 Date: ________

Chapter Goals:

- Understand how to use basic summation formulas to evaluate more complex sums.
- Understand how to compute limits of rational functions at infinity.
- Understand how to use the basic summation formulas and the limit rules you learned in this chapter to evaluate some definite integrals.

Assignments: Assignment 19 Assignment 20

The rules and formulas given below allow us to compute fairly easily Riemann sums where the number n of subintervals is rather large. We can also get compact and manageable expressions for the sum so that we can readily investigate what happens as n approaches infinity.

► **Summation rules:**

[**sr**$_1$] $\displaystyle\sum_{k=1}^{n} c = n\,c$ [**sr**$_2$] $\displaystyle\sum_{k=1}^{n}(c\,a_k) = c\sum_{k=1}^{n} a_k$ [**sr**$_3$] $\displaystyle\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Note: The summations rules are nothing but the usual rules of arithmetic rewritten in the Σ notation. For example, [**sr**$_2$] is nothing but the distributive law of arithmetic

$$c\,a_1 + c\,a_2 + \cdots + c\,a_n = c\,(a_1 + a_2 + \cdots + a_n)$$

[**sr**$_3$] is nothing but the commutative law of addition

$$(a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_n \pm b_n) = (a_1 + a_2 + \cdots + a_n) \pm (b_1 + b_2 + \cdots + b_n)$$

► **Summation formulas:**

[**sf**$_1$] $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ [**sf**$_2$] $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: In the case of [**sf**$_1$], let S denote the sum of the integers $1, 2, 3, \ldots, n$. Let us write this sum S twice: we first list the terms in the sum in increasing order whereas we list them in decreasing order the second time:

$$\begin{array}{ccccccccc} S & = & 1 & + & 2 & + & \cdots & + & n \\ S & = & n & + & n-1 & + & \cdots & + & 1 \end{array}$$

If we now add the terms along the vertical columns, we obtain

$$2S = \underbrace{(n+1) + (n+1) + \quad\cdots\quad + (n+1)}_{n \text{ times}} = n(n+1).$$

This gives our desired formula, once we divide both sides of the above equality by 2.

In the case of [**sf**$_2$], let S denote the sum of the integers $1^2, 2^2, 3^2, \ldots, n^2$. The *trick* is to consider the sum $\displaystyle\sum_{k=1}^{n}[(k+1)^3 - k^3]$. On the one hand, this new sum collapses to

$$(\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \cdots + (\cancel{n^3} - \cancel{(n-1)^3}) + ((n+1)^3 - \cancel{n^3}) = (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using our summation rules together with [**sf**$_1$] gives us

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = \sum_{k=1}^{n}[3k^2 + 3k + 1] = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3S + 3\,\frac{n(n+1)}{2} + n$$

Equating the right hand sides of the above identities gives us: $3S + 3\,\dfrac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n.$

If we solve for S and properly factor the terms, we obtain our desired expression.

$\boxed{\textbf{More summation rules:}}$ The next formulas can be verified in a sequential order using the same type of trick used in the proof for [$\mathbf{sf_2}$]. The proofs get increasingly more tedious.

$$[\mathbf{sf_3}] \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad\qquad [\mathbf{sf_4}] \quad \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$\boxed{\textbf{Example 1:}}$ Evaluate the sum $\displaystyle\sum_{k=1}^{9}(5k+8)$.

$$\sum_{k=1}^{9}(5k+8) = \underbrace{\sum_{k=1}^{9}(5k) + \sum_{k=1}^{9} 8}_{\text{use property } [sr_3]} = \underbrace{5\sum_{k=1}^{9} k}_{\text{use property } [sr_2]} + \sum_{k=1}^{9} 8 = 5 \cdot \underbrace{\frac{9\cdot(9+1)}{2}}_{\text{use } [sf_1]} + \underbrace{9\cdot 8}_{\text{use } [sr_1]}$$

$$= 5\cdot 45 + 72$$
$$= \boxed{297} \longleftarrow$$

$\boxed{\textbf{Example 2:}}$ Evaluate the sum $\displaystyle\sum_{k=1}^{8}(5k^2+8k+1)$.

$$= 5\left(\sum_{k=1}^{8} k^2\right) + 8\left(\sum_{k=1}^{8} k\right) + \sum_{k=1}^{8} 1$$

$$= 5 \cdot \frac{8\cdot(8+1)(16+1)}{6} + 8\cdot\frac{8\cdot(8+1)}{2} + 8\cdot 1$$

$$= 5\cdot 204 + 8\cdot 36 + 8$$

$$= \boxed{1{,}316} \longleftarrow$$

$\boxed{\textbf{Example 3:}}$ Evaluate the sum $\displaystyle\sum_{k=7}^{12}(k+1)$.

$$= \sum_{k=7}^{12} k + \underbrace{\sum_{k=7}^{12} 1}_{\text{we add 1 exactly } (12-7+1) \text{ times}}$$

trick

$$= \left(\sum_{k=1}^{12} k - \sum_{k=1}^{6} k\right) + 6\cdot 1$$

$$= \left(\frac{12\cdot 13}{2} - \frac{6\cdot 7}{2}\right) + 6 = (78-21)+6 = \boxed{63} \longleftarrow$$

Example 4: Evaluate the sum $\sum_{k=3}^{100}(2+5k)$.

$$\Rightarrow$$

$$= \left(\sum_{k=3}^{100} 2\right) + 5\left(\sum_{k=3}^{100} k\right)$$

$$= 2\cdot(100-3+1) + 5\left(\sum_{k=1}^{100} k - \sum_{k=1}^{2} k\right)$$

$$= 2\cdot 98 + 5\left(\frac{100\cdot(101)}{2} - \frac{2\cdot 3}{2}\right)$$

$$= 2\cdot 98 + 5\,(5050-3) = \boxed{25{,}431} \leftarrow$$

Example 5: Evaluate the sum $1+5+10+15+20+\cdots+245$.

Rewrite the sum as

$$= 1 + 5(1+2+3+4+\cdots+49)$$

$$= 1 + 5\cdot\left(\sum_{k=1}^{49} k\right) = 1 + 5\cdot\frac{49\cdot(50)}{2}$$

$$= 1 + 5\cdot 1{,}225 = \boxed{6{,}126} \leftarrow$$

Example 6: Evaluate the sum $24+27+30+33+36+\cdots+90$.

Observe that the terms are of the form:

$$3\cdot 8 + 3\cdot 9 + 3\cdot 10 + 3\cdot 11 + 3\cdot 12 + \cdots + 3\cdot 30$$

$$= 3\,(8+9+10+11+12+\cdots+30)$$

$$= 3\left(\sum_{k=8}^{30} k\right) = 3\left(\sum_{k=1}^{30} k - \sum_{k=1}^{7} k\right)$$

$$= 3\cdot\left(\frac{30\cdot 31}{2} - \frac{7\cdot 8}{2}\right) = 3\,(465-28) = \boxed{1{,}311} \leftarrow$$

Example 7: Evaluate the sum $-5-4-3-2-1+0+1+2+3+\cdots+30$.

Observe that we can rewrite the sum as

$$-(5+4+3+2+1)+0+(1+2+3+\cdots+30)$$

$$=-\left(\sum_{k=1}^{5} k\right)+0+\left(\sum_{k=1}^{30} k\right)$$

$$=-\frac{5\cdot 6}{2}+0+\frac{30\cdot 31}{2}=-15+0+465=\boxed{450}\leftarrow$$

Example 8: If we write $\displaystyle\sum_{k=1}^{n} k^4=\frac{4n(n+1)(2n+1)(3n^2+3n-1)}{A}$. What is the value of A?
(**Hint:** Substitute a convenient value of n to help you evaluate A.)

The formula has to be true for any n. Let's substitute for example $n=1$. We get

$$1^4=\frac{4\cdot(1)(1+1)(2\cdot 1+1)(3\cdot 1^2+3\cdot 1-1)}{A}$$

or $\quad 1=\dfrac{4(2)(3)(5)}{A}\quad$ or $\quad\boxed{A=120}\leftarrow$

Example 9: If $\displaystyle\sum_{k=1}^{n}(k^2-k)=\frac{2n(n+1)(n-1)}{A}$, find A.

As in the previous problem the formula has to be true for any n.

But for $n=1$ we get $\quad 1^2-1=\dfrac{2\cdot 1(1+1)(1-1)}{A}$

so $\quad 0=\dfrac{0}{A}\quad$ it doesn't give us any info

Try $\boxed{n=2}$ We get

$$(1^2-1)+(2^2-2)=\frac{2\cdot(2)(2+1)(2-1)}{A}\quad\text{or}\quad 2=\frac{12}{A}$$

$$\therefore\ \boxed{A=6}\leftarrow$$

▶ **Limits at infinity:** We need to be able to evaluate limits of the form $\boxed{\lim_{n\to\infty} \frac{p(n)}{q(n)},}$ where $p(n)$ and $q(n)$ are both polynomials in n. E.g., how does $\lim_{n\to\infty} \frac{n^3 - 3n^2 + 2n - 1}{5n^3 + 4n^2 + 3n + 1}$ behave?

There is a general principle that makes computing these limits easy. The **idea** is that, for large values of n, the term with the highest power of n has the most influence on the behavior of the polynomial. In other words, when n is very large, the term with the highest power dominates the other terms.

Theorem: Let $p(n)$ and $q(n)$ be polynomials. Then $\lim_{n\to\infty} \frac{p(n)}{q(n)} = \lim_{n\to\infty} \frac{\text{highest order term of } p(n)}{\text{highest order term of } q(n)}$.

Example 10: Find the limit as n tends to infinity.

$$\lim_{n\to\infty} \frac{8n^2 + 7n + 9}{4n^2 + 2n + 1}$$

We already encountered limits at infinity in Chapter 3

$$= \lim_{n\to\infty} \frac{8n^2}{4n^2} = \frac{8}{4} = \boxed{2}$$

Example 11: Find the limit as n tends to infinity.

$$\lim_{n\to\infty} \frac{(2n+1)^2}{5n^2 + 2n + 1}$$

$$= \lim_{n\to\infty} \frac{4n^2 + 4n + 1}{5n^2 + 2n + 1} = \lim_{n\to\infty} \frac{4n^2}{5n^2} = \lim_{n\to\infty} \frac{4}{5} = \boxed{\frac{4}{5}}$$

Example 12: Find the limit as n tends to infinity.

$$\lim_{n\to\infty} \frac{n^4 + n^2 + 13}{n^3 + 8n + 9}$$

$$\lim_{n\to\infty} \frac{n^4}{n^3} = \lim_{n\to\infty} n = +\infty \quad \text{or} \quad \underline{\text{DNE}}$$

▶ **Computing limits of Riemann sums:** Let f be a positive valued function defined on an interval $[a, b]$. In Chapter 8 we started studying the problem of finding the area of the region in the xy-plane underneath the graph of the function f and lying above the x-axis. We first partitioned the interval $[a, b]$ into n subintervals of lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, respectively. For $k = 1, \ldots, n$ we picked representative points $p_1, p_2, \ldots, p_n$ in each of the n subintervals in which $[a, b]$ has been partitioned. We then formed the Riemann sum

$$\sum_{k=1}^{n} f(p_k) \cdot \Delta x_k.$$

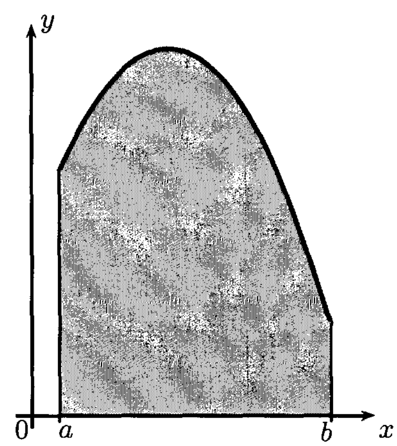
The definite integral of f from a to b was defined as $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(p_k) \cdot \Delta x_k$, if the limit exists.

Alternatively, if we set $\|P\| = \max\limits_{1\le i\le n} \{\Delta x_i\}$ we can write the above limit as $\displaystyle\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(p_k)\cdot\Delta x_k$.
This means that we are taking the limit as the length of the longest subinterval of the partition of $[a, b]$ is approaching zero.

As we observed in Chapter 8, it is computationally easier to partition the interval $[a, b]$ into n subintervals of equal length: $\Delta x = (b - a)/n$. If we then use the right endpoints of this regular partition we have seen that:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k \cdot \Delta x) \cdot \Delta x.$$

Definite integrals and areas:

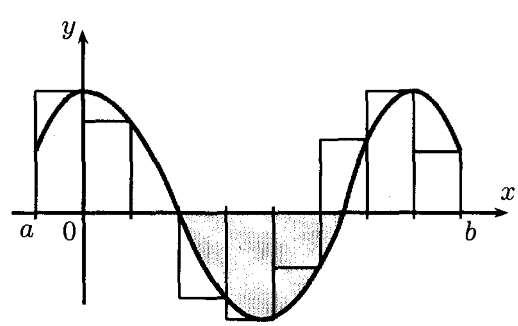


We stress again the fact that if the function f takes on only positive values then the definite integral is nothing but the area of the region underneath the graph of f, lying above the x-axis, and bounded by the vertical lines $x = a$ and $x = b$.

Distance traveled by an object:

If the positive valued function under consideration is the velocity $v(t)$ of an object at time t, then the area underneath the graph of the velocity function and lying above the t-axis represents the total distance traveled by the object from $t = a$ to $t = b$.

What if the function takes on also negative values?



If f happens to take on both positive and negative values, then the Riemann sum is the sum of the areas of rectangles that lie above the x-axis and the negatives of the areas of rectangles that lie below the x-axis.

Passing to the limit, we obtain that, in general, a definite integral can be interpreted as a difference of areas:

$$\boxed{\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]}$$

Example 13: Evaluate the limit as n tends to infinity. Note that you will have to use the summation formulas to first simplify.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k+9}{n}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \left(\frac{1}{n} \left(\sum_{k=1}^{n} (k+9) \right)\right)\right] = \lim_{n\to\infty} \left[\frac{1}{n^2} \left(\left(\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} 9\right)\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \left(\frac{n(n+1)}{2} + 9\cdot n \right)\right] = \lim_{n\to\infty} \left[\frac{1}{n^2} \cdot \left(\frac{n(n+1)+18n}{2} \right)\right]$$

$$= \lim_{n\to\infty} \frac{n^2+n+18n}{2n^2} = \lim_{n\to\infty} \frac{n^2+19n}{2n^2} = \lim_{n\to\infty} \frac{n^2}{2n^2} = \boxed{\frac{1}{2}}$$

Example 14: Evaluate the limit as n tends to infinity. Note that you will have to use the summation formulas to first simplify.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(k\frac{7}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \left(\frac{7^2}{n^2} k^2 \right) = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \frac{49}{n^2} \sum_{k=1}^{n} k^2 \right) =$$

$$= \lim_{n\to\infty} \frac{49}{n^3} \cdot \left(\sum_{k=1}^{n} k^2 \right) = \lim_{n\to\infty} \frac{49}{n^3} \cdot \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \frac{49 \cdot [2n^3 + \cdots]}{6n^3} = \lim_{n\to\infty} \frac{49 \cdot 2n^3}{6n^3} = \boxed{\frac{49}{3}}$$

OBSERVE THAT $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(k \cdot \frac{7}{n}\right)^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{7} \left(k \cdot \underbrace{\frac{7}{n}}_{\Delta x}\right)^2 \cdot \underbrace{\frac{7}{n}}_{\Delta x} \right) = \boxed{\int_0^7 \frac{1}{7} x^2 \, dx}$

Example 15: The integral $\int_0^5 x^2\,dx$ is computed as the limit of the sum $\sum_{k=1}^{n} \frac{A}{n}\left(k\frac{A}{n}\right)^2$. What value of A must appear in the sum?

A must be "b − a" ie. $5 - 0 = \boxed{5 = A}$ Good practice!

So $\int_0^5 x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{5}{n} \cdot \left(k\frac{5}{n}\right)^2 = \lim_{n\to\infty} \frac{5^3}{n^3} \sum_{k=1}^{n} k^2 =$

$$= \lim_{n\to\infty} \frac{5^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{5^3(2n^3+\cdots)}{6n^3} = \frac{5^3 \cdot 2}{6} = \boxed{\frac{125}{3}}$$

Example 16: The integral $\int_7^{10} x^2\,dx$ is computed as the limit of the sum $\sum_{k=1}^{n} \frac{3}{n}\Big(\underbrace{A + k\,\frac{3}{n}}_{x_k}\Big)^2$.
What value should be used as A?

$$x_k = 7 + k \cdot \Delta x = 7 + k \cdot \frac{10-7}{n} = 7 + k\,\frac{3}{n}$$

$k = 1, \dots, n$ $\qquad \therefore \boxed{A = 7}$

Example 17: The limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n}\left(\frac{n+k}{n}\right)^2$$

is obtained by applying the definition of the integral to

$$\int_1^2 f(x)\,dx.$$

What is the function $f(x)$?

Rewrite the limit as

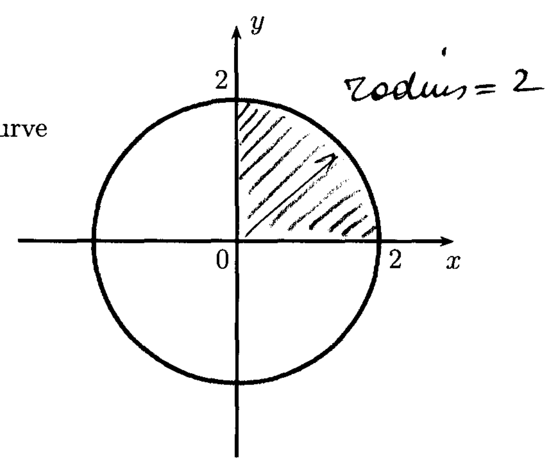
$$\lim_{n\to\infty} \sum_{k=1}^{n} 3 \cdot \Big(\underbrace{1 + k \cdot \frac{1}{n}}_{1+k\,\Delta x}\Big)^2 \cdot \underbrace{\frac{1}{n}}_{\Delta x}$$

$$= \int_1^2 3x^2\,dx \qquad \text{So} \qquad \boxed{f(x) = 3x^2}$$

Example 18: Evaluate the limit

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \sqrt{4 - \left(k\,\frac{2}{n}\right)^2}$$

Hint: What limit would you compute to evaluate the area under the curve $y = \sqrt{4 - x^2}$ for x between 0 and 2 ? What is this area in geometric terms?



We are computing

$$\int_0^2 \sqrt{4 - x^2}\,dx$$

= area of the circle in the first quadrant

$$= \frac{1}{4} \cdot \underbrace{(\pi \cdot 2^2)}_{\text{area of the whole circle}} = \pi$$

$$y^2 + x^2 = 4$$

or

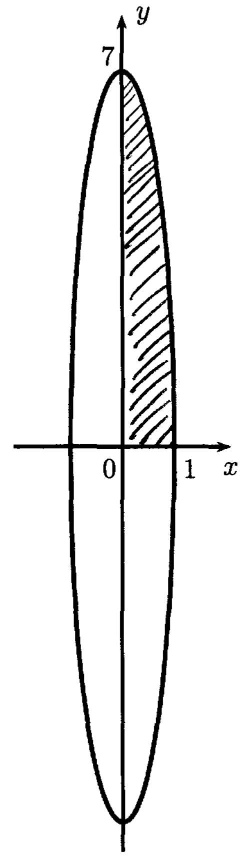
$$y = \pm\sqrt{4 - x^2}$$

Example 19:

Given that the area of the ellipse $49x^2 + y^2 = 49$ is 7π, evaluate the integral

$$\int_0^1 \sqrt{49 - 49x^2}\, dx$$

(**Hint:** Think of the definite integral as an area.)



$\int_0^1 \sqrt{49-49x^2}\,dx$ represents the area under the graph of the ellipse in the 1^{st} quadrant

$$\therefore \quad = \frac{1}{4} \text{ (total area of the ellipse)}$$

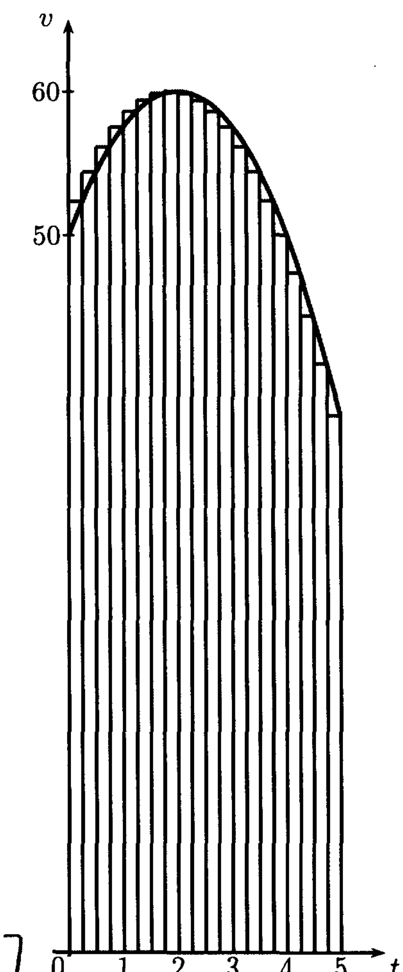
$$= \frac{7\pi}{4}$$

Example 20: A car is traveling due east.

Its velocity (in miles per hour) at time t hours is given by

$$v(t) = -2.5t^2 + 10t + 50.$$

How far did the car travel during the first five hours of the trip?



Riemann sum with $n = 20$ terms

distance traveled $= \int_0^5 (-2.5t^2 + 10t + 50)\, dt$

points in subdivision $x_k = 0 + k\frac{5-0}{n} = k\frac{5}{n}$

$$\text{so} = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\left(-2.5\left(k\frac{5}{n}\right)^2 + 10\cdot\left(k\frac{5}{n}\right) + 50\right)}_{v(x_k)} \cdot \underbrace{\frac{5}{n}}_{\Delta x}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\frac{2.5\cdot 25}{n^2}\cdot k^2 + \frac{50}{n}\cdot k + 50\right)\cdot\frac{5}{n}$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n}\left(-\frac{2.5\cdot125}{n^3}k^2 + \frac{250}{n^2}k + \frac{250}{n}\right)\right] =$$

$$= \lim_{n\to\infty} \left[-\frac{2.5\cdot125}{n^3}\sum_{k=1}^{n} k^2 + \frac{250}{n^2}\sum_{k=1}^{n} k + \frac{250}{n}\sum_{k=1}^{n}(1)\right]$$

$$= \lim_{n\to\infty} \left[-\frac{2.5\cdot125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{250}{n^2}\cdot\frac{n(n+1)}{2} + \frac{250}{n}\cdot n\right]$$

$$= \frac{-2.5\cdot125\cdot2}{6} + \frac{250}{2} + 250 \;\; \cancel{129} = \boxed{270.8\overline{3}}$$