Contents

CHAPTER 11

Combined Parametric Motions

Sometimes we are interested in the motion of a point P which has a simple motion relative to some other point C, which, in turn, also is moving in a particular way. It is not hard to write parametric formulas $x(t)$, $y(t)$ for the point P, if we proceed carefully in a step-by-step manner.

EXAMPLE 11.1. (The baton)

Suppose we want to describe the motion of a tip of a baton as it rotates after being thrown into the air. More precisely, suppose that at time $t = 0$ a 2-foot long baton is situated vertically with the tip we are interested in at height 6 ft above ground level and the other tip at height 4 ft. At that moment the baton is tossed at 20 ft/sec at an angle of 45° above the horizontal, and it is also set spinning at 3 rev/sec clockwise in the plane of its trajectory. (It could have been set spinning in a different plane, but that would have made this into a *three*-dimensional problem, and that would be too complicated for our present purposes.)

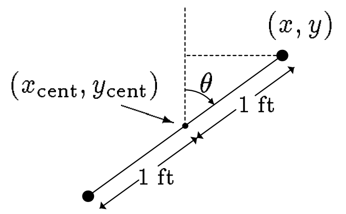
To set up the parametric equations for the motion of the tip of the baton, the basic procedure is to divide the motion into two parts: **the motion of the center of the baton** and **the rotation of the tip around the center**. The first of these is a falling body problem, and the second is a circular motion problem. We shall take the y-axis to be the vertical line along which the baton is situated at time $t = 0$, and we shall take the x-axis to be the line on the ground under the baton's trajectory.

According to a principle of physics, the center of the baton follows the same path as a small object (say, a ball) that is thrown with the same initial velocity. In other words, the first half of our procedure — the motion of the center of the baton — is like Example 10.6 of the last chapter. Let $(x_{\text{cent}}(t), y_{\text{cent}}(t))$ denote the x- and y-coordinates of the center of the baton at time t. Repeating the procedure used in Example 10.6 — with $g = 32$ ft/sec^2, $v_{0,\text{horiz}} = 20\cos 45° = 14$ ft/sec (rounded to the nearest ft/sec), $v_{0,\text{vert}} = 20\sin 45° = 14$ ft/sec — we obtain:

$$x_{\text{cent}}(t) = 14t, \qquad y_{\text{cent}} = -16t^2 + 14t + 5.$$

(Notice that $y_0 = 5$ ft, because the initial location of the center is half-way between 4 and 6.)

Now let $(x(t), y(t))$ denote the x- and y-coordinates of the tip of the baton at time t. The second half of our procedure consists in determining how much $x(t)$ differs from $x_{\text{cent}}(t)$ and how much $y(t)$ differs from $y_{\text{cent}}(t)$, i.e., how far the tip is from the center in the x- and y-directions. Let θ be the angle in radians through which the baton has spun at time t. We have the following diagram:



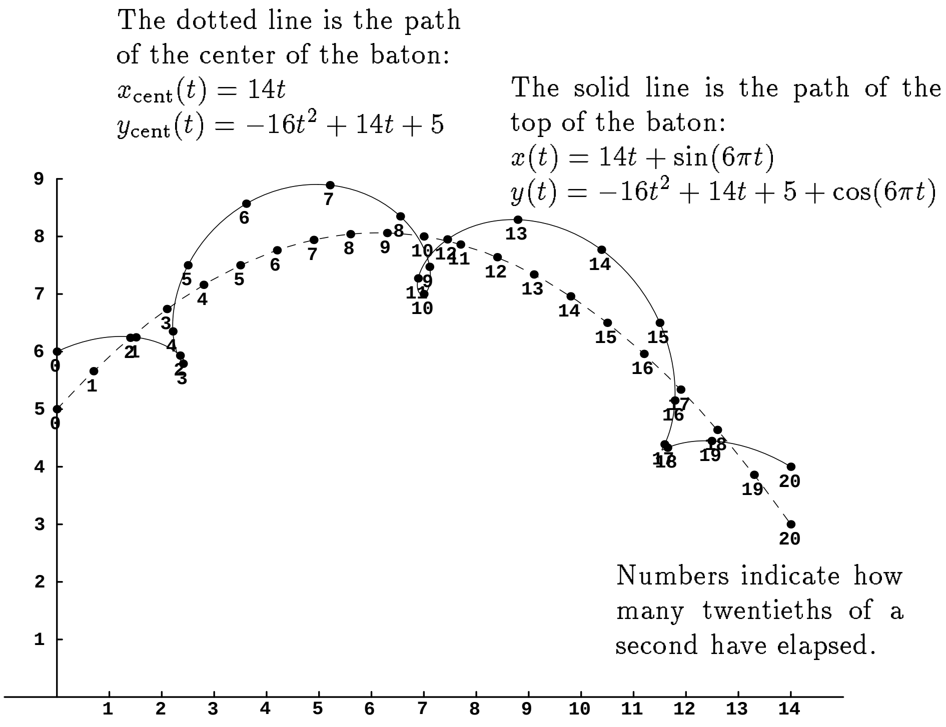
Since the tip is 1 ft from the center, the hypotenuse of this triangle is 1. Hence $x - x_{\text{cent}} = \sin\theta$ and $y - y_{\text{cent}} = \cos\theta$, i.e.,

$$x(t) = 14t + \sin\theta, \qquad y(t) = -16t^2 + 14t + 5 + \cos\theta.$$

It remains to write θ in terms of t. We do this as in Example 10.3 of the last chapter, obtaining $\theta = 6\pi t$. We conclude that

$$x(t) = 14t + \sin(6\pi t), \qquad y(t) = -16t^2 + 14t + 5 + \cos(6\pi t).$$

(Notice that here it is the x-coordinate that involves the sine and the y-coordinate that involves the cosine, the reverse of what we got in our earlier examples of circular motion. The reason is that in this problem it was convenient to measure the angle θ clockwise from the positive y-direction rather than counterclockwise from the positive x-direction. This different way of defining the angle accounts for the reversal of the sine and cosine.)



The graph above shows the trajectory of the center of the baton (dotted line) and the trajectory of the tip of the baton (solid line) between $t = 0$ and $t = 1$ sec. The numbered points from 0 to 20 give the location at intervals of 0.05 seconds. If you join the numbers on the dotted line with the corresponding numbers on the solid line, you will see the spinning of (the top half of) the baton.

Even though the equations for $x(t)$ and $y(t)$ are not tremendously complicated, the path of the tip of the baton is a rather elaborate curve. In this example we cannot find an algebraic or trig formula for y in terms of x, because we cannot eliminate t. So the parametric method is the only reasonable way to describe this curve.

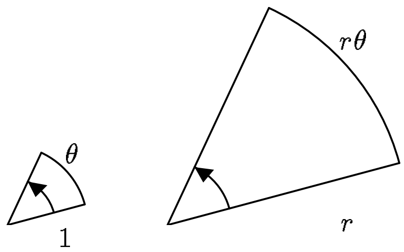
EXAMPLE 11.2. (The cycloid)

Suppose a bicycle wheel of radius r is rolling along the x-axis. A pebble lying at the origin is picked up in the tread of the tire as it rolls over the origin. We would like to describe the path of the pebble as it moves along the x-axis, spinning around the center of the wheel as the wheel turns. This path is called a **cycloid**.

As in Example 11.1, the procedure is to split the problem into two parts: describe the motion of the center of the wheel, and then determine the location of the pebble relative to the center. The first part is easy, because the center of the wheel is simply moving along the straight line $y = r$ (parallel to the x-axis at a height of r); the second part is a circular motion problem.

We shall express the coordinates of the center $(x_{\text{cent}}, y_{\text{cent}})$ and the coordinates of the pebble (x, y) in terms of the following parameter: the angle θ in radians through which the wheel has rotated after picking up the pebble at the origin. We shall also introduce another letter s to denote the linear distance the wheel has traveled. s is the distance along the x-axis from the origin to the point of contact of the wheel, and s is also the x-coordinate of the center of the wheel.

There is a simple relation between the distance s the wheel has traveled and the angle θ in radians through which it has rotated. Namely, the distance traveled along the x-axis is also equal to the distance measured around the circumference through which the wheel has turned (here we are assuming that the wheel rolls without slipping or sliding). For example, every time the wheel rotates through 1 revolution (i.e., 2π radians), the wheel moves a distance equal to the circumference $2\pi r$. More generally, if the wheel rotates through an angle of θ radians, the wheel moves a distance equal to θr. This is because the distance along the circumference of the unit circle of the part of the circumference corresponding to an angle of θ radians is simply θ, by the definition of the measure of an angle in radians. In a circle of radius r — i.e., the unit circle magnified by a factor of r — the distance along the circumference will be r times as great:



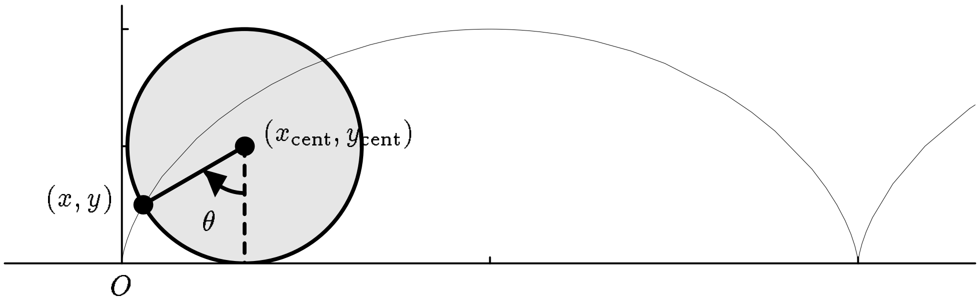
To summarize, we have the relation:

$$s = r\,\theta.$$

The center of the wheel always has the same y-coordinate r (a constant), and its x-coordinate is $s = r\,\theta$. Thus, we have

$$x_{\text{cent}}(\theta) = r\theta, \qquad y_{\text{cent}}(\theta) = r.$$

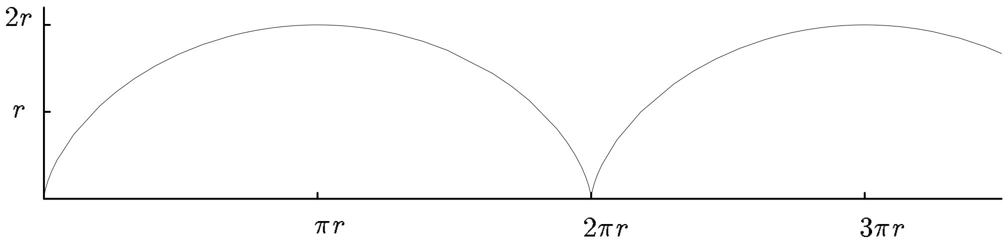
The second part of our procedure is to find the difference between the coordinates (x, y) of the pebble and the coordinates $(x_{\text{cent}}, y_{\text{cent}})$ of the center. Since the wheel is rolling from left to right, and the radius from the center to the pebble is pointing down when we start out at the origin, it is convenient to measure the angle θ clockwise from the negative y-direction. We have the following diagram:



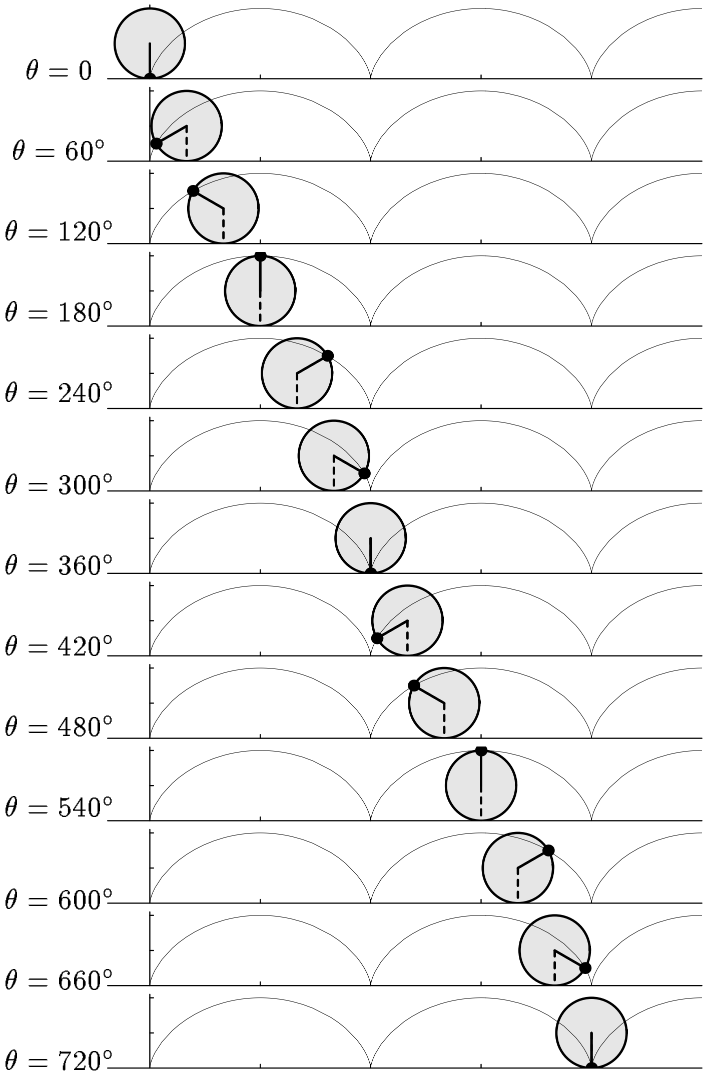
If θ is an acute angle, as in the diagram, then $\sin\theta$ and $\cos\theta$ are positive. Since the pebble is to the *left* and *below* the center, we must *subtract* the horizontal leg of the triangle from x_{cent} to get x, and we must *subtract* the vertical leg of the triangle from y_{cent} to get y. Using the trig functions of θ to get the horizontal and vertical legs of the triangle (whose hypotenuse is equal to r), we end up with

$$x(\theta) = x_{\text{cent}}(\theta) - r\sin\theta = r\theta - r\sin\theta, \qquad y(\theta) = y_{\text{cent}}(\theta) - r\cos\theta = r - r\cos\theta.$$

This parametric curve is drawn below. Each time θ goes through 2π radians, the wheel turns around once, and the pebble goes through one motion from cusp to cusp.



The figure below shows the point on the curve for several values of θ.



Now suppose that we know how the bicycle is moving with time. For example, suppose that it is traveling at constant speed v, and it passes the origin at time $t = 0$. That is, $ds/dt = v$. To express the position of the pebble in terms of t, all we have to do is express θ in terms of t, and then substitute in the above equations for (x, y). Since θ has a simple relation to s — namely, $\theta = \frac{1}{r}\, s$ — it follows that $d\theta/dt$ has a simple relation to ds/dt, namely (taking d/dt of both sides):

$$\frac{d\theta}{dt} = \frac{1}{r}\,\frac{ds}{dt} = \frac{1}{r}\, v.$$

Because θ is 0 at time $t = 0$, and it increases as the constant rate v/r, it follows that we have: $\theta = (v/r)t$. Thus, when the bicycle is traveling at constant velocity v, we obtain the following equations for the motion of the pebble:

$$x(t) = r(v/r)t - r\sin((v/r)t) = vt - r\sin(vt/r), \qquad y(t) = r - r\cos(vt/r).$$

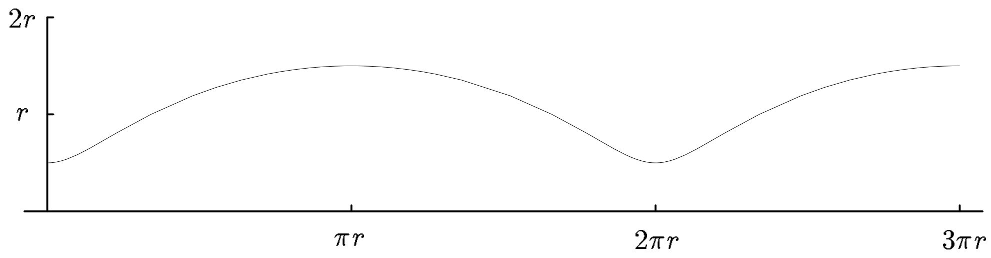
EXAMPLE 11.3. (The trochoid)

The same as in Example 11.2, except that when the wheel rolls over the origin, instead of getting stuck in the tread on the outside of the tire, the pebble flies up and gets stuck in a spoke at a distance a directly below the center (where $a < r$). Thus, the pebble again rotates around the center, but at a distance a rather than r from the center. The path of this pebble is called a **trochoid**.

In this problem everything is just like in Example 11.2, except that in the circular motion part the hypotenuse of the triangle (which is the line from the center to the pebble after the wheel has rotated through an angle of θ radians) is now a rather than r. This leads to the formulas

$$x(\theta) = x_{\text{cent}}(\theta) - a\sin\theta = r\theta - a\sin\theta, \qquad y(\theta) = y_{\text{cent}}(\theta) - a\cos\theta = r - a\cos\theta.$$

Here is a picture of the trochoid path in the case when $a = r/2$:



HOMEWORK 11.

1. Suppose that the wheel in Example 11.2 has radius 1.

(a) Graph the cycloid by plotting the points where the pebble is located after the wheel rotates through all multiples of 30° from 0 to 360° (i.e., θ goes through all multiples of $\pi/6$ from 0 to 2π).

(b) Use the chain rule in the form

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

to find a formula for dy/dx in terms of θ.

(c) Draw the tangent lines to the curve in part (a) at all of the plotted points from $\theta = \pi/6$ to $\theta = 11\pi/6$, and use part (b) to find the slope of each tangent line.

2. A spring is causing an object to oscillate up and down sinusoidally between $y = 3$ and $y = -3$. At $t = 0$ the object is at maximum height $y = 3$, and it makes one full motion (from 3 to -3 and back to 3) twice a second. Meanwhile, the whole apparatus is moving at constant velocity 2 units/sec down the x-axis, starting at 0 at time $t = 0$.

(a) Find $x(t)$.

(b) Find $y(t)$. Namely, use the information given to (i) graph y as a sinusoidal function of time in the ty-plane, and (ii) find A, B, C and D in the formula $y = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D$.

(c) Graph the path of the object in the xy-plane. Find the formula for $y = f(x)$ in two ways: (i) Use the graph of the path to find a, b, c and d in $y = a\sin\left(\frac{2\pi}{b}(x - c)\right) + d$ (where we're using small letters for the constants, so as to avoid confusion with the sinusoidal function in part (b)). (ii) Eliminate t, and use part (b) to write y directly in terms of x. Use both methods, and check that your answers agree.

3. The same as Problem 2, except suppose that the whole apparatus is moving down the x-axis according to the formula $x(t) = t^2$. Notice that the vertical motion $y(t)$ as a function of time is unaffected by this change. However, in the horizontal direction the object is **accelerating**, i.e., its velocity, rather than being constant, is steadily increasing according to the formula $v(t) = \dot{x}(t) = 2t$; and the acceleration $a(t) = \ddot{x}(t) = 2$ is constant. Here $x(t) = t^2$ can be thought of as $\frac{1}{2}gt^2$, where the constant acceleration g here is 2 units/sec^2 (and of course, unlike gravity, which pulls in the negative y-direction, here the constant acceleration is pulling in the positive x-direction). Also notice that your graph of the path $y = f(x)$ is no longer sinusoidal: the distance between peaks becomes greater as the object accelerates to the right. The formula for $y = f(x)$ can be obtained by substituting $t = \sqrt{x}$ in place of t in the formula in part (b).

4. A wheel of radius 2 ft is centered at the point $(0, 100\text{ ft})$ and is rotating clockwise at the constant rate of one revolution every 2 seconds. At time $t = 0$ a pebble on the perimeter of the wheel is located at the point $(0, 102)$. At time $t = 0$ the wheel is dropped. Write down formulas for the x- and y-coordinates of the pebble as functions of time, and graph its trajectory for t between 0 and 2.5 sec (plot points at intervals of 1/4 sec).

5. A wheel rolls down the x-axis from left to right without slipping at constant speed 6 m/sec. The diameter of the wheel is 1.5 m. As it passes over the origin at time $t = 0$ a pebble gets stuck in a spoke right under the center at a distance 25 cm from the rim. Find $x(t)$ and $y(t)$ for the pebble.

6. In an episode of the TV program "Northern Exposure," the hero Chris and a mathematician named Amy send his motorcycle careening over a cliff. We want to describe the path of the point P at the top of one of the wheels at the instant $t = 0$ when the wheel starts over the cliff. Suppose that the motorcycle wheel has a radius 0.25 m, and until time $t = 0$ the motorcycle is traveling at 10 m/sec. Neglect air resistance, and take $g = 9.8$ m/sec^2. Suppose that the wheel is traveling in the xy-plane, the motorcycle is moving horizontally (along the negative x-axis) until time $t = 0$, and the origin is the point at the edge of the cliff where the wheel is touching at the instant $t = 0$. Find the coordinates $x(t)$, $y(t)$ of P at time t (while the motorcycle is falling).