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Stability of 2nd Hilbert Points of Canonical Curves

Maksym Fedorchuk[1] **and David Jensen**[2]

[1]Department of Mathematics, Boston College, Chestnut Hill, MA 02467, USA and [2]Department of Mathematics, SUNY Stony Brook, Stony Brook, NY 11794, USA

Correspondence to be sent to: maksym.fedorchuk@bc.edu

We establish Geometric Invariant Theory (GIT) semistability of the 2nd Hilbert point of every Gieseker–Petri general canonical curve by a simple geometric argument. As a consequence, we obtain an upper bound on slopes of general families of Gorenstein curves. We also explore the question of what replaces hyperelliptic curves in GIT quotients of the Hilbert scheme of canonical curves.

1 Introduction

The log minimal model program for the moduli space of stable curves, also known as the *Hassett–Keel program*, offers a promising approach to understanding the birational geometry of $\bar{M}_g$. The goal of this program is to find a functorial interpretation of the log canonical models

$$\bar{M}_g(\alpha) = \mathrm{Proj} \bigoplus_{m\geq 0} \mathrm{H}^0(\bar{\mathcal{M}}_g, \lfloor m(K_{\bar{\mathcal{M}}_g} + \alpha\delta)\rfloor).$$

Such an interpretation could then be used to study properties of rational contractions $\bar{M}_g \dashrightarrow \bar{M}_g(\alpha)$ and to obtain structural results about effective divisors on $\bar{M}_g$, in particular, the Mori chamber decomposition of the effective cone.

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Hassett and Hyeon proved that the first two log canonical models of $\bar{M}_g$ are Geometric Invariant Theory (GIT) quotients of asymptotically linearized Hilbert schemes of tricanonical and bicanonical curves [14, 15]. It is widely expected that further progress in the Hassett–Keel program will require GIT stability analysis of finite (i.e., nonasymptotic) Hilbert points of bicanonical and canonical curves; see [3, 12, 19, 20]. The case of canonical curves is of particular interest because it should lead to birational contractions of $\bar{M}_g$ affecting the interior M_g.

Only recently it was shown that finite Hilbert points of general canonical curves are semistable in all genera [2]. Still, the question of *which* smooth canonical curves have (semi)stable mth Hilbert points for a given m is widely open. Here we make partial progress toward answering this question. Our main result is a geometric proof of semistability of the 2nd Hilbert point of a general canonical curve, which gives a sufficient condition for semistability, something that the previous results lack.

Theorem 1.1. Let C be a Gieseker–Petri general smooth curve of genus $g \geq 4$. Then its canonical embedding $C \hookrightarrow \mathbb{P}\mathrm{H}^0(C, K_C)$ has semistable 2nd Hilbert point. □

This result strengthens and complements the results of [2] in the case of 2nd Hilbert points of canonical curves. Not only do we show that the GIT quotient of the variety of 2nd Hilbert points of canonical curves is nonempty, but also that this GIT quotient parameterizes all curves whose linear systems behave generically. Assuming the expected *stability* of the general canonical curve, this GIT quotient is an interesting projective birational model of $\bar{M}_g$:

If G is the quotient in question, then the map $f : \bar{M}_g \dashrightarrow G$ is not a local isomorphism along the locus of curves of low Clifford index. As we show in this paper, f is not regular along the hyperelliptic locus $\bar{H}_g \subset \bar{M}_g$ (see Section 4). In addition, f is not regular along the locus $\mathrm{Trig}_g(+)$ of trigonal curves with positive Maroni invariant and contracts the locus $\mathrm{Trig}_g(0)$ of trigonal curves with Maroni invariant 0 to a point (this locus is nonempty only for even g); see Corollary 3.2. We also observe that f is not regular along the bielliptic locus for $g \geq 7$. Finally, when $g = 6$, the rational map f contracts both the bielliptic locus (see Proposition 5.3) and the locus of plane quintics (see Corollary 3.5).

In addition to studying the indeterminacy locus of the map $f : \bar{M}_g \dashrightarrow G$, we also examine the indeterminacy locus of its inverse $f^{-1} : G \dashrightarrow \bar{M}_g$. To this end, we show that G parameterizes many different types of singular curves, a large number of which are enumerated in Theorem 3.3. Each of these singular curves is predicted to play a role in a functorial interpretation of $\bar{M}_g(\alpha)$; see [1] for precise predictions. As a consequence of our analysis, we discover a class of curves, the A_{2g}-*rational curves*, which lie in the total transform under f of the hyperelliptic locus $\bar{H}_g$.

Finally, we include an important application of our semistability result, providing an upper bound on slopes of one-parameter families of Gorenstein curves with a sufficiently general generic fiber.

Theorem 1.2. Let B be a complete curve. Consider a flat and proper family $\mathcal{C} \to B$ of Gorenstein curves with a relatively ample dualizing sheaf. Suppose that the generic fiber is a canonically embedded curve with semistable 2nd Hilbert point. Then the degree λ of the Hodge bundle and the degree δ of the discriminant divisor satisfy the inequality

$$\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}. \tag{1.1}$$

□

This theorem is an extension of a celebrated result of Cornalba and Harris [8], also independently obtained by Xiao [24], saying that the slope of any generically smooth family of Deligne–Mumford stable curves of genus g is at most $8 + 4/g$. In the case of trigonal fibrations, a result analogous to that of Theorem 1.2 was obtained by Barja and Stoppino [6].

We prove Theorem 1.2 in Section 5, where we explain the assumptions and give precise definitions of λ and δ. We note, in particular, that the condition that the generic fiber is canonically embedded and has GIT semistable 2nd Hilbert point implies that it is neither hyperelliptic nor trigonal with positive Maroni invariant. It has long been expected that lower bounds on the slope of a family of curves should depend on the Clifford index and other geometric properties of the generic fiber; see, for example [5, 17, 21].

We work over the field of complex numbers $\mathbb{C}$.

2 Semistability of 2nd Hilbert Points

We briefly recall the necessary definitions. Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a canonically embedded smooth curve of genus $g \geq 4$. Using Max Noether's theorem on projective normality of canonical curves [4, p. 117], we define the 2nd Hilbert point of $C \hookrightarrow \mathbb{P}^{g-1}$ to be the quotient

$$[\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2)) \to \mathrm{H}^0(C, \mathcal{O}_C(2)) \to 0] \in \mathrm{Grass}\left(3g-3, \binom{g+1}{2}\right).$$

We denote by $\overline{\mathrm{Hilb}}_g^2$ the closure of the locus of 2nd Hilbert points of canonically embedded curves in the Grassmannian $\mathrm{Grass}(3g-3, \binom{g+1}{2})$ and endow $\overline{\mathrm{Hilb}}_g^2$ with the linearization $\mathcal{O}(1)$ coming from the Plücker embedding of the Grassmannian into $\mathbb{P}\bigwedge^{3g-3}\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$. Finally, we set

$$G := \overline{\mathrm{Hilb}}_g^{2,ss}//\mathrm{SL}(g) = \mathrm{Proj}\bigoplus_{m\geq 0}\mathrm{H}^0(\overline{\mathrm{Hilb}}_g^2, \mathcal{O}(m))^{\mathrm{SL}(g)}$$

to be the resulting GIT quotient.

One reason that this construction is of particular interest is that the map

$$f : \bar{M}_g \dashrightarrow G$$

is not an isomorphism on the interior $M_g \subset \bar{M}_g$. More precisely, we show that curves of Clifford indices 0 and 1 are outside of the locus where this map is locally an isomorphism. We consider hyperelliptic, trigonal, and bielliptic curves in the later sections of the paper.

We proceed to state the main result of our paper in its greatest generality and to record its most important corollaries.

Theorem 2.1. A canonically embedded curve not lying on a quadric of rank 3 or less has semistable 2nd Hilbert point. □

Proof. Our key geometric tool is the $\mathrm{SL}(g)$-invariant effective divisor $\mathcal{D} \subset \mathrm{Grass}(3g-3, \binom{g+1}{2})$ defined as the locus of $(3g-3)$-dimensional quotients of $\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$ whose kernel contains a quadric of rank at most 3. The fact that $\mathcal{D}$ is a divisor follows directly from the fact that the locus of quadrics of rank at most 3 has dimension $3g-3$ in $\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$. Since $\mathrm{Grass}(3g-3, \binom{g+1}{2})$ is smooth and has Picard number 1, the divisor $\mathcal{D}$ is defined by a global section of some power of $\mathcal{O}(1)$. Since $\mathrm{SL}(g)$ has no nontrivial characters, this section is $\mathrm{SL}(g)$-invariant. It follows that any curve whose Hilbert point is not contained in $\mathcal{D}$ is semistable. ■

Recall that a complete smooth curve C is said to be *Gieseker–Petri general* if it satisfies the Petri condition that

$$\mu : \mathrm{H}^0(C, L) \otimes \mathrm{H}^0(C, K_C - L) \to \mathrm{H}^0(C, K_C)$$

is injective for all $L \in \mathrm{Pic}(C)$. That a general curve in M_g is Gieseker–Petri general was proved by Gieseker [13], as well as Eisenbud and Harris [10] using degeneration arguments.

Lemma 2.2. The canonical embedding of a Gieseker–Petri general curve does not lie on a rank 3 quadric. □

Proof. Suppose a canonically embedded curve lies on a quadric of rank 3 whose vertex is a linear space Λ of dimension $g-3$. The projection away from Λ maps C onto a conic $R \simeq \mathbb{P}^1$ in $\mathbb{P}^2$. It follows that there is a decomposition $K_C = 2L + B$. Here, B is an effective divisor with $\mathrm{Supp}(B) = \Lambda \cap C$, and L is a pullback of $\mathcal{O}(1)$ from R. In particular, we have $h^0(C, L) \geq 2$. Let s_0 and s_1 be two distinct nonzero global sections of L, and s_0' and s_1' be the same rational functions considered now as sections of $L + B$. Then

$$\mu(s_0 \otimes s_1' - s_1 \otimes s_0') = 0,$$

violating the Petri condition. ■

Proof of Theorem 1.1: Theorem 1.1 follows from Theorem 2.1 using Lemma 2.2. ■

3 Degenerations to Rational Normal Surface Scrolls

Aside from canonical curves, there is another variety of interest in $\mathbb{P}^{g-1}$ with ideal generated by $\binom{g-2}{2}$ quadrics, namely a rational normal surface scroll. Recall that for nonnegative integers a and b satisfying $a + b = g - 2$, a rational normal surface scroll $S_{a,b} \subset \mathbb{P}^{g-1}$ is the join of two rational normal curves of degrees a and b whose linear spans do not intersect.

Rational normal surface scrolls are of particular interest to us because the linear system of quadrics containing a smooth trigonal canonical curve $C \subset \mathbb{P}^{g-1}$ cuts out precisely such a surface. Namely, by the geometric Riemann–Roch the g^1_3's on C are collinear in $\mathbb{P}^{g-1}$, and the resulting lines sweep out a rational normal surface scroll $S_{a,b}$. The difference $|a-b|$ is classically known as the *Maroni invariant* of C. An important fact is that the ideal of the rational normal surface containing C is generated by the quadrics containing C [4]. It follows that the 2nd Hilbert point of a smooth trigonal canonical curve $C \subset \mathbb{P}^{g-1}$ coincides with the 2nd Hilbert point of the rational normal scroll containing it.

In this section, we show that a rational normal surface scroll $S_{a,b}$ is semistable if and only if $a=b$, that is, if it is a $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by the complete linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, a)|$ in $\mathbb{P}^{2a+1}$.

Proposition 3.1. A rational normal surface $S_{a,b}$ has semistable 2nd Hilbert point if and only if $a=b$. □

Proof. The scroll $S_{a,a}$ in $\mathbb{P}^{2a-1}$ is the image of the homogeneous space $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the complete linear system $|\mathcal{O}(a-1, 1)|$. The fact that its mth Hilbert point is semistable now follows from Kempf's stability results [16, Corollary 5.3].

Suppose now that $a \neq b$. To see that the scroll $S_{a,b}$ is nonsemistable, recall that the ideal of $S_{a,b}$ is generated by the determinants of the 2×2 minors of the following matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{a-1} & y_0 & y_1 & \cdots & y_{b-1} \\ x_1 & x_2 & \cdots & x_a & y_1 & y_2 & \cdots & y_b \end{array}\right).$$

We consider the one-parameter subgroup ρ of $\mathrm{Aut}(S_{a,b}) \subset \mathrm{SL}(g)$ acting with weight $-(b+1)$ on x_i's and weight $a+1$ on y_i's. The ideal of $S_{a,b}$ becomes homogeneous with respect to ρ. It follows that every monomial basis of $\mathrm{H}^0(S_{a,b}, \mathcal{O}(m))$ has the same ρ-weight. For $m=2$, we compute that the ρ-weight of $\mathrm{H}^0(S_{a,b}, \mathcal{O}(2))$ is

$$2(b+1)\binom{a}{2} - 2(a+1)\binom{b}{2} - ab(a-b) = (a-b)(a+b-1) \neq 0.$$

Since the ρ-weight is nonzero, we conclude that $S_{a,b}$ is nonsemistable. ■

As a corollary, we obtain the following two results.

Corollary 3.2. Trigonal curves with positive Maroni invariant have nonsemistable 2nd Hilbert points. Trigonal curves with Maroni invariant 0 are strictly semistable and are identified in $G = \overline{\mathrm{Hilb}}_g^{2,ss}//\mathrm{SL}(g)$ with the point corresponding to the balanced rational normal surface scroll in $\mathbb{P}^{g-1}$. □

Proof. This follows immediately from Proposition 3.1 and the fact that the 2nd Hilbert point of a trigonal curve of genus g and Maroni invariant r coincides with the 2nd Hilbert point of the scroll $S_{(g+r)/2-1,(g-r)/2-1}$. ■

We note that nonsemistability of trigonal curves with a positive Maroni invariant reflects the fact that the locus in $\bar{M}_g$ of trigonal curves contained in an unbalanced scroll is covered by families of slope strictly greater than $7+\frac{6}{g}$; in particular, when g is odd, Trig_g is covered by families of slope $7+\frac{20}{3g+1}$ [9].

The second corollary of Proposition 3.1 shows that $G=\overline{\mathrm{Hilb}}_g^{2,ss}//\mathrm{SL}(g)$ parameterizes curves with numerous singularities, as predicted by Alper et al. [1].

Theorem 3.3. Suppose $g\geq 6$ is even. There exist nontrigonal canonical curves of genus g with semistable 2nd Hilbert point and possessing the following classes of singularities:

(a) All A_n singularities with $n\leq 2g+1$.
(b) All D_n singularities with $n\leq 2g$.
(c) If $g=6m-2$, the singularity $y^3=x^{g+2}$ and its deformations.
(d) If $g=6m-4$, the singularity $y^3=x^{g+1}$ and its deformations.
(e) If $g=6m$, the singularity $y^3=x^{g+1}$ and its deformations. □

Proof. Let $g=2k$. We begin by constructing a curve C, with a desired singularity $p\in C$, in the class $(3,k+1)$ on $\mathbb{P}^1\times\mathbb{P}^1$. Next, we embed C via the restriction of the complete linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,k-1)|$, which is evidently a canonical linear system on C. The 2nd Hilbert point of the canonical embedding of C will then be the 2nd Hilbert point of the balanced normal scroll $S_{k-1,k-1}$, hence semistable by Proposition 3.1. We can then deform C out of the scroll preserving singularities of C and the semistability of its 2nd Hilbert point.

Construction of the singular curve (C,p) *on the scroll*:

(a) Consider a smooth rational curve C_1 in the class $(2,1)$ on $\mathbb{P}^1\times\mathbb{P}^1$. Since $h^0(\mathcal{O}_{C_1}(-1,k-1))=h^1(\mathcal{O}_{C_1}(-1,k-1))=0$, the restriction map $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,k)|\to|\mathcal{O}_{C_1}(2k+1)|$ is bijective. It follows that for every $p\in C_1$, there exists a unique divisor $C_2\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,k)|$ such that $(C_1\cdot C_2)_p=(2k+1)$. Evidently, such a divisor is smooth if p is not a ramification point of the projection $C_1\to\mathbb{P}^1$ onto the second factor.

It follows that for the general point $p\in C_1$, there is a smooth rational curve $C_2\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,k)|$ such that C_1 and C_2 are maximally tangent at p. Namely, we have $(C_1\cdot C_2)_p=(2k+1)$. It follows that $C:=C_1\cup C_2$ is a curve of class $(3,k+1)$ on $\mathbb{P}^1\times\mathbb{P}^1$ with a unique singularity of type A_{2g+1}. The complete linear system $|\mathcal{O}(1,k-1)|$ embeds $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^{g-1}$, mapping C_1 and C_2 to rational normal curves, meeting in a singularity of type A_{2g+1} at p. Thus, the image of C under this embedding is an A_{2g+1}-rational curve of Definition 4.1.

(b) A curve with a D_{2g} singularity is obtained by taking a nodal curve C_1 of class $(2, 2)$ and a curve C_2 of class $(1, k-1)$ that is tangent with multiplicity $2k-1$ to one of the branches at the node of C_1.

(c) If $k = 3m-1$, then we take three rational curves in the class $(1, m)$, all meeting at a single point where they pairwise intersect with multiplicity $2m$. The resulting singularity is analytically isomorphic to $y^3 = x^{g+2}$.

(d) This part may be proved analogously to Part (c). Specifically, when $k = 3m-2$, we may take three maximally tangent rational curves in the classes $(1, m)$, $(1, m)$, and $(1, m-1)$. The resulting singularity is analytically isomorphic to $y^3 = x^{g+1}$.

(e) We exhibit an explicit curve in the class $(3, 3m+1)$ with singularity analytically isomorphic to $y^3 = x^{6m+1}$. Namely, consider

$$(y - x^m)^3 - x^{3m+1} y^3 = 0. \tag{3.1}$$

This curve has a rational parameterization $x = t^3$, $y = t^{3m}/(1 - t^{3m+1})$. Evidently, under this parameterization $x = t^3$ and $y - x^m = t^{6m+1} + \cdots$. The claim follows.

Having established the existence of a curve C, with a desired singularity $p \in C$, in the class $(3, k+1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, we must now show that there exists a nontrigonal canonical curve with the same singularity and semistable 2nd Hilbert point. To do this, we observe that a general equisingular deformation of C in $\mathbb{P}^{g-1}$ is nontrigonal. More precisely, the deformations of C as a subscheme of $\mathbb{P}^{g-1}$ and the deformations of C as a $(3, k+1)$ divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ both surject smoothly onto the deformation space of the singularity $p \in C$. Since the dimension of the Hilbert scheme of canonical curves is $(3g-3) + (g^2 - 2g)$, the dimension of the $\mathrm{SL}(g)$-orbit of the scroll is $g^2 - 2g - 6$, and the dimension of the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, k+1)|$ is $2g+7$, we conclude that the general equisingular deformation of (C, p) does not lie on the scroll if and only if $3g - 3 > 2g + 1$, or $g > 4$. This concludes the proof. ■

In the specific case of $g = 6$, there is another surface of interest—the Veronese surface: If $C \subset \mathbb{P}^5$ is a smooth canonical curve of genus 6 that admits a g^2_5, then any five points in a g^2_5 are coplanar by the geometric Riemann–Roch. It follows that each of the quadrics containing C also contains the conic through these five points. The resulting two-dimensional family of conics sweeps out the Veronese surface in $\mathbb{P}^5$. Moreover, the ideal of the Veronese surface is generated by the quadrics containing C.

Proposition 3.4. The Veronese surface in $\mathbb{P}^5$ has semistable 2nd Hilbert point. □

Proof. This also follows immediately from [16, Corollary 5.3], as the Veronese surface is simply $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ via the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. ■

Corollary 3.5. A canonically embedded plane quintic has semistable 2nd Hilbert point, coinciding with the 2nd Hilbert point of a Veronese surface in $\mathbb{P}^5$. □

4 An Answer to the Riddle

What is the limit of the canonical model of a smooth curve as it degenerates to a hyperelliptic curve? This is the question that opens a well-known paper of Bayer and Eisenbud [7]. In this section, we aim to show that their answer—a ribbon—is only part of the story. In fact, there is a larger class of curves, the A_{2g}-rational curves, that give a canonical answer to this question, at least from the point of view of GIT for canonical curves.

Definition 4.1. A complete connected reduced curve of genus g with a unique singularity of type A_{2g} ($y^2 = x^{2g+1}$) is called an *A_{2g}-rational curve*. A complete connected reduced curve of genus g with a unique singularity of type A_{2g+1} ($y^2 = x^{2g+2}$) is called an *A_{2g+1}-rational curve*. □

Note that the genera of the singularities A_{2g+1} and A_{2g} both equal g. Therefore, an A_{2g}-rational curve is necessarily irreducible and its normalization is isomorphic to $\mathbb{P}^1$. Similarly, an A_{2g+1}-rational curve necessarily has two irreducible components, each isomorphic to $\mathbb{P}^1$. We will denote an A_{2g+1}-rational curve C with the singularity $\hat{\mathcal{O}}_{C,p} \simeq \mathbb{C}[[x, y]]/(y^2 - x^{2g+2})$ by (C, p).

Isomorphism classes of A_{2g+1}-rational curves with a fixed pointed normalization are in bijection with closed points of $\mathbb{C}^* \times \mathbb{C}^{g-1}$. Indeed, let the pointed normalization of an A_{2g+1}-rational curve be a disjoint union of two-pointed rational curves $(\mathbb{P}^1, p_1)$ and $(\mathbb{P}^1, p_2)$, where the uniformizer at p_1 is x and at p_2 is y. Then the isomorphism class of a (parameterized) A_{2g+1}-curve is specified by a gluing datum $y \mapsto a_1 x + \cdots + a_g x^g$, where $a_1 \neq 0$, that defines an isomorphism

$$\mathbb{C}[y]/(y^{g+1}) \to \mathbb{C}[x]/(x^{g+1})$$

along which the two length $g+1$ subschemes supported at p_1 and p_2, respectively, are glued. We call $(a_1, a_2, \dots, a_g) \in \mathbb{C}^* \times \mathbb{C}^{g-1}$ the *crimping*, and refer the reader to [23] for a systematic treatment of crimping for singular curves.

Suppose that C is an A_{2g+1}-rational curve given by the gluing datum $y \mapsto a_1 x + \cdots + a_g x^g$. Since C is a local complete intersection curve, it admits a dualizing line bundle ω_C. While there are numerous ways to obtain a handle on this line bundle, we will only consider the one that, to us, is the most explicit. Namely, we use the defining property which says that ω_C is the unique line bundle that restricts to $\mathcal{O}(g-1)$ on each irreducible rational component and has g global sections. It follows that we can identify K_C with the triple $(\mathcal{O}(g-1), \mathcal{O}(g-1), \kappa_C)$ where

$$\kappa_C = 1 + k_1 x + \cdots + k_g x^g \in (\mathbb{C}[x]/(x^{g+1}))^*$$

is a gluing datum for a line bundle on C. Thus, the determination of ω_C reduces to computing κ_C.

Proposition 4.2. The canonical line bundle ω_C is defined by

$$\kappa_C = 1 + k_1 x + \cdots + k_{g-1} x^{g-1} + k_g x^g,$$

where $k_g = 0$ and k_i, $1 \le i \le g-1$, are (uniquely determined) polynomials in $a_1, (a_1)^{-1}, a_2, \ldots, a_g$. □

Proof. Since $\omega_C|_{C_1} \simeq \mathcal{O}(g-1)$ has exactly g global sections $1, y, \ldots, y^{g-1}$, all of them have to lift to global sections of ω_C. This means that each of the elements $\kappa_C, \kappa_C y, \ldots, \kappa_C y^{g-1}$ of $\mathbb{C}[x]/(x^{g+1})$ must be a linear combination of $1, x, \ldots, x^{g-1}$. From this, we immediately obtain that $k_g = 0$. Next, setting to 0 the coefficient of x^g in

$$\kappa_C y^n = (1 + k_1 x + \cdots + k_{g-1} x^{g-1})(a_1 x + \cdots + a_g x^g)^n,$$

we obtain

$$a_1^n k_{g-n} + n a_1^{n-1} a_2 k_{g-n-1} + \cdots = 0. \tag{4.1}$$

Setting $n = g-1$, this gives $k_1 = -n a_2 / a_1$, which determines k_1 uniquely. The assertion for $k_2, \ldots, k_{g-1}$ follows by induction by applying (4.1) for $n = g-2, \ldots, 1$ repeatedly. ■

Example 4.3 (See [11, Section 2.3.7]). Up to projectivities, there is a unique canonically embedded A_9-curve $C \subset \mathbb{P}^3$. It can be defined by the crimping datum $y \mapsto x + x^2 + x^3 +$

x^4. A quick computation shows that the gluing datum of ω_C is $\kappa_C = 1 - 3x + 5x^2 - 5x^3$. It follows that the normalization of C is given by

$$x_0 = \begin{pmatrix} 1 \\ 1 - 3x + 5x^2 - 5x^3 \end{pmatrix}, \quad x_1 = \begin{pmatrix} y \\ x - 2x^2 + 2x^3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} y^2 \\ x^2 - x^3 \end{pmatrix}, \quad x_3 = \begin{pmatrix} y^3 \\ x^3 \end{pmatrix}. \quad \square$$

Example 4.4. Suppose $g = 2k + 1$. Consider the crimping datum

$$y \mapsto x - tx^{k+2},$$

where $t \neq 0$ is a parameter. One easily computes that $\kappa_C = 1 + tkx^{k+1}$ and that the following is a basis of $\mathrm{H}^0(C, \omega_C)$:

$$\omega_i = (x^i + t(k-i)x^{k+1+i}, y^i), \quad i = 0, \dots, k-1, \qquad \omega_i = (x^i, y^i), \quad i = k, \dots, 2k. \quad \square$$

We recall the definition of the *balanced canonical ribbon* R of genus $g = 2k + 1$ from [2]: R is a canonical ribbon obtained by gluing $\operatorname{Spec} \mathbb{C}[u, \epsilon]/(\epsilon^2)$ and $\operatorname{Spec} \mathbb{C}[v, \eta]/(\eta^2)$ via the isomorphism

$$u \mapsto v^{-1} - v^{-k-2}\eta,$$
$$\epsilon \mapsto v^{-g-1}\eta$$

of distinguished open affines $\operatorname{Spec} \mathbb{C}[u, u^{-1}, \epsilon]/(\epsilon^2)$ and $\operatorname{Spec} \mathbb{C}[v, v^{-1}, \eta]/(\eta^2)$.

Lemma 4.5. The flat limit as $t \to 0$ of the A_{2g+1}-curve in Example 4.4 is the balanced canonical ribbon R of genus $g = 2k + 1$. $\square$

Proof. Recall from [2, Lemma 3.1] that there is a basis of $\mathrm{H}^0(R, \omega_R)$ whose elements can be identified with the following polynomials in u and ϵ (here $\epsilon^2 = 0$):

$$z_i = u^i, \quad 0 \leq i \leq k, \qquad z_i = u^i + (i-k)u^{i-k-1}\epsilon, \quad k+1 \leq i \leq 2k.$$

Keeping the notation of Example 4.4, we note that if we set $\psi_i := \omega_i/(x^{2k}, y^{2k})$ and $w := 1/x$, then

$$\psi_i = (w^i, y^{-i}), \quad 0 \leq i \leq k, \qquad \psi_i = (w^i + (i-k)w^{i-k-1}t, y^{-i}), \quad k+1 \leq i \leq 2k.$$

To prove the lemma, it suffices to show that any quadratic relation among the z_i's is a flat limit of a quadratic relation among the ψ_i's as $t \to 0$. If we evaluate a quadratic relation among z_i's on ψ_i, we obtain an expression of the form $(f(w)t^2, 0)$, where $\deg f(w) \leq 2k-2$. It remains to show that $(w^i t, 0)$ can be obtained as a quadratic polynomial in the ψ's for $0 \leq i \leq 2k-2$. Indeed, we have

$$\begin{aligned}(w^i t, 0) &= (w^{k+i+1} + (i+1)w^i t, y^{-k-i-1})(1,1) - (w^{k+i} + iw^{i-1}t, y^{-k-i})(w, y^{-1}) \\ &= \psi_{k+i+1}\psi_0 - \psi_{k+i}\psi_1 \quad \text{for } 0 \leq i \leq k-1\end{aligned}$$

and

$$\begin{aligned}(w^i t, 0) &= (w^{2k} + kw^{k-1}t, y^{-2k})(x^{i-k+1}, y^{-i+k-1}) - (w^{2k-1} + iw^{k-2}t, y^{-2k+1})(w^{i-k+2}, y^{-i+k-2}) \\ &= \psi_{2k}\psi_{i-k+1} - \psi_{2k-1}\psi_{i-k+2} \quad \text{for } k \leq i \leq 2k-2.\end{aligned}$$ ■

We conclude with an observation that the general A_{2g+1}-rational curve is semistable. We would prefer the stronger statement that such a curve is in fact stable, but at present we have no proof.

Proposition 4.6. A general A_{2g+1}-rational curve has semistable 2nd Hilbert point. □

Proof. By the above, the variety of A_{2g+1}-rational curves in $\mathbb{P}^{g-1}$ is irreducible. Thus, it suffices to find a single A_{2g+1}-rational curve with semistable 2nd Hilbert point. When g is even, this is already done by Theorem 3.3(a). In the case of odd genus, the balanced canonical ribbon R has semistable 2nd Hilbert point by Alper et al. [2, Theorem 4.1]. Since R deforms flatly to A_{2g+1}-rational curves by Lemma 4.5, we are done. ■

Corollary 4.7. A general A_{2g}-rational curve is semistable. □

Proof. The general A_{2g}-rational curve is a deformation of the general A_{2g+1}-rational curve. The statement now follows from Proposition 4.6. ■

5 A Slope Inequality Après Cornalba and Harris

In this section, we prove Theorem 1.2. To set notation, let B be a complete smooth curve and consider a flat proper family $\pi : \mathcal{C} \to B$ of Gorenstein curves of arithmetic genus

$g \geq 2$. Suppose that the relative dualizing line bundle $\omega := \omega_{\mathcal{C}/B}$ is relatively ample. Then $\pi_*(\omega^m)$ is a vector bundle of rank g if $m = 1$ and rank $(2m-1)(g-1)$ if $m \geq 2$. We set

$$\lambda := c_1(\pi_*\omega), \quad \lambda_2 := c_1(\pi_*\omega^2).$$

After a finite base change, we will assume that λ is divisible by g in $\mathrm{Pic}(B)$ and we let $\tilde{\omega} := \omega(-\pi^*(\lambda/g)$. Then the *normalized Hodge bundle* $\mathbb{E} := \pi_*\tilde{\omega}$ has a trivial determinant, that is, the transition matrices of $\mathbb{E}$ are given by elements of $\mathrm{SL}(g, \mathcal{O}_B)$.

5.1 Line bundles on the moduli stack of Gorenstein curves

Consider the stack of all complete Gorenstein curves of arithmetic genus g with an ample dualizing sheaf. Let $\mathcal{U}_g$ be its irreducible component parameterizing smoothable curves. Then λ and λ_2 are well-defined line bundles on $\mathcal{U}_g$. We formally define $\delta := 13\lambda - \lambda_2$. Note that $\bar{\mathcal{M}}_g \subset \mathcal{U}_g$ is an open substack and that the line bundle δ on $\bar{\mathcal{M}}_g$ has a geometric interpretation as the line bundle associated to the Cartier divisor of nodal curves. Under certain conditions this geometric interpretation can be extended to a larger open substack of $\mathcal{U}_g$. To do this, we consider the regular locus $\mathcal{U}_g^{\mathrm{reg}} \subset \mathcal{U}_g$ and define $\Delta := \mathcal{U}_g^{\mathrm{reg}} \smallsetminus \mathcal{M}_g$ to be the locus parameterizing singular curves. Let Δ' be the union of those irreducible components of Δ whose generic points parameterize worse than nodal curves. Then on $\mathcal{U}_g^{\circ} := \mathcal{U}_g^{\mathrm{reg}} \smallsetminus \Delta'$ the irreducible components of Δ are Cartier divisors whose generic points parameterize nodal curves. By construction, the locus of worse than nodal curves in $\mathcal{U}_g^{\circ}$ is of codimension at least two. Thus, the relation $\mathcal{O}(\Delta) = 13\lambda - \lambda_2$ extends from $\bar{\mathcal{M}}_g$ to $\mathcal{U}_g^{\circ}$. We conclude that at least on $\mathcal{U}_g^{\circ}$, the formally defined line bundle δ is the associated line bundle of the Cartier divisor $\Delta \subset \mathcal{U}_g^{\circ}$ parameterizing singular curves.

5.2 Slopes of families of Gorenstein curves

Given a family $\mathcal{C} \to B$ as above, we define its *slope* to be $(\delta \cdot B)/(\lambda \cdot B)$. We proceed to prove Theorem 1.2, which is a generalization of a special case of a well-known result of Cornalba and Harris regarding divisor classes associated to families of Hilbert (semi)stable varieties [8]. In the case of curves, the Cornalba–Harris theorem says that the slope of an *arbitrary* generically smooth family of stable curves of genus g is at most $8 + 4/g$ [8, Theorem 1.3]. This result was proved independently by Xiao [24] for the wider class of fibered algebraic surfaces using a vector bundle argument and more recently by Moriwaki [18] using semistability of a certain vector bundle on $\bar{\mathcal{M}}_g$.

Note that the Cornalba–Harris theorem is sharp: the general family of hyperelliptic curves of genus g has slope precisely $8+4/g$, while there exist families of bielliptic curves of slope 8 [5, Theorem 2.1] and there are families of trigonal curves of slope $36(g+1)/(5g+1)$ by [1, 21].

Our proof of Theorem 1.2 follows closely the original argument of Cornalba and Harris, which relies on GIT. We also note that the Cornalba–Harris GIT approach was recently generalized to more general families by Stoppino [22].

Proof of Theorem 1.2. The key input in Cornalba–Harris method [8] is the *asymptotic* Hilbert semistability of the canonically embedded general fiber of $\mathcal{C}\to B$. Our assumption that the general fiber C has semistable 2nd Hilbert point is much stronger than asymptotic semistability and so leads to a stronger inequality. On the other hand, not every smooth canonical curve has a semistable 2nd Hilbert point (see Proposition 5.3), so while our inequality is stronger, not every family will satisfy it.

To begin, GIT-semistability of the 2nd Hilbert point of C is equivalent to the existence of an $\mathrm{SL}(g)$-invariant polynomial $f\in \mathrm{H}^0(\mathbb{P}W,\mathcal{O}(d))$, where $W=\bigwedge^{3g-3}\mathrm{Sym}^2\mathrm{H}^0(C,\omega_C)$, that does not vanish at the point

$$\bigwedge^{3g-3}\mathrm{Sym}^2\mathrm{H}^0(C,\omega_C)\to\bigwedge^{3g-3}\mathrm{H}^0(C,\omega_C^2)\to 0$$

of $\mathbb{P}W$. Under the usual identification $\mathrm{H}^0(\mathbb{P}W,\mathcal{O}(d))\simeq\mathrm{Sym}^dW$, the polynomial f corresponds to a section of Sym^dW that maps to a nonzero section of $\mathrm{Sym}^d\bigwedge^{3g-3}\mathrm{H}^0(C,\omega_C^2)$.

Consider now the family $\pi:\mathcal{C}\to B$ as in the statement of the theorem. Let $\mathbb{E}=\pi_*(\tilde{\omega})$ be the normalized Hodge bundle, so $\det\mathbb{E}\simeq\mathcal{O}_B$. Since f is $\mathrm{SL}(g)$-invariant, it defines a section F of $\mathrm{Sym}^d\bigwedge^{3g-3}\mathrm{Sym}^2\mathbb{E}$ that restricts to f on C. By construction, F maps to a generically nonvanishing section of $\mathrm{Sym}^d\bigwedge^{3g-3}\pi_*(\tilde{\omega}^2)$. Since $\bigwedge^{3g-3}\pi_*(\tilde{\omega}^2)$ is a line bundle on B, we conclude that

$$c_1\left(\bigwedge^{3g-3}\pi_*(\tilde{\omega}^2)\right)\geq 0.$$

It follows that $c_1(\pi_*(\tilde{\omega}^2))\geq 0$.

Since $c_1\pi_*(\tilde{\omega}^2)=\lambda_2-2\lambda(3g-3)/g$, we conclude

$$13\lambda-\delta=\lambda_2\geq 2(3g-3)\lambda/g,$$

which gives the desired inequality

$$\frac{\delta}{\lambda} \le 7 + \frac{6}{g}.$$ ■

As an immediate consequence of Theorem 1.2, we obtain the following result.

Corollary 5.1. Suppose $\pi : \mathcal{C} \to B$ is a relatively minimal fibration of a smooth projective surface over a smooth complete curve such that the generic fiber is a Gieseker–Petri general curve of genus $g \ge 4$. Then

$$\frac{\delta \cdot B}{\lambda \cdot B} \le 7 + \frac{6}{g}. \tag{5.1}$$

□

Proof. Note that the fibers of π are Gorenstein curves by adjunction. The general fiber of π is a smooth nonhyperelliptic curve whose canonical embedding has semistable 2nd Hilbert point by Theorem 1.1. Passing to the relative canonical model of π, we can assume that π has relatively ample dualizing sheaf. The claim now follows from Theorem 1.2. ■

5.3 Bielliptic curves

As [24, Example 4.3] shows, certain double covers of trivial families of elliptic curves give rise to families of bielliptic curves of genus g and slope 8; for reader's convenience, we recall this construction below. Interestingly, Barja proved that any family of curves with a bielliptic generic fiber has slope at most 8 and those of slope 8 are necessarily double covers of isotrivial families of smooth elliptic curves [5, Theorem 2.1].

Example 5.2 (Bielliptic family of slope 8, cf. [24, Example 4.3])**.** Let E be a curve of genus one. Consider a constant family $X := E \times B$ and a divisor $D \subset X$ of relative degree $(2g-2)$ over B. Since $K_X = \pi^* K_B$, adjunction gives $K_D - \pi^* K_B = (K_X + D) \cdot D - \pi^* K_B \cdot D = D^2$. Thus, the number of branch points of $D \to B$ is D^2 by Riemann–Hurwitz formula. Consider now the double cover $Y \to X$ branched over D. The singular fibers of $Y \to B$ correspond to branch points of $D \to B$. Assuming the branch points are simple, we conclude that $\delta_{Y/B} = D^2$. On the other hand, by Mumford's formula

$$12\lambda_{Y/B} - \delta_{Y/B} = \kappa_{Y/B} = 2(\omega_{X/B} + D/2)^2 = D^2/2.$$

It follows that

$$\lambda_{Y/B} = D^2/8 = \delta/8. \qquad \square$$

We now contrast the computation of Example 5.2 with Theorem 1.2. Since $8 > 7 + \frac{6}{g}$ for $g \geq 7$, Theorem 1.2 implies that the canonically embedded general bielliptic curve of genus $g \geq 7$ must have a nonsemistable 2nd Hilbert point. In fact, we have a more precise result.

Proposition 5.3. The 2nd Hilbert point of a canonically embedded smooth bielliptic curve of genus $g \geq 7$ is nonsemistable. The 2nd Hilbert point of a canonically embedded smooth bielliptic curve of genus $g = 6$ is strictly semistable. $\square$

Proof. Consider a genus one curve $E \subset \mathbb{P}^{g-2}$ embedded by a degree $g-1$ complete linear system. There are $\binom{g}{2} - (2g-2) = \binom{g+1}{2} - 3(g-1) - 1$ quadrics containing E. It follows that a projective cone $\mathrm{Cone}(E)$ over E in $\mathbb{P}^{g-1}$ is cut out by one less quadric than a smooth canonical curve. In fact, any smooth quadric section of $\mathrm{Cone}(E)$ is a canonically embedded bielliptic curve of genus g, as can be easily verified using adjunction, and conversely every canonically embedded bielliptic curve lies on such a cone. If $C \in |\mathcal{O}_{\mathrm{Cone}(E)}(2)|$, then there are $\binom{g+1}{2} - 3(g-1) - 1$ quadrics in $\mathrm{H}^0(C, \mathcal{I}_C(2))$ that are singular at the vertex of $\mathrm{Cone}(E)$. Suppose the vertex has coordinates $[0:0:\ldots:0:1]$. Now, if ρ is the one-parameter subgroup of $\mathrm{SL}(g)$ acting with weights $(-1, -1, \ldots, -1, g-1)$, then the ρ-weight of any monomial basis of $\mathrm{H}^0(C, \mathcal{I}_C(2))$ is at most

$$-2\left(\binom{g+1}{2} - 3(g-1) - 1\right) + 2(g-1) = -g^2 + 7g - 6 = -(g-1)(g-6).$$

Thus a bielliptic curve has a nonstable 2nd Hilbert point for all $g \geq 6$, and a nonsemistable 2nd Hilbert point for all $g \geq 7$.

It remains to establish the semistability of a canonically embedded smooth bielliptic curve of genus 6. Every such curve is a quadric section of a projective cone over a genus one curve of degree 5 in $\mathbb{P}^4$. Hence, a canonically embedded smooth bielliptic curve can be degenerated isotrivially to a double hyperplane section of a projective cone over an elliptic curve in $\mathbb{P}^4$. The semistability of this nonreduced curve follows from Kempf's results [16]. $\blacksquare$

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