SCROLLAR INVARIANTS OF TROPICAL CURVES

DAVID JENSEN AND KALILA JOELLE SAWYER

ABSTRACT. We define scrollar invariants of tropical curves with a fixed divisor of rank 1. We examine the behavior of scrollar invariants under specialization, and compute these invariants for a much-studied family of tropical curves. Our examples highlight many parallels between the classical and tropical theories, but also point to some substantive distinctions.

1. INTRODUCTION

In this note, we initiate a study of scrollar invariants of tropical curves. Classically, every canonically embedded trigonal curve is contained in a unique rational normal surface. Any such surface is isomorphic to a Hirzebruch surface

$$\mathbb{F}_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)),$$

and the integer m is known as the *Maroni invariant* of the trigonal curve.

More generally, given a curve X and a dominant map $\pi : X \to \mathbb{P}^1$ of degree k, one defines the *Tschirnhausen bundle* of π to be the dual of $\mathcal{E}^\vee = \pi_* \mathcal{O}_X / \mathcal{O}_{\mathbb{P}^1}$. The Tschirnhausen bundle is a vector bundle of rank $k - 1$ on $\mathbb{P}^1$. As such, it splits into a direct sum of line bundles $\mathcal{E} = \oplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$. The integers a_i are natural invariants of the k-gonal curve X, known as the *scrollar invariants*. There are many open questions concerning scrollar invariants. For example, there is no known classification of which sequences of integers a_i arise as scrollar invariants of k-gonal curves. Even in cases where a curve with given scrollar invariants is known to exist, it is unknown whether the space of such curves is irreducible, or what its dimension is.

In a family of curves of gonality k, the scrollar invariants are not lower semicontinuous, but a related set of invariants is. If we order the scrollar invariants

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1},$$

we define the *composite scrollar invariant* σ_j to be the sum of the first j scrollar invariants:

$$\sigma_j = a_1 + a_2 + \cdots + a_j.$$

Of course, the scrollar invariants themselves can be recovered from the set of composite scrollar invariants. The composite scrollar invariants are known to be lower semicontinuous.

In this article, we define tropical analogues of composite scrollar invariants. Key to our study is the observation that the scrollar invariants are determined by the ranks of the line bundles $\pi^* \mathcal{O}_{\mathbb{P}^1}(c)$. Combining this observation with the Baker-Norine theory of divisors on tropical curves, we obtain definitions of tropical composite scrollar invariants. We refer the reader to Section 2 for precise definitions.

We prove that composite scrollar invariants cannot increase under specialization.

Theorem 1.1. *Let X be a curve over a nonarchimedean field with skeleton isometric to Γ, and let D be a divisor of degree k and rank 1 on X. Then*

$$\sigma_j(X, D) \geq \sigma_j(\Gamma, \mathrm{Trop}\, D) \text{ for all } j.$$

Having established this relationship between the composite scrollar invariants of a curve and those of its tropicalization, we then compute composite scrollar invariants of certain metric graphs. Of primary interest to us are the *chains of loops*, a much-studied family of metric graphs that has played a central role in tropical proofs of the Brill-Noether Theorem [CDPR12] and the Gieseker-Petri Theorem [JP14], as well as establishing new results such as the Maximal Rank Conjecture for quadrics [JP16, JP17] and an analogue of the Brill-Noether Theorem for curves of fixed gonality [Pfl17a, JR17, CPJ19].

By varying the edge lengths, we obtain chains of loops of various gonalities. More precisely, the divisor theory of a chain of loops is determined by its *torsion profile*. We refer the reader to Definition 2.5 for a definition. In order for a chain of loops to be hyperelliptic, it must have a specific torsion profile. The torsion profiles corresponding to trigonal chains of loops of genus g are determined by a pair of integers a and b between 1 and g, as described in Corollary 4.4. Given such a pair of integers, let

$$\ell = \left\lceil \frac{b - a + 4}{2} \right\rceil,$$

and let n be the smallest integer such that

$$g \leq \left\lfloor \frac{3}{2}n + \frac{1}{2}(\ell - 1) \right\rfloor.$$

If $a \neq b$, then the corresponding chain of loops possesses a unique divisor of degree 3 and rank 1, which we denote $D_{a,b}$.

Theorem 1.2. *Let Γ be the trigonal chain of loops corresponding to the integers a and b, and let $D_{a,b}$ be the divisor of degree 3 and rank 1 on Γ. Then*

$$\sigma_1(\Gamma, D_{a,b}) = \left\lfloor \frac{n + \ell}{2} \right\rfloor.$$

Combining Theorems 1.1 and 1.2, we see that if X is a curve over a nonarchimedean field with skeleton isometric to Γ, and D is a divisor of rank 1 on X that specializes to $D_{a,b}$, then

$$\sigma_1(X, D) \geq \left\lfloor \frac{n + \ell}{2} \right\rfloor.$$

Indeed, we will see that ℓ is the smallest positive integer such that $\mathrm{rk}(\ell D_{a,b}) > \ell$. It follows from Baker's Specialization Lemma that ℓ is a lower bound for $\sigma_1(X, D)$. In general, however, this lower bound is not tight. The integer n has a similar interpretation – it is the smallest positive integer such that $K_\Gamma - nD_{a,b}$ is not effective. It follows from Baker's Specialization Lemma that n is a lower bound for $a_2(X, D)$. Again, this lower bounds is typically not tight. On the curve X, the invariants σ_1 and a_2 satisfy the relationship $a_2 = g + 2 - \sigma_1$, but on the metric graph Γ, the invariants ℓ and n do not satisfy this relationship. Theorem 1.2 shows that we can obtain a stronger bound on $\sigma_1(X, D)$ by *averaging* the two invariants ℓ and n.

As the gonality increases, so too does the number of torsion profiles for which the corresponding chain of loops has the given gonality. In these cases, we do not

have a closed formula for composite scrollar invariants analogous to Theorem 1.2. Nevertheless, given a torsion profile, we can algorithmically compute the composite scrollar invariants, and we have implemented this algorithm in a Sage program, which can be found on the second author's website:

https://github.com/kalilajo/numberboxes.

If X is an algebraic curve and D is a divisor of degree k and rank 1 on X, then the datum of the scrollar invariants is equivalent to that of the sequence of ranks $rk(cD)$. More precisely, the sequence of ranks $rk(cD)$ is a convex, piecewise linear function in c, and the scrollar invariants correspond to the "bends" between domains of linearity (see Eq. (1)). For a tropical curve, however, the sequence of ranks is not necessarily convex. This is perhaps most striking in the trigonal case – that is, when $k = 3$. In this case, the sequence of ranks $rk(cD)$ exhibits substantively different behavior.

Proposition 1.3. *Let Γ be the trigonal chain of loops corresponding to the integers a and b, and let $D_{a,b}$ be the divisor of degree 3 and rank 1 on Γ. Then for $0 \leq i < n$, we have*

$$rk((\ell + i)D_{a,b}) = \begin{cases} rk((\ell + i - 1)D_{a,b}) + 1 \ \text{if } i \text{ is odd} \\ rk((\ell + i - 1)D_{a,b}) + 2 \ \text{if } i \text{ is even}. \end{cases}$$

It is our hope that the study initiated here could be used to resolve outstanding questions concerning scrollar invariants of classical curves. In order to do this, we would need a lifting result for scrollar invariants. We pose this as an open question.

Question 1.1. *Let Γ be a chain of loops, and let D be a divisor of degree k and rank 1 on Γ. Under what circumstances does there exist a curve X, over a nonarchimedean field, with skeleton Γ and a rank 1 divisor D_X on X specializing to D, such that $\sigma_j(X, D_X) = \sigma_j(\Gamma, D)$?*

2. Preliminaries

2.1. The Maroni Invariant and Scrollar Invariants.
Let X be a curve of genus g and $\pi : X \to \mathbb{P}^1$ a dominant map of degree $k \geq 3$. The map π induces a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_* \mathcal{O}_X \to \mathcal{E}^\vee \to 0.$$

The sheaf $\mathcal{E}$ is a vector bundle of rank $k - 1$ on $\mathbb{P}^1$, called the *Tschirnhausen bundle* of the map π. Since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, we may write

$$\mathcal{E} = \bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

The integers a_i are known as the *scrollar invariants* of the map π. We order them so that

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1}.$$

We define the jth *composite scrollar invariant* to be the sum of the first j scrollar invariants:

$$\sigma_j = a_1 + a_2 + \cdots + a_j.$$

The scrollar invariants determine, and are determined by, the sequence of integers $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c))$. Setting $a_0 = 0$, this can be seen by the following calculation:

$$(1) \qquad h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c)) = h^0(\mathbb{P}^1, \pi_*\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1}(c))$$

$$= \sum_{i=0}^{k-1} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c - a_i))$$

$$= \sum_{i=0}^{k-1} \max\{0, c + 1 - a_i\}$$

$$= \max\{(c+1)(j+1) - \sigma_j\}.$$

Note in particular that $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c))$ is convex as a function in c.

Because $h^0(X, \mathcal{O}_X) = 1$, we see that each of the scrollar invariants a_i is strictly positive. Moreover, for c sufficiently large, we have $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c)) = ck - g + 1$, so we see that $\sigma_{k-1} = g + k - 1$.

When $k = 3$, the scrollar invariants are determined by the single value $|a_2 - a_1|$, which is known as the *Maroni invariant* of the trigonal curve. The parity of the Maroni invariant agrees with that of g. The space of trigonal curves with given Maroni invariant m is known to be irreducible and, except in the case $m = 0$, it has codimension $m - 1$ in the space of all trigonal curves.

When $k \geq 4$, the situation is more mysterious. One defines the *Maroni locus* $M(\mathcal{E})$ to be the space of k-gonal curves with Tschirnhausen bundle isomorphic to $\mathcal{E}$. In general, given a vector bundle $\mathcal{E}$, it is not even known whether $M(\mathcal{E})$ is empty.

2.2. **Divisor Theory of Metric Graphs.** In this section, we review the theory of divisors on metric graphs. For more details, we refer the reader to [Bak08].

Recall that a *metric graph* is a compact, connected metric space Γ obtained by identifying the edges of a graph G with line segments of fixed positive real length.

Definition 2.1. A *divisor* D on a metric graph Γ is a finite formal $\mathbb{Z}$-linear combination of points of Γ. That is, $D = \sum_{v \in \Gamma} D(v) \cdot v$, where $D(v) \in \mathbb{Z}$ is zero for all but finitely many v.

The group of all divisors on a metric graph Γ is simply the free abelian group on points of the metric space Γ, called the *divisor group* $\mathrm{Div}(\Gamma)$ of Γ. Divisors on metric graphs should be thought of as the tropical analogues of divisors on algebraic curves. We now define the tropical analogues of rational functions.

Definition 2.2. A *rational function* on a metric graph Γ is a continuous piecewise-linear function $\varphi : \Gamma \to \mathbb{R}$ with integer slopes. The rational functions on Γ form a group under pointwise addition, denoted $\mathrm{PL}(\Gamma)$. Given $\varphi \in \mathrm{PL}(\Gamma)$ and $v \in \Gamma$, we define the *order of vanishing* of φ at v, $\mathrm{ord}_v(\varphi)$, to be the sum of the incoming slopes of φ at v.

Note that $\mathrm{ord}_v(\varphi)$ is nonzero for only finitely many points $v \in \Gamma$. We define the divisor associated to φ as

$$\mathrm{div}(\varphi) = \sum_{v \in \Gamma} \mathrm{ord}_v(\varphi) \cdot v.$$

Definition 2.3. We say that two divisors D and D' on a metric graph Γ are *linearly equivalent* if their difference $D - D'$ is equal to $\operatorname{div}(\varphi)$ for some rational function $\varphi \in \operatorname{PL}(\Gamma)$.

It is straightforward to show that linear equivalence is in fact an equivalence relation. For our purposes, it suffices to consider linear equivalence classes of divisors.

A basic invariant of a divisor D is its *degree*, defined to be the integer

$$\deg(D) = \sum_{v \in \Gamma} D(v).$$

In analogy with divisors on algebraic curves, we say that a divisor D is *effective* if $D(v) \geq 0$ for all $v \in \Gamma$. Similarly, we say that a divisor D is *special* if both D and $K_\Gamma - D$ are equivalent to effective divisors, where K_Γ is the canonical divisor

$$K_\Gamma = \sum_{v \in \Gamma} (\operatorname{val}(v) - 2)v.$$

Perhaps the most important invariant of a divisor on a metric graph is its Baker-Norine rank.

Definition 2.4. A divisor D has *rank* at least r if $D - E$ is equivalent to an effective divisor for all effective divisors E of degree r.

2.3. **Divisors on Chains of Loops.** In Sections 4 and 5, we will consider equivalence classes of special divisors on the metric graph pictured in Figure 1. This graph, known as the *chain of loops*, has appeared in several articles that use tropical techniques to develop results in algebraic geometry [CDPR12, JP14, JP16, JP17, Pfl17b, Pfl17a, JR17, CPJ19]. We denote by v_k the point where the kth loop meets a bridge on the left and by w_k the point where the kth loop meets a bridge on the right. We label edges by their initial and terminal vertices when traversing the loop counter-clockwise. For example, $w_2 v_2$ denotes the top edge of the second loop.
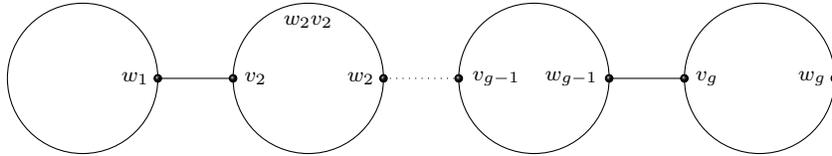

FIGURE 1. A Chain of loops Γ

In this section we summarize the main result of [Pfl17b] and draw a few corollaries.

Definition 2.5. Let ℓ_i denote the length of the ith cycle, and let $\ell(w_i v_i)$ denote the length of the counterclockwise edge from w_i to v_i. If $\ell(w_i v_i)$ is an irrational multiple of ℓ_i, then the ith *torsion order* m_i is 0. Otherwise, m_i is the minimum positive integer such that $m_i \cdot \ell(w_i v_i)$ is an integer multiple of ℓ_i. We record the torsion order of each loop as the vector $\underline{m} = (m_1, m_2, \ldots, m_g)$, called the *torsion profile* of Γ.

To represent divisors on chains of loops, we use the fact that the Picard group $\operatorname{Pic}(\Gamma)$ has a natural coordinate system. Denote by $\langle x \rangle_i$ the point on the ith loop

of Γ located $x \cdot \ell(w_i v_i)$ units clockwise from w_i. Note that $\langle x \rangle_i = \langle y \rangle_i$ if and only if $x \equiv y \pmod{m_i}$.

By the Tropical Abel-Jacobi theorem [BF11], every divisor class D of degree d on Γ has a unique *break divisor* representative

$$D \sim (d-g)w_g + \sum_{i=1}^{g} \langle \xi_i(D) \rangle_i$$

for some $\xi_i(D) \in \mathbb{R}/m_i\mathbb{Z}$. These divisors are our primary object of study. We also define a helpful combinatorial object.

Definition 2.6. An $\underline{m}$-*displacement tableau* on a partition λ is a function $t : \lambda \to \{1, \ldots, g\}$ such that:

   (1) t increases across each row and column of λ, and
   (2) if $t(x,y) = t(x',y') = i$, then $y - x \equiv y' - x' \pmod{m_i}$.

Each such tableau t defines a locus $\mathbb{T}(t) \subseteq \mathrm{Pic}^d(\Gamma)$ homeomorphic to a torus of dimension equal to g minus the number of symbols appearing in t. Specifically,

$$\mathbb{T}(t) = \{D \in \mathrm{Pic}^d(\Gamma) | \xi_{t(x,y)}(D) \equiv y - x \pmod{m_{t(x,y)}} \text{ for all } (x,y) \in \lambda\}.$$

Note that if the function t is not surjective, then there is a symbol i not appearing in the tableau, and a corresponding value ξ_i upon which no restrictions are placed.

Recall that $W_d^r(\Gamma)$ is the set of all divisor classes of degree d and rank at least r on Γ. Pflueger's main result in [Pfl17b] is the following.

Theorem 2.7. [Pfl17b] *Let Γ be a chain of loops of genus g and torsion profile $\underline{m}$, and let r and d be positive integers with $r > d - g$. Let λ be the rectangular partition of dimensions $(r+1) \times (g-d+r)$. Then*

$$W_d^r(\Gamma) = \bigcup_t \mathbb{T}(t),$$

where t ranges over all $\underline{m}$-displacement tableaux on λ.

Corollary 2.8. *A chain of loops with torsion profile $\underline{m}$ has gonality k if and only if there is an $\underline{m}$-displacement tableau on a rectangle λ of dimensions $(g-k+1) \times 2$ and no such tableau on a rectangle of dimensions $(g-k+2) \times 2$.*

The following lemma will prove to be a crucial step in our analysis of trigonal chains of loops in Section 4.

Lemma 2.9. *Given a divisor D on Γ, denote by $\xi_i^c := \xi_i^c(D)$ the coordinate on the ith loop of Γ in the break divisor representative of cD. Then $\xi_i^{c+1} = \xi_i^c + \xi_i^1 - (i-1)$. It follows by induction on c that $\xi_i^c = c\,\xi_i^1 - (c-1)(i-1)$.*

Proof. By [Pfl17b, Remark 3.4], the function

$$\widetilde{\xi_i} := \xi_i - (i-1)$$

is linear. This gives

$$
\begin{aligned}
\xi_i^{c+1} &= i - 1 + \widetilde{\xi_i}^{c+1} \\
&= i - 1 + \widetilde{\xi_i}^c + \widetilde{\xi_i}^1 \\
&= i - 1 + \xi_i^c - (i-1) + \xi_i^1 - (i-1) \\
&= \xi_i^c + \xi_i^1 - (i-1).
\end{aligned}
$$

$\square$

2.4. **Specialization.** The theory of divisors on metric graphs informs that of algebraic curves via *specialization*. Here, we recall the basic properties of specialization. We refer the reader to [Bak08] for details. Let K be an algebraically closed field that is complete with respect to a nontrivial valuation

$$\mathrm{val} : X \to \mathbb{R}^*.$$

Let X be an algebraic curve over K. A *skeleton* of X is a certain type of subset of the set of valuations on the function field $K(X)$ that extend the given valuation on K. A skeleton of X is endowed with a topology, giving it the structure of a metric graph. There is a natural map from X to its skeleton Γ. Extending linearly yields the tropicalization map on divisors

$$\mathrm{Trop} : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma).$$

The tropicalization map satisfies an important property, known as Baker's Specialization Lemma.

Lemma 2.10. [Bak08] *Let D_X be a divisor on X. Then*

$$rk(D_X) \leq rk(\mathrm{Trop}\, D_X).$$

3. Specialization for Composite Scrollar Invariants

We now define composite scrollar invariants of divisors on metric graphs.

Definition 3.1. Let Γ be a metric graph and D a divisor of degree k and rank 1 on Γ. We define the jth *composite scrollar invariant* of the pair (Γ, D) to be

$$\sigma_j(\Gamma, D) := \min\{m | \mathrm{rk}(cD) \geq (c+1)(j+1) - (m+1) \text{ for all } c\}.$$

Note that $\mathrm{rk}(cD) \geq c$ for all c, with equality if $c = 0$, so $\sigma_0 = 0$. By Riemann-Roch, we have $\mathrm{rk}(cD) \geq ck - g$ with equality if c is sufficiently large, so $\sigma_{k-1} = g + k - 1$.

We note that there are several other ways we could define tropical analogues of these invariants. For example, we could define σ_1 to be the minimum value of c such that $\mathrm{rk}(cD) > c$. For algebraic curves, these two definitions of σ_1 agree because the rank sequence $\mathrm{rk}(cD)$ is convex as a function in c. For metric graphs, however, the rank sequence is not necessarily convex, so these two definitions do not agree.

We now prove a specialization lemma for composite scrollar invariants.

Theorem 3.2. *Let X be a curve over a nonarchimedean field with skeleton isometric to Γ, and let D be a divisor of degree k and rank 1 on X. Then*

$$\sigma_j(X, D) \geq \sigma_j(\Gamma, \mathrm{Trop}\, D) \text{ for all } j.$$

Proof. By Eq. (1), for any value of j we have

$$\mathrm{rk}(cD) \geq (c+1)(j+1) - (\sigma_j(X, D) + 1).$$

Simultaneously, by Baker's Specialization Lemma, we have

$$\mathrm{rk}(cD) \leq \mathrm{rk}(c\,\mathrm{Trop}\, D) \text{ for all } c.$$

It follows that

$$\mathrm{rk}(c\,\mathrm{Trop}\, D) \geq (c+1)(j+1) - (\sigma_j(X, D) + 1) \text{ for all } c.$$

Since $\sigma_j(\Gamma, \mathrm{Trop}\, D)$ is defined to be the minimum value of m such that

$$\mathrm{rk}(c\,\mathrm{Trop}\, D) \geq (c+1)(j+1) - (m+1) \text{ for all } c,$$

we see that

$$\sigma_j(\Gamma, \operatorname{Trop} D) \leq \sigma_j(X, D).$$

$\square$

4. Gonality Three

For the remainder of the paper, we compute composite scrollar invariants for a specific family of tropical curves, the chains of loops. In this section, we classify chains of loops of gonality three. Given a chain of loops Γ and a divisor D on Γ of degree 3 and rank 1, we compute $\operatorname{rk}(cD)$ for all values of c. We begin with the following observation.

Lemma 4.1. *The following is the unique tableau Λ on the rectangular partition $(g-1) \times 2$.*

1	2
2	3
3	4
⋮	⋮
g-2	g-1
g-1	g

Proof. The boxes of Λ must contain integers between 1 and g so that the entries strictly increase in each row and column. There cannot be a g in the zeroth column, since the box to the right of it must contain a larger number. Similarly, there cannot be a 1 in the first column. This leaves exactly $g-1$ distinct symbols that may appear in each column, which must appear in increasing order. This yields the above tableau. $\square$

By Lemma 4.1, we see that there is a unique hyperelliptic chain of loops.

Corollary 4.2. *A chain of loops Γ is hyperelliptic if and only if its torsion profile (termwise) divides $\underline{m} = (0, 2, 2, \ldots, 2, 0)$. In this case, there is a divisor D on Γ of degree 2 and rank 1 whose corresponding tableau is Λ.*

Proof. By Corollary 2.8, Γ is hyperelliptic if and only if there is an $\underline{m}$-displacement tableau on a rectangle of dimensions $(g-1) \times 2$.

By Lemma 4.1, we see that Λ is the unique tableau on a $(g-1) \times 2$ rectangle. Since the symbols 1 and g appear only once, Λ imposes no conditions on the torsion of the first or last loops of Γ. Each symbol i in the range $1 < i < g$ appears twice in Λ, in boxes $(0, i-1)$ and $(1, i-2)$, which are lattice distance 2 from each other. Thus we must have $m_i = 2$ and the torsion profile of Γ is as above. $\square$

We will denote by $\lambda_{a,b}$ the tableau on the rectangular partition $(g-2) \times 2$ obtained by deleting boxes $(1, a-2)$ and $(0, b-1)$ from Λ. Note that the symbols appearing in these boxes are a and b, respectively. This defines a tableau if and only if $b \geq a-1$. Tableaux of the form $\lambda_{a,b}$ are of interest for the following reason.

Proposition 4.3. *All tableaux on a rectangle λ of dimensions $(g-2) \times 2$ are of the form $\lambda_{a,b}$ for some $b \geq a-1$.*

Proof. Let t be a displacement tableau on λ. We must show that $t = \lambda_{a,b}$ for some $b \geq a - 1$. Note that t has $g - 2$ distinct entries in each column, which must be between 1 and g. As in Corollary 4.2, there may not be a g in the zeroth column or a 1 in the first column, so there is exactly one integer "missing" from each column. Let b be the integer that is missing from the zeroth column, and let a be the integer that is missing from the first column. Moreover, note that the missing box in the first column may not be strictly below the missing box in the zeroth column. From this, we achieve the desired result. $\square$

Proposition 4.3 allows us to classify trigonal chains of loops.

Corollary 4.4. *A chain of loops Γ is trigonal if and only if it is not hyperelliptic, and has torsion profile that (termwise) divides*

$$\underline{m} = (0, 2, \ldots, 2, 0, 3, \ldots, 3, 0, 2, \ldots, 2, 0).$$

Proof. By Corollary 2.8, Γ is trigonal if and only if there is an $\underline{m}$-displacement tableau on a rectangle λ of dimensions $(g - 2) \times 2$ and none on a rectangle of dimensions $(g - 1) \times 2$. By Proposition 4.3, every tableau on λ is of the form $\lambda_{a,b}$ for some a and b. The tableau $\lambda_{a,b}$ imposes no conditions on the torsion of loops 1, a, b, and g, but the torsion of each other loop is determined by the tableau. In particular, if $i < a$, the symbol i appears twice in $\lambda_{a,b}$, both in boxes $(0, i - 1)$ and $(1, i - 2)$. These boxes are lattice distance 2 from each other, so we must have $m_i = 2$. In the same way, $m_i = 2$ for symbols i in the range $b < i < g$. Similarly, if $a < i < b$, the symbol i appears in boxes $(0, i - 1)$ and $(1, i - 3)$, which are lattice distance 3 apart, so $m_i = 3$. $\square$

Having classified trigonal chains of loops, we now turn to the problem of computing their scrollar invariants. Given a chain of loops Γ and a divisor D on Γ of degree 3 and rank 1, our goal is to compute the rank of cD for all c. Note that if $a \neq b$, then there is a unique divisor class $D_{a,b} \in \mathbb{T}(\lambda_{a,b})$. For the remainder of this section, we fix integers a and b, and assume both that $D_{a,b} \in \mathbb{T}(\lambda_{a,b})$ and that Γ has the corresponding torsion profile.

Given this setup, we define the integer

$$\ell := \left\lceil \frac{b - a + 4}{2} \right\rceil.$$

Note that ℓ depends only on the number of torsion 3 loops, $b - a - 1$. Let n be the smallest integer such that

$$g \leq \left\lfloor \frac{3}{2}n + \frac{1}{2}(\ell - 1) \right\rfloor.$$

Remark 4.5. We will see in Corollary 4.8 below that ℓ is the smallest positive integer such that $\mathrm{rk}(\ell D_{a,b}) > \ell$. Similarly, we will see in Corollary 4.8 that n is the smallest positive integer such that $K_\Gamma - nD_{a,b}$ is not effective. On an algebraic curve, the integers ℓ and n defined in this way satisfy a natural relationship. Specifically, in the classical case, we would have $\ell = \sigma_1$ and $n = a_2 = g + 2 - \sigma_1$. These tropical invariants, however, do not satisfy this relationship.

By Theorem 2.7, the divisor $cD_{a,b}$ has rank at least r if and only if there exists a tableau $\lambda_{a,b}^c$ on a rectangle with $r + 1$ columns and $g - 3c + r$ rows such that $cD_{a,b} \in$

$\mathbb{T}(\lambda_{a,b}^c)$. By Lemma 2.9, $cD_{a,b} \in \mathbb{T}(\lambda_{a,b}^c)$ if and only if, whenever $\lambda_{a,b}^c(x,y) = i$, we have

$$(2) \qquad y - x = \begin{cases} i - 1 \pmod{m_i} & \text{if } i \leq b, \\ i - 1 - 3c \pmod{m_i} & \text{if } i > b. \end{cases}$$

Our goal is therefore to construct the largest possible $\underline{m}$-displacement tableau satisfying the above congruence conditions. Note in particular that if $i \leq b$, then the congruence conditions above are independent of c.

We will proceed in two steps. First, we will construct a tableau $\lambda_{a,b}^c$ satisfying the congruence conditions above. After constructing this tableau, we will then prove that there does not exist a larger tableau satisfying the congruence conditions.

Definition 4.6. Let $\alpha(y) \in \{-1, 0, 1\}$ be congruent to $y - a \pmod 3$. Let $\gamma(c) \in \{0, 1\}$ be congruent to $c - \lfloor \frac{a-b}{2} \rfloor \pmod 2$. We define the tableau $\lambda_{a,b}^c$ as follows.

$$\lambda_{a,b}^c(x,y) = \begin{cases} x + y + 1 & \text{if } x + y + 1 < a \\ 2x + y + 1 & \text{if } y \geq \max\{a - 4, a - x - 1\} \text{ and} \\ & \qquad 2x + y + 1 < b \\ 2x + 2y - (a - 4) - \alpha(y) & \text{if } y < a - 4 \text{ and} \\ & \qquad a < 2x + 2y - (a - 4) - \alpha(y) < b \\ x + y + c + 1 & \text{if } 2x + y + 1 \geq b \text{ and} \\ & \qquad x \leq c < \ell \\ x + y + \ell + 1 + \gamma(c) & \text{if } c \geq \ell \text{ and} \\ & \qquad b \leq \min\{2x + y + 1, 2x + 2y - (a - 4) - \alpha(y)\} \\ g & \text{if } c < \ell, \ x = c, \text{ and } y = g - 2c - 1, \\ & \qquad \text{or if } \ell \leq c < n, \\ & \qquad x = \left\lceil \frac{3}{2}c - \frac{1}{2}(\ell - 1) \right\rceil, \text{ and} \\ & \qquad y = g - 1 - \left\lfloor \frac{3}{2}c + \frac{1}{2}(\ell - 1) \right\rfloor. \end{cases}$$

In order to help the reader understand the formula above, we also describe it algorithmically. To assist the reader in navigating this algorithm, we note that the cases in the statement correspond (in order) to the six regions pictured in Fig. 2.

To produce the tableau $\lambda_{a,b}^c$ as described, we first fill in the triangle above the $(a-1)$st diagonal by placing the symbols 1 through $a-1$ on successive diagonals. More precisely, we place the symbol s in every box (x,y) along the diagonal $x + y + 1 = s$.

We then place the symbols a through $b-1$ in regions 2 and 3, as shown in Fig. 3. Each of these symbols appears in every column of region 2. Specifically, we place the symbol s in the box $(0, s - 1)$, and then make "knight moves" to the right 1 box and up 2 boxes, placing the symbol s until we exit region 2. Region 3 is filled similarly, except that we alternate between knight moves to the right 2 boxes and up 1 box, and knight moves to the right 1 box and up 2 boxes.

Next, we place the symbols b through $g-1$ in regions 4 and 5. As in region 1, we place these symbols along an entire diagonal, starting with the first diagonal that contains an empty box.

Finally, we place the symbol g in a single box. Like the symbols a through $b-1$, the symbol g first makes knight moves to the right 1 box and up 2 boxes, until it crosses the line $y = a - 4$. At this point, we alternate between knight moves to the right 2 boxes and up 1 box, and knight moves to the right 1 box and up 2 boxes.
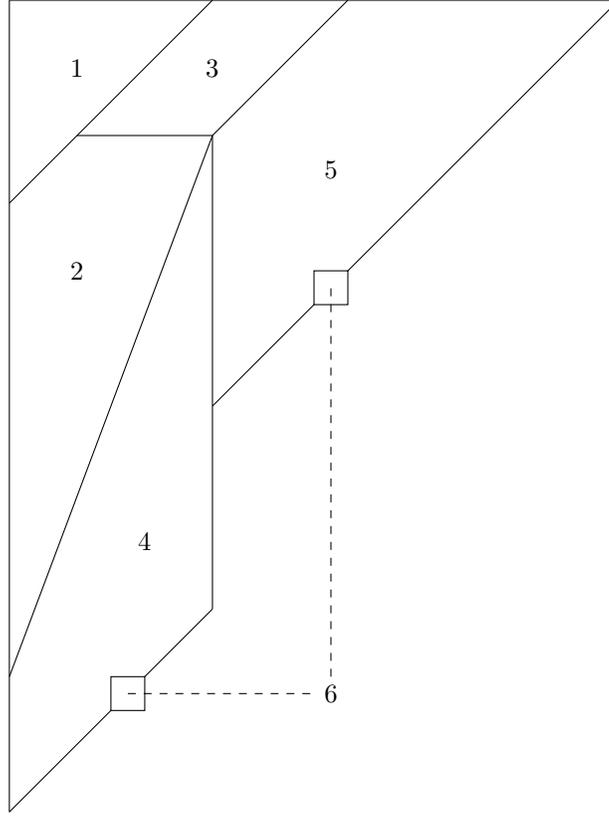
FIGURE 2. The characteristic regions of $\lambda_{a,b}^c$

We now show that this is the most efficient way to construct a tableau satisfying Eq. (2).

Theorem 4.7. *Suppose that $a \neq b$ and $m_a = m_b = m_g = 0$. Let t be a tableau such that $cD_{a,b} \in \mathbb{T}(t)$. Then $t(x,y) \geq \lambda_{a,b}^c(x,y)$ for all x, y.*

Proof. We prove this by induction. The base case, that $t(0,0) \geq 1$, is immediate. We assume that $t(x',y') \geq \lambda_{a,b}^c(x',y')$ for all x', y' satisfying either $x' < x$, $y' \leq y$ or $x' \leq x$, $y' < y$, and we show that $t(x,y) \geq \lambda_{a,b}^c(x,y)$.

We prove this for each region separately. To begin, if (x,y) is in region 1, then $t(x,y) \geq x + y + 1$ because the rows and columns of a tableau are increasing.

Similarly, in regions 2 and 3 we fill column 0 with consecutive integers, which is clearly optimal. If $x > 0$ and $x + y + 1 = a$, then since $m_a = 0$, we see that $t(x,y) > a$. If $t(x,y) < \lambda_{a,b}^c(x,y)$, then the symbols in this region correspond to torsion 3 loops, so we must have $t(x,y) \equiv y - x + 1 \pmod{3}$. It follows that $t(x,y) \geq a + 2 - \alpha(y)$. Otherwise, if $x > 0$ and $x + y + 1 > a$, we must have

$$t(x,y) \geq t(x-1,y) + 1.$$

But if $t(x,y) < \lambda_{a,b}^c(x,y)$, then again, the symbols in this region correspond to torsion 3 loops, and $y - x \not\equiv y - (x-1) + 1 \pmod{3}$. It follows that we may not have

							a+2
						a+1	a+3
					a+3	a+5	
				a+2	a+4		
			a+1	a+3	a+5		
		a+3	a+5				
	a+2	a+4					
a+1	a+3	a+5					
a	a+2	a+4					
a+1	a+3	a+5					
a+2	a+4						

FIGURE 3. Filling Regions 2 and 3 with torsion 3 symbols

equality in the displayed equation above. In other words, $t(x, y) \geq t(x - 1, y) + 2$. Since equality holds for $\lambda_{a,b}^c$, we see that $\lambda_{a,b}^c$ is optimal in these regions.

After filling regions 1,2, and 3, we find the empty box (x, y) that minimizes $x + y$. Because $b + 1$ and $b + 2$ correspond to torsion 2 loops, one of $\{b, b + 1, b + 2\}$ can be placed in this box, and we make the minimal choice. This is clearly optimal. If (x, y) is in region 4 or 5 and does not minimize $x + y$, then since

$$t(x, y) > t(x, y - 1) \geq \lambda_{a,b}^c(x, y - 1) = \lambda_{a,b}^c(x, y) - 1,$$

we see that $t(x, y) \geq \lambda_{a,b}^c(x, y)$.

Finally, if $\lambda_{a,b}^c(x, y) = g$, then $\lambda_{a,b}^c(x - 1, y) = g - 1$. Since $t(x, y) > t(x - 1, y) \geq \lambda_{a,b}^c(x - 1, y)$, we see that $t(x, y) \geq g$ as well. □

Corollary 4.8. *We have*

$$rk(cD_{a,b}) = r(c) := \begin{cases} c & \text{if } c < \ell \\ \left\lceil \frac{3}{2}c - \frac{1}{2}(\ell - 1) \right\rceil & \text{if } \ell \leq c < n \\ 3c - g & \text{if } c \geq n. \end{cases}$$

Proof. By Theorem 2.7, the divisor $cD_{a,b}$ has rank at least r if and only if there exists a tableau t on a rectangle with $r + 1$ columns and $g - 3c + r$ rows such that

$cD_{a,b} \in \mathbb{T}(t)$. The tableau $\lambda_{a,b}^c$ has $r(c)+1$ columns and $g-3c+r(c)$ rows, where $r(c)$ is as defined above. It follows that $\mathrm{rk}(cD_{a,b}) \geq r(c)$.

Now, if $\mathrm{rk}(cD_{a,b}) > r(c)$, then there exists a tableau t with $r(c)+2$ columns and $g-3c+r(c)+1$ rows such that $cD_{a,b} \in \mathbb{T}(t)$. By Theorem 4.7, we have $t(x,y) \geq \lambda_{a,b}^c(x,y)$ for all (x,y). In particular, $t(r(c), g-3c+r(c)-1) \geq g$. This is impossible, because this implies that $t(r(c), g-3c+r(c)) > g$, but there is no symbol larger than g to place in this box. Thus $\mathrm{rk}(cD_{a,b}) \leq r(c)$, and the result follows. $\qquad\square$

We note the following consequence of Corollary 4.8, which shows that the sequence of integers $\mathrm{rk}(cD_{a,b})$ is not convex, as it is in the classical case.

Corollary 4.9. *For $0 \leq i \leq n - \ell$,*

$$rk((\ell+i)D_{a,b}) = \begin{cases} rk((\ell+i-1)D_{a,b}) + 1 & \text{if } i \text{ is odd} \\ rk((\ell+i-1)D_{a,b}) + 2 & \text{if } i \text{ is even.} \end{cases}$$

Proof. This is a direct consequence of Corollary 4.8. $\qquad\square$

We now compute the composite scrollar invariant σ_1.

Theorem 4.10. *We have*

$$\sigma_1(\Gamma, D_{a,b}) = \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

Proof. By Corollary 4.8, we have

$$2(n-1) + 1 - \mathrm{rk}((n-1)D_{a,b}) = 2(n-1) + 1 - \left\lceil \frac{3}{2}(n-1) - \frac{1}{2}(\ell-1) \right\rceil$$

$$= 1 + \left\lfloor \frac{1}{2}(n-1) + \frac{1}{2}(\ell-1) \right\rfloor = \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

Thus, by the definition of σ_1, we have

$$\sigma_1(\Gamma, D_{a,b}) \geq \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

It therefore suffices to show that

$$\mathrm{rk}(cD_{a,b}) \geq 2c + 1 - \left\lfloor \frac{n+\ell}{2} \right\rfloor \text{ for all } c.$$

By Corollary 4.8, if $i > 0$, then

$$\mathrm{rk}((n-i)D_{a,b}) \geq \mathrm{rk}((n-1)D_{a,b}) - 2(i-1) = 2(n-i) + 1 - \left\lfloor \frac{n+\ell}{2} \right\rfloor,$$

and if $i \geq 0$, then

$$\mathrm{rk}((n+i)D_{a,b}) \geq \mathrm{rk}((n-1)D_{a,b}) + 2(i+1) = 2(n+i) + 1 - \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

$\qquad\square$

5. Higher Gonality Generalizations

In this section, we imitate our approach in the trigonal case in order to provide an algorithm for computing the scrollar invariants of a divisor on a k-gonal chain of loops. We begin with a natural generalization of of Proposition 4.3.

Proposition 5.1. *Every tableau on $(g - k + 1) \times 2$ may be obtained by removing $k - 2$ boxes from each column of Λ (as defined in Lemma 4.1) in such a way that, above any row, the number of boxes deleted from the left column of Λ does not exceed the number of boxes deleted from the right column.*

Proof. Consider the result λ of removing $k - 2$ boxes from each column of Λ as described and sliding the remaining boxes together vertically. This forms a rectangle of dimensions $(g - k + 1) \times 2$, and the condition on removed boxes guarantees that the entries in each row are increasing.

It remains to show that every displacement tableau t on λ can be obtained in this way. For any such tableau, note that each column of t must have $g - k + 1 = g - 1 - (k - 2)$ distinct entries, which must be between 1 and g. By the definition of tableau, there may not be a 1 in the first column of t or a g in the zeroth column, so each column contains all but $k - 2$ of the symbols that appear in the corresponding column of Λ. In other words, the entries in each column may be obtained by deleting $k - 2$ of the entries in the corresponding column of Λ. Requiring the entries in each row to increase exactly recovers our condition on the boxes removed, and the result follows. $\square$

Example 5.2. Fig. 4 illustrates this process for the tableau in Example 5.10.
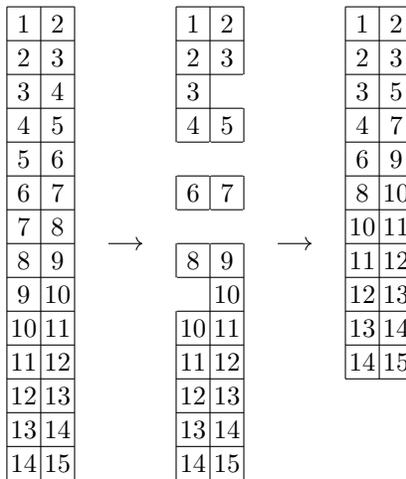
FIGURE 4. Making λ_D from Λ

This construction provides a natural classification of the tableaux corresponding to divisors of degree k and rank 1 on chains of loops. We use similar notation to the trigonal case, denoting by λ_D the tableau obtained in this manner corresponding to a divisor D on Γ. We associate a Dyck word (which we represent with matched sets of parentheses) to each tableau λ_D as follows: delete boxes from Λ to form D,

from top to bottom. As each box is deleted, add a $($ or a $)$ to the end of the word if the box is deleted from the zeroth or first column, respectively.

We say two tableaux are of the same *combinatorial type* if they have the same associated Dyck word. Since it is known that Dyck words are enumerated by the Catalan numbers, the following is immediate.

Corollary 5.3. *The number of combinatorial types of tableaux corresponding to divisors of degree k and rank 1 on chains of loops is equal to the $(k-2)^{nd}$ Catalan number, C_{k-2}.*

This result has significant computational implications. In the trigonal case, all tableaux have the same combinatorial type, which allows us to define the tableau $\lambda_{a,b}^c$ representing $cD_{a,b}$ in Definition 4.6 with a (relatively) small number of cases. In higher gonality cases, the tableau λ_D^c depends on the *i-blocks* of $\underline{m}$, which we now define.

Definition 5.4. Let $i > 1$ be an integer. A collection $\{a+1, \ldots, b-1\}$ of consecutive integers in $\{1, \ldots, g\}$ is called an *i-block* if

 (1) i is a multiple of m_j for $a < j < b$, and
 (2) i is not a multiple of m_a or m_b.

Each combinatorial type of λ_D corresponds to a different distribution of i-blocks. In particular, if the symbol i appears only once in the tableau λ_D, then the ith torsion torsion order m_i is arbitrary. Otherwise, the ith torsion order m_i must divide

$$2 + \#\,(\,\text{symbols } < i \text{ missing from column } 0)$$
$$-\#\,(\,\text{symbols } < i \text{ missing from column } 1)\,.$$

Definition 5.5. Let λ_D be a rectangular tableau of dimensions $(g - k + 1) \times 2$ containing each of the symbols in $\{1, \ldots, g\}$. We say that the torsion profile $\underline{m}$ is *nondegenerate* if it satisfies the following conditions:

 (1) if i appears only once in the tableau λ_D, then $m_i = 0$, and
 (2) otherwise,

$$m_i = 2 + \#\,(\,\text{symbols } < i \text{ missing from column } 0)$$
$$-\#\,(\,\text{symbols } < i \text{ missing from column } 1)\,.$$

Corollary 5.3 implies that the number of combinatorial types grows exponentially with respect to k. It is therefore unfeasible to describe λ_D^c for every combinatorial type. Instead, we use the tools developed in Section 4 to construct λ_D^c recursively for each value of c. Recording the widths of the tableaux λ_D^c is equivalent to recording the rank sequence of our tropical divisor, and is therefore sufficient to calculate the sequence of composite scrollar invariants.

As in the trigonal case, Theorem 2.7 gives that the divisor cD has rank at least r if and only if there exists a tableau λ_D^c on a rectangle with $r + 1$ columns and $g - kc + r$ rows such that $cD \in \mathbb{T}(\lambda_D^c)$. Again, by Lemma 2.9, $cD \in \mathbb{T}(\lambda_D^c)$ if and only if, whenever $\lambda_D^c(x, y) = i$, we have

$$(3) \qquad\qquad\qquad y - x \equiv \xi_i^c \pmod{m_i}\,.$$

To produce the largest possible $\underline{m}$-displacement tableau satisfying this congruence condition, we make use of some original SAGE code available at

$$\texttt{https://github.com/kalilajo/numberboxes}.$$

In the remainder of this section, we describe the algorithm implemented by this code, prove that the resulting tableaux are optimal, and provide a few corollaries.

5.1. Algorithm for constructing λ_D^c from λ_D^{c-1}.

Definition 5.6. For $c \geq 2$, let

$$j := k - (\mathrm{rk}(cD) - \mathrm{rk}((c-1)D)).$$

In other words, λ_D^c has j fewer rows and $k - j$ more columns than λ_D^{c-1}.

Note that identifying $j \in \{1, \ldots, k-1\}$ is the overall goal of our calculation. Given λ_D^{c-1}, we construct λ_D^c recursively as follows.

Step 1: Set $j = 1$. We begin by setting $j = 1$, and we attempt to construct λ_D^c so that it has j fewer rows and $k - j$ more columns than λ_D^{c-1}.

Step 2: Start with the diagonal $x + y = 0$. To construct λ_D^c, we "traverse" each diagonal defined by fixing the sum of the coordinates, beginning with $x + y = 0$.

Step 3: Traverse the diagonal. When traversing a diagonal, we start with its leftmost box. Each time we arrive at a new box (x, y), we fill it with the smallest $s \in \{1, \ldots, g\}$ that is larger than both the entry $\lambda_D^c(x, y-1)$ above it and the entry $\lambda_D^c(x-1, y)$ to the left of it, and such that Eq. (3) is satisfied. If there is no value of s such that these conditions hold, we increase the value of j by 1 and return to Step 2.

If we fill the box (x, y), we proceed to the box $(x+1, y-1)$ above and to the right of the current box, along the same diagonal. If the box (x, y) is the rightmost box on this diagonal, we increase the sum $x + y$ by 1 and repeat Step 3. If (x, y) is the bottom right corner of the rectangle, terminate the algorithm and output the rectangular tableau λ_D^c.

5.2. Verifying the algorithm.

We apply this algorithm recursively to find the largest tableau λ_D^c such that $cD \in \mathbb{T}(\lambda_D^c)$ for each value of c. It remains to show the tableaux generated by this algorithm are optimal.

Proposition 5.7. *Suppose that the symbols removed to form λ_D as in Proposition 5.1 are distinct. Let t be a tableau such that $cD \in \mathbb{T}(t)$. Then $t(x, y) \geq \lambda_D^c(x, y)$ for all x, y.*

Proof. As in the proof of Theorem 4.7, we proceed by induction. The base case, $t(0, 0) \geq 1$ is again immediate. We assume that $t(x', y') \geq \lambda_D^c(x', y')$ for all x', y' such that either $x' < x$, $y' \leq y$ or $x' \leq x$, $y' < y$ and show that $t(x, y) \geq \lambda_D^c(x, y)$. By construction, $\lambda_D^c(x, y)$ is the smallest symbol greater than both $\lambda_D^c(x-1, y)$ and $\lambda_D^c(x, y-1)$ that satisfies Eq. (3). Our inductive hypothesis implies that $t(x, y)$ must satisfy these conditions as well. We must therefore have $t(x, y) \geq \lambda_D^c(x, y)$. □

We make a simple observation on the output of our algorithm. We show that a row of λ_D^c contains only every $(i-1)^{st}$ symbol in an i-block.

Lemma 5.8. *Suppose that the torsion profile $\underline{m}$ is nondegenerate. If $\lambda_D^c(x, y)$ and $\lambda_D^c(x, y) + i - 1$ are in the same i-block, then*

$$\lambda_D^c(x+1, y) \geq \lambda_D^c(x, y) + i - 1.$$

Proof. By definition, we have

$$y - x \equiv \xi_{\lambda_D^c(x,y)} \equiv c\xi_{\lambda_D(x,y)}^1 - (c-1)(i-1) \pmod{i}.$$

Since $\underline{m}$ is nondegenerate and $\lambda_D(x,y)$ and $\lambda_D(x,y)+i-1$ are in the same i-block, we see that

$$\xi_{\lambda_D(x,y)+j}^1 = \xi_{\lambda_D(x,y)}^1 + j \text{ for all } 0 \leq j \leq i-1,$$

so

$$\xi_{\lambda_D(x,y)+j}^c \equiv \xi_{\lambda_D(x,y)}^c + j \pmod{i}$$

for all j in the same range. It follows that $i-1$ is the smallest value of j such that

$$\xi_{\lambda_D^c(x,y)+j} \equiv \xi_{\lambda_D^c(x,y)} - 1 \pmod{i}.$$

We therefore see that $\lambda_D^c(x+1,y) \geq \lambda_D^c(x,y) + i - 1$. $\qquad\square$

As a consequence, we see that there is a torsion profile that maximizes the composite scrollar invariants. The torsion profile below corresponds to the tableau where the symbols $g - k + 2, \ldots, g$ are missing from column zero, and the symbols $1, \ldots, k-1$ are missing from column one. We note that this torsion profile has been used in several papers to examine the behavior of general curves of gonality k [Pfl17a, JR17, CPJ19]. Corollary 5.9 provides further evidence that this chain of loops behaves like a general curve of gonality k, as it has the scrollar invariants of a general curve.

Corollary 5.9. *Suppose*

$$\underline{m} = (0, \ldots, 0, k, \ldots, k, 0, \ldots, 0).$$

Then $rk(cD) = c$ for all c such that $g > c(k-1)$. In other words, we have

$$\sigma_j(\Gamma, D) = \left\lceil \frac{j(g+k-1)}{k-1} \right\rceil \text{ for all } j.$$

Proof. Suppose that λ_D^c has more than $c+1$ columns. By Lemma 5.8, $\lambda_D^c(c+1,0) \geq (c+1)(k-1)$. It follows that

$$\lambda_D^c(c+1, g - c(k-1)) \geq g - c(k-1) + (c+1)(k-1) = g + k - 1 > g,$$

which is impossible. It follows that λ_D^c has at most $c+1$ columns, and $rk(cD) = c$. $\qquad\square$

On the other hand, if the torsion profile is more exotic, then the composite scrollar invariants can vary in interesting ways. We illustrate this phenomenon using an example.

Example 5.10. Let $g = 15$, $k = 5$, and let λ_D be the tableau constructed in Fig. 4 by removing the symbols 5, 7, and 9 from the zeroth column and 4, 6, and 8 from the first column. The output of the SAGE code can be seen in Fig. 9. We reproduce these results manually by using the algorithm in Definition 5.6 as follows.

First, we build λ_D^2 (labeled 2D in the figure) from λ_D. We naively assume λ_D^2 has $j = 1$ fewer rows and more columns than λ_D. We traverse and fill the diagonals as in steps 2 and 3 of the algorithm.

While doing this, we may only place symbol s in box (x, y) if Eq. (3) is satisfied; we list the relevant values in Fig. 5. Using this data and Eq. (3), we traverse and fill the diagonals of a 10×6 tableau as we are able. The result is shown in Fig. 6.

We see that this attempt was unsuccessful, as there were not enough symbols to fill the whole tableau. We therefore repeat this process with a tableau of dimensions

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ξ_s^2	0	1	0	3	-2	3	-2	3	-2	1	0	1	0	1	4
m_s	0	2	2	0	0	0	0	0	0	2	2	2	2	2	0

FIGURE 5. Relevant data for placing s in λ_D^2

1	2	3	10	11	12
2	3	10	11	12	13
3	10	11	12	13	14
4	11	12	13	14	
11	12	13	14		
12	13	14			
13	14	15			
14					

FIGURE 6. Attempting to build λ_D^2 with $j = 1$

9×5, that is, assuming $j = 2$. This is similarly unsuccessful, as is letting $j = 3$. Both tableaux are shown in Fig. 7. Note that each tableau is the restriction of the previous one to a rectangle of smaller dimensions. Specifically, each rectangle has one fewer row and one fewer column than the previous one. Our procedure restricts to smaller and smaller rectangles until every box is filled.

1	2	3	10	11
2	3	10	11	12
3	10	11	12	13
4	11	12	13	14
11	12	13	14	
12	13	14		
13	14	15		
14				

1	2	3	10
2	3	10	11
3	10	11	12
4	11	12	13
11	12	13	14
12	13	14	
13	14	15	
14			

FIGURE 7. Attempting to build λ_D^2 with $j = 2$ and $j = 3$

Finally, when $j = 4$, we succeed in building the rectangular tableau shown in Fig. 8. We label this tableau by $2D$ in Fig. 9. We then repeat this process from the beginning to obtain the tableaux λ_D^3 and λ_D^4.

For the benefit of the reader, we have chosen an example where the genus is relatively small in comparison to the gonality. Because of this, the tropical rank sequence happens to be convex. In examples of larger genus, this is typically not the case.

1	2	3
2	3	10
3	10	11
4	11	12
11	12	13
12	13	14
13	14	15

FIGURE 8. λ_D^2, attained when $j = 4$

```
The genus is:15
Enter a_1 through a_k-2 as a list of numbers separated by spaces:4 6 8
Enter b_1 through b_k-2 as a list of numbers separated by spaces:5 7 9
_m_= [2, 2, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 0]
k= 5
D
  1  2
  2  3
  3  5
  4  7
  6  9
  8 10
 10 11
 11 12
 12 13
 13 14
 14 15
2 D
  1  2  3
  2  3 10
  3 10 11
  4 11 12
 11 12 13
 12 13 14
 13 14 15
3 D
  1  2  3 11 12
  2  3 11 12 13
  3  8 12 13 14
  4 10 13 14 15
4 D
  1  2  3 10 11 12 13 14
  2  3  8 11 12 13 14 15

The rank sequence is: [0, 1, 2, 4, 7]
The scrollar invariants are: {0: 0, 1: 3, 2: 7, 3: 13, 4: 19}
```

FIGURE 9. A sample calculation in SAGE

References

[Bak08]   Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008. With an appendix by Brian Conrad.

[BF11]   Matthew Baker and Xander Faber. Metric properties of the tropical Abel-Jacobi map. *J. Algebraic Combin.*, 33(3):349–381, 2011.

[CDPR12]   Filip Cools, Jan Draisma, Sam Payne, and Elina Robeva. A tropical proof of the Brill-Noether theorem. *Adv. Math.*, 230(2):759–776, 2012.

[CPJ19]   Kaelin Cook-Powell and David Jensen. Components of Brill-Noether loci for curves with fixed gonality. preprint arXiv:1907.08366v1, 2019.

[JP14]   David Jensen and Sam Payne. Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem. *Algebra Number Theory*, 8(9):2043–2066, 2014.

[JP16]   David Jensen and Sam Payne. Tropical independence II: The maximal rank conjecture for quadrics. *Algebra Number Theory*, 10(8):1601–1640, 2016.

[JP17]   David Jensen and Sam Payne. Combinatorial and inductive methods for the tropical maximal rank conjecture. *J. Combin. Theory Ser. A*, 152:138–158, 2017.

[JR17]   David Jensen and Dhruv Ranganathan. Brill-Noether theory for curves of a fixed gonality. preprint arXiv:1701.06579, 2017.

[Pfl17a]   Nathan Pflueger. Brill-Noether varieties of k-gonal curves. *Adv. Math.*, 312:46–63, 2017.

[Pfl17b]   Nathan Pflueger. Special divisors on marked chains of cycles. *J. Combin. Theory Ser. A*, 150:182–207, 2017.