent to asking whether the divisor D has nonnegative rank. By Riemann-Roch, if $\deg D \geq g$, then $\mathrm{rank}(D) \geq 0$. In other words, if the total number of chips is at least the genus of the graph, then we can always eliminate the debt.

3. What's the Connection?

The preceding discussion suggests that there is some deep connection between the theory of divisors on graphs and that of divisors on algebraic curves. The main idea behind this connection is that of *degeneration*. While degeneration arguments have been standard in algebraic geometry for more than a century, many early-to-mid 20th century developments were required in order to give such arguments a rigorous foundation. Given a curve, we perturb it in an extreme and violent fashion, and then study this "degenerate curve." It is often possible to discern geometric properties of the original curve from the degenerate one. Greater detail on the results of this section can be found in [Bak08] or the survey [BJ16].

For example, let $F(x, y, z)$ be a homogeneous polynomial of degree 4, and consider the following family of curves, parameterized by t:

$$C_t = \{(x, y, z) \in \mathbb{P}^2 \mid t \cdot F(x, y, z) + xyz(x + y + z) = 0\}.$$

For almost all values of t, the curve $C_t \subset \mathbb{P}^2$ is a smooth plane quartic, such as the one illustrated in Figure 1. When $t = 0$, however, the family of curves degenerates to a singular union of lines, pictured in Figure 4.

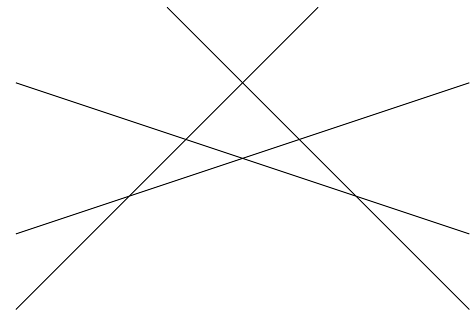

Figure 4. The curve C_0 is a union of four lines.

The strength of this approach is that the degenerate curve can be understood via combinatorics. The irreducible components of C_0 are just lines, and have no interesting geometry. The geometry of C_0 is completely determined by the combinatorial data of which pairs of lines meet. This information is recorded by the *dual graph* G of the curve C_0. The vertices of the dual graph G correspond to components of C_0, and there is an edge between two vertices for every point in the intersection of the corresponding components. In this example, the dual graph has four vertices, corresponding to the four lines, and each pair of lines intersect in exactly one point, so the dual graph has one edge between every pair of vertices. The dual graph is therefore the complete graph K_4, pictured in Figure 5.

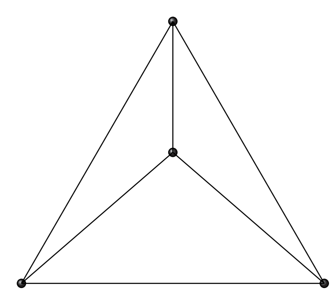

Figure 5. The dual graph of C_0 is the complete graph K_4.

More generally, we may consider any one-parameter family of curves C_t. If the *special fiber* C_0 is a union of lines (or more generally, rational curves), no three of which mutually intersect and no two of which intersect nontransversally, then one can construct the dual graph G of C_0 as above. In order to discern properties of the curves C_t from analogous properties of the graph G, the family must satisfy certain technical hypotheses. For the technicality-inclined reader, we let C be a smooth curve over an arbitrary discretely valued field K, and we let R be the corresponding discrete valuation ring. (In our example above, we have $K = \mathbb{C}((t))$ and $R = \mathbb{C}[[t]]$.) A curve $\mathcal{C}$ over R is called a *strongly semistable model* for C if $\mathcal{C}$ is regular, it is proper and flat over R, its general fiber is C, and its special fiber C_0 is reduced and has only ordinary double points as singularities. The curve C is the generic member of this family—if certain properties hold for C, then they hold for C_t for all t in a dense open set.

By moving from the curve C to the special fiber C_0, and then to the dual graph G, we lose an extraordinary amount of information. For example, the isomorphism class of a plane quartic depends on several continuous parameters, whereas a graph is determined by discrete data. It is remarkable that we could learn anything about our original curve by studying the graph. As we shall see, however, the graph retains substantial information about divisors on C.

Now, consider a 1-parameter family of points on our 1-parameter family of curves C_t. The family of points yields a section as in Figure 6. Our technical hypotheses are restrictive enough to imply that any such section meets the special fiber C_0 in a smooth point. (More precisely, if $\mathcal{C}$ is a strongly semistable model for C, then every K-point on C specializes to a smooth point of C_0.) Since every smooth point of C_0 lies on a unique component, and the components of C_0 correspond to vertices of the dual graph G, we obtain a well-defined map from $C(K)$ to $V(G)$. Extending linearly, we obtain a map from the group of K-divisors on the curve C to the group of divisors on the graph G. This map preserves linear equivalence, so it descends to a map

$$\mathrm{Trop} : \mathrm{Pic}_K(C) \to \mathrm{Pic}(G).$$

Note that this map preserves the degree. It is natural to ask how our other invariant, the rank, behaves under this map. This question has a simple answer.

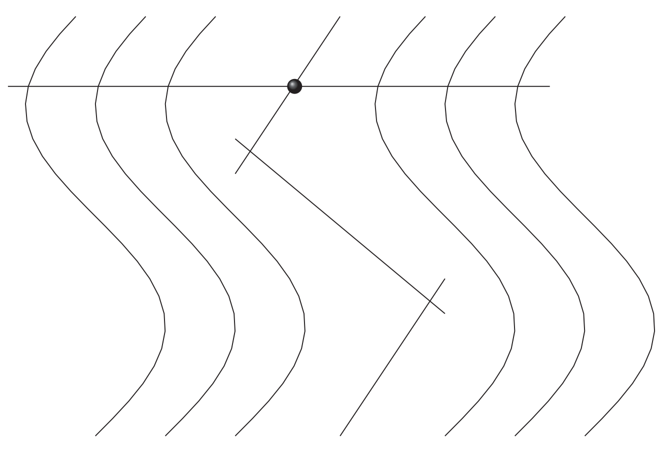

Figure 6. A section of a 1-parameter family of curves.

Baker's Specialization Lemma ([Bak08]). *Let D be a K-divisor on C. Then*

$$\mathrm{rank}(D) \leq \mathrm{rank}(\mathrm{Trop}\, D).$$

Returning to our example, consider the divisor D pictured in Figure 1, given by the intersection of the plane quartic C with a line. We saw above that this divisor has degree 4 and rank 2. Assume that the line is in sufficiently general position, so that it intersects each component of C_0 transversally. Fixing the line and letting C degenerate to C_0, the intersection becomes a union of four points, one on each of the four lines. The divisor D therefore specializes to the divisor on K_4 with one chip on each of the four vertices. By Baker's Specialization Lemma, Trop D must have rank at least 2. Indeed, Trop D is the canonical divisor on K_4, and since K_4 has genus 3, the canonical divisor has rank 2 by Riemann-Roch.

In a different direction, we encourage the reader to check that the complete graph K_4 is not hyperelliptic. In other words, no divisor of degree 2 on K_4 has positive rank. By Baker's Specialization Lemma, if the curve C is hyperelliptic, then so is the graph K_4, because the divisor of degree 2 and rank at least 1 on C must specialize to a divisor of degree 2 and rank at least 1 on the graph. It follows that the curve C is not hyperelliptic. In this way, we see that the geometry of the curve is reflected by the combinatorics of the graph, and vice versa.

A word of caution is in order here. The argument just given shows only that there is no divisor of degree 2 and rank 1 on C that is defined over the field K. It is theoretically possible that such a divisor exists over some finite extension of K. This is an issue for algebraic geometers, who often work over fields that are algebraically closed. One way to handle this problem is via *refinement*. Given a graph G and a positive integer e, define $\frac{1}{e}G$ to be the eth *refinement* of G, obtained by subdividing each edge of G into e edges, as pictured in Figure 7. If K' is a valued field extension of K of degree e, then there exists a strongly semistable model for $C_{K'}$ such that the dual graph of the special fiber is the eth refinement of the original dual graph. To complete the proof that C is not hyperelliptic over any finite extension, therefore, it suffices to show that the graph $\frac{1}{e}K_4$ is not hyperelliptic for every positive integer e. We again leave this to the reader.

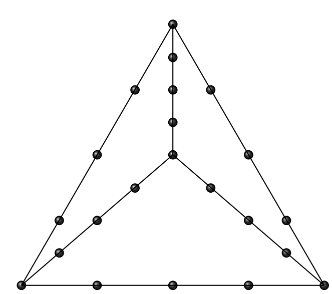

Figure 7. The graph $\frac{1}{4}K_4$.

A different approach to the problem is to use *metric graphs*, also known as *tropical curves*. While this perspective is somewhat beyond the scope of the present article, it is the one more commonly used in applications, such as those discussed in the next section. A metric graph is a metric space obtained from a graph by identifying each edge with an interval of some positive length. Given a (discrete) graph G, consider the metric graph Γ where all edges have length 1. In Γ, the points of rational distance from the vertices can be naturally identified with vertices of refinements of G. In this way, metric graphs can be thought of as the limits of discrete graphs under refinement.

4. Going Further

We now return to Questions 1.2 and 2.3: on a curve C or graph G, for which values of r and d does there exist a divisor of degree d and rank r? In these cases, what is the dimension of $W^r_d(C)$? The answers to these questions certainly depend on the curve or graph. As we have seen, for example, some curves of genus 3 are hyperelliptic, and some are not. In other words, there exists a curve C of genus 3 such that $W^1_2(C)$ is empty, and there also exists a curve C' of genus 3 such that $W^1_2(C')$ is nonempty. One might ask which situation is more typical—if one were to select a genus 3 curve "at random," is that curve more likely to be hyperelliptic or nonhyperelliptic? More generally, what is the expected gonality of a curve of genus g? Note that one can ask the same question about a graph: what is the expected gonality of a random graph? (The answer depends on what one means by a random graph; see [DJKM16, DJ18].)

More precisely, there is a *moduli space* $\mathcal{M}_g$ parameterizing isomorphism classes of curves of genus g. When we say that a property is satisfied by a general curve of genus g, we mean that the property is satisfied by all curves in a dense open subset of the moduli space $\mathcal{M}_g$. In the particular example above, the moduli space $\mathcal{M}_3$ has dimension 6, and the space of hyperelliptic curves is a subvariety of dimension 5. In this sense, almost all curves of genus 3 are nonhyperelliptic. More generally, the answer to Question 1.2 for general curves is given by the celebrated Brill-Noether Theorem of Griffiths and Harris.

Brill-Noether Theorem. *Let C be a general curve of genus g, and define*

$$\rho(g, r, d) := g - (r + 1)(g - d + r).$$

1. *If $\rho(g, r, d) < 0$, then $W^r_d(C)$ is empty.*
2. *If $\rho(g, r, d) \geq 0$, then $\dim W^r_d(C) = \rho(g, r, d)$.*

Chip firing allows us to translate questions about the geometry of general curves into combinatorial problems about graphs. In [CDPR12], an alternate proof of the Brill-Noether Theorem is given, using chip firing. To see how this works, let us briefly return to the case of hyperelliptic curves in $\mathcal{M}_3$. The set of nonhyperelliptic curves is open in $\mathcal{M}_3$, and is therefore dense in $\mathcal{M}_3$ if and only if it is nonempty. To show that the general curve of genus 3 is nonhyperelliptic, it therefore suffices to find one nonhyperelliptic curve of genus 3. By Baker's Specialization Lemma, it's actually enough to find a nonhyperelliptic *graph* of genus 3. Indeed, we found such a graph in the previous section, and used it to show that the curve C from Figure 1 is itself nonhyperelliptic.

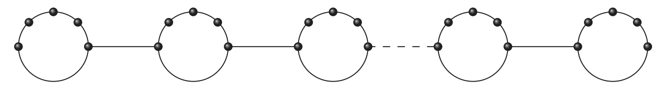

Figure 8. The chain of loops.

More generally, to prove the first statement above, fix integers g, r, and d such that $\rho(g, r, d)$ is negative. By Baker's Specialization Lemma, it suffices to exhibit a family of graphs of genus g, closed under refinement, that possess no divisor of degree d and rank at least r. This property is satisfied by the *chain of loops* pictured in Figure 8, provided that the ratio of the number of edges on the top of each loop to the number on the bottom is sufficiently large. More precisely, [CDPR12] provides a complete answer to Question 2.3 when the graph G is a sufficiently general chain of loops. It remains an open problem to find other infinite families of graphs with no divisors of degree d and rank r, when $\rho(g, r, d) < 0$.

With a little more work, one can also use this graph to obtain the second part of the Brill-Noether Theorem. Indeed, this graph has proven to be a remarkably useful tool for studying the Brill-Noether theory of general curves. It can also be used to study the tangent space to $W^r_d(C)$ [JP14] and the Hilbert functions of divisor classes on general curves [JP16]. If we relax the condition that the ratio of the number of edges on the top of each loop to the number on the bottom is sufficiently large, then we can use the graph to study general curves in the *Hurwitz space* $\mathcal{H}_{g,k}$, parameterizing curves of genus g and gonality k [Pfl17, JR21, CJ20]. The chain of loops has also been used in [LU21, CLRW21] to study the Brill-Noether theory of general Prym curves.

These results demonstrate the potential for using graph theory to establish results in algebraic geometry. But it is also possible to go in the other direction, using algebraic geometry to learn more about graphs. For example, in a related (and historically earlier) direction, it was shown independently by Kempf and Kleiman-Laksov that if C is *any* curve of genus g, then

$$\dim W^r_d(C) \geq \rho(g, r, d).$$

This has the following, purely combinatorial, consequence.

Theorem 4.1 ([Bak08]). *Let G be a graph of genus g, and fix integers r and d such that $\rho(g, r, d) \geq 0$. Then there exists a* *positive integer e such that the refinement $\frac{1}{e}G$ possesses a divisor of degree d and rank at least r.*

Although Theorem 4.1 is a completely combinatorial statement, there is currently no known proof of this theorem that does not use algebraic geometry. A purely combinatorial proof has thus far remained elusive. It is also unknown whether Theorem 4.1 holds without passing to a refinement—that is, whether the integer e can always be taken to be 1. This is known as the Brill-Noether Existence Problem, and it has been solved only in small genus cases.

Many open questions remain concerning the geometry of general curves. Chip firing provides a promising approach to these problems using combinatorial tools. At the same time, ideas from geometry can be applied to conjectures in graph theory. By developing connections between two diverse areas of mathematics, we heighten our understanding of each.

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