CHAPTER 1

Improper Integrals

An improper integral is the area of a shape that extends out to infinity in some direction. In many cases, such a shape has infinite area, and we say that the improper integral is ∞. However, in certain cases the total area is finite, because the part extending out infinitely far is "pinched closely together," so that the very narrow part going out infinitely far does not contribute much area. This all sounds rather vague. The best way to understand improper integrals is through examples.

Example 1.1. Find the area under the reciprocal cube curve $y = 1/x^3$ to the right of the line $x = 1$.

First, suppose that we bound the area on the right at $x = b$, where b is a very large upper limit of integration. Then we compute

$$\int_1^b \frac{dx}{x^3} = -\frac{1}{2}x^{-2} \Big|_1^b = \frac{\left(1 - b^{-2}\right)}{2}.$$

For example, the area from $x = 1$ to $x = 100$ is exactly 0.49995, the area from $x = 1$ to $x = 1000000$ is exactly 0.4999999999995, and so on. Clearly, the limit as $b \longrightarrow \infty$ is 0.5. This is what we mean by the area out to infinity:

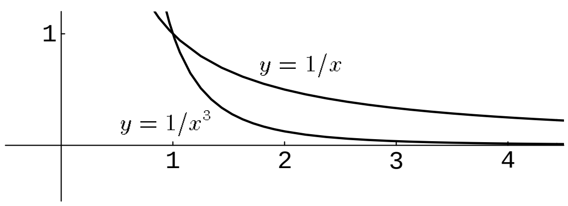
$$\int_1^\infty \frac{dx}{x^3} = \lim_{b \longrightarrow \infty} \int_1^b \frac{dx}{x^3} = \lim_{b \longrightarrow \infty} \frac{\left(1 - b^{-2}\right)}{2} = 0.5.$$

Example 1.2. Find the area under the reciprocal curve $y = 1/x$ to the right of the line $x = 1$.

Here we have

$$\lim_{b \longrightarrow \infty} \int_1^b \frac{dx}{x} = \lim_{b \longrightarrow \infty} \ln b = \infty.$$

Roughly speaking, the reciprocal cube function pinches together with the x-axis a lot faster than does the reciprocal function, and so we get finite area with $1/x^3$ but not with $1/x$.

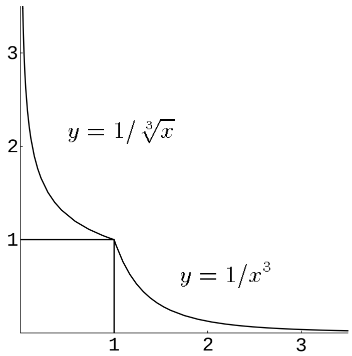


Example 1.3. Let $y = 1/x^n$, where n is a positive constant not equal to 1. Find the area under this curve to the right of the line $x = 1$.

Here we want the limit as $b \longrightarrow \infty$ of $\int_1^b x^{-n} dx = \frac{1}{-n+1} x^{-n+1} \Big|_1^b = \frac{1}{n-1}\left(1 - b^{1-n}\right)$. What happens as $b \longrightarrow \infty$ depends on the exponent of b, which is $1 - n$. If this is positive, i.e., if $n < 1$, then the limit is infinite. On the other hand, if $n > 1$, then b^{1-n} approaches 0, and so we obtain the finite area $1/(n-1)$. We can summarize the situation as follows (note that in Example 1.2 we saw that the area is infinite in the special case $n = 1$):

$$\int_1^\infty \frac{dx}{x^n} = \begin{cases} \frac{1}{n-1} & \text{if } n > 1; \\ \infty & \text{if } n \le 1. \end{cases}$$

EXAMPLE 1.4. Find the area bounded by the x- and y-axes, the line $x = 1$, and the reciprocal cube root curve $y = 1/\sqrt[3]{x}$.



Here the region extends out to infinity in the y-direction. But the curve comes very close to the y-axis (its asymptote), so there is very little area enclosed when y is large. For this reason, as we shall see, the total area is finite. The area is the improper integral $\int_0^1 x^{-1/3}dx$. To evaluate this, let us compute the ordinary (*not* improper) integral $\int_a^1 x^{-1/3}dx$ for $a > 0$, and then see what happens when $a \longrightarrow 0$. That is,

$$\int_0^1 x^{-1/3}dx = \lim_{a\longrightarrow 0}\int_a^1 x^{-1/3}dx = \lim_{a\longrightarrow 0}\left(\frac{3}{2}x^{2/3}\,\Big|_a^1\right) = \lim_{a\longrightarrow 0}\left(\frac{3}{2}\left(1 - a^{2/3}\right)\right) = 3/2,$$

because a raised to any positive power approaches 0 as $a \longrightarrow 0$.

In Example 1.4, suppose we remove from our region the square of unit side that is bounded by the x- and y-axes, the line $x = 1$, and the line $y = 1$. The remaining area is the area to the left of the curve $x = 1/y^3$ above the line $y = 1$. This is exactly the same area as in Example 1.1, except that the roles of x and y are reversed. Thus, we expect the area in Example 1.4 to equal 1 plus the area in Example 1.1. Thus, the value of 3/2 in the last paragraph agrees with the value of 1/2 in Example 1.1.

In the same way as in Example 1.4, we can compute the area bounded by the x- and y-axes, the line $x = 1$, and the curve $y = 1/x^n$. The computation is just like the special case when $n = 1/3$. The result is:

$$\int_0^1 \frac{dx}{x^n} = \begin{cases} \frac{1}{1-n} & \text{if } n < 1; \\ \infty & \text{if } n \ge 1. \end{cases}$$

EXAMPLE 1.5. Find $\int_{-2}^2 dx/\sqrt{4-x^2}$.

Here the integrand $(4-x^2)^{-1/2}$ is infinite at *both* endpoints of the interval of integration — the region goes out to infinity along the line $x = -2$ and along the line $x = 2$. To get a region that does not extend to infinity, we integrate between $-a$ and a, where a is a little less than 2. Then we take the limit as $a \longrightarrow 2$. In evaluating the integral, we use Example **??** (with r set equal to 2) of the section on the Arcsin. Thus,

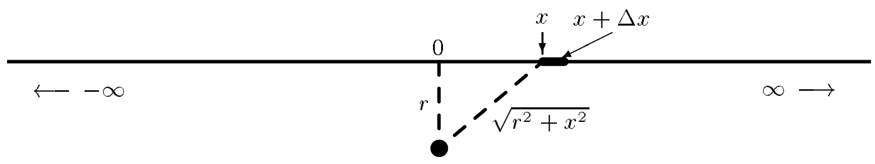
$$\begin{aligned}\int_{-2}^2 \frac{dx}{\sqrt{4-x^2}} &= \lim_{a\longrightarrow 2}\int_{-a}^a \frac{dx}{\sqrt{4-x^2}} = \lim_{a\longrightarrow 2}\arcsin(x/2)\,\Big|_{-a}^a \\ &= \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.\end{aligned}$$

EXAMPLE 1.6. The region in the fourth quadrant bounded by the y-axis, the line $y = -1$, and the curve $y = -1/x$ is rotated around the y-axis to obtain a container (an infinitely deep funnel). (a) Find the volume of the funnel. (b) Show that the funnel has infinite surface area. Thus, one could fill the funnel entirely with water, but if one cuts the side of the funnel into small pieces and spreads them out on a table, there is no table of finite size that would hold all of the pieces!

(a) We first write x as a function of y: $x = -1/y$. Then in the volume integral $\pi \int_{-\infty}^{-1} dy/y^2$, we make the change of variables $u = -y$, so that $du = -dy$. This transforms the volume to $-\pi \int_{\infty}^{1} dy/y^2 = \pi \int_{1}^{\infty} du/u^2$ (we used the fact that the sign of a definite integral changes when you reverse the upper and lower limits of integration). The last integral was computed in Example 9.3 (take $n = 2$): it has finite value 1. Thus, the volume of the funnel is π.

(b) We have not developed the formula for the surface area of a volume of revolution. However, we can answer the question in part (b) by observing that the surface area is greater than the area of the flattened surface contained in the third and fourth quadrants below $y = -1$ between the curves $x = 1/y$ and $x = -1/y$. This flat area is equal to $\int_{-\infty}^{-1} \frac{-2}{y} dy = 2\int_{1}^{\infty} du/u$ (where we make the same substitution $u = -y$ as before). But the latter improper integral is infinite, as we saw in Example 1.2. Therefore, the funnel has infinite surface area.

EXAMPLE 1.7. Determine how the noise from a freeway drops off as you move farther away. Make the following simplifying assumptions: (1) the amount of noise generated by a segment of the freeway is proportional to the segment's length (i.e., the traffic is uniformly distributed along the freeway); (2) the contribution of a small segment of the freeway to the amount of noise you hear is inversely proportional to the square of your distance from that segment; (3) the freeway is a straight line extending infinitely far in both directions. (We are also assuming that the noise from different parts of the freeway add up without "canceling each other," and we are using a linear scale of noise, rather than a logarithmic scale such as the decibel system.)



Let r be our perpendicular distance from the freeway. Take the freeway to be the x-axis, with 0 located at the point closest to us. We shall find the total noise by the same type of procedure that we used to find definite integral formulas for the area under a curve, the volume of a solid of revolution, and the total moment of a uniform plate. In those cases we had the integral from a to b of the contribution (to area, volume, moment, or whatever) coming from the part between x and $x + \Delta x$ (where we replace Δx by dx when we go from the Riemann sum to the integral).

For the sake of review, the chart on the next page shows the contribution from the part between x and $x + \Delta x$ and the integral that resulted.

TYPE OF PROBLEM	CONTRIBUTION	FORMULA
area under curve	$\approx f(x)\Delta x$	$\int_a^b f(x)dx$
vol of revolution	$\approx \pi f(x)^2\Delta x$	$\pi\int_a^b f(x)^2 dx$
moment of plate	$\approx D\,x\,f(x)\Delta x$	$D\int_a^b x\,f(x)dx$

So in the freeway problem we want to find the contribution to noise that comes from the tiny segment $[x, x+\Delta x]$. According to assumptions (1) and (2), the noise you hear from this segment is proportional to the length Δx of the segment and inversely proportional to the square of your distance from the segment. In computing this distance, we take the distance from the left endpoint of the segment (since Δx is very small, this is very close to the distance from any other point on the segment); by the Pythagorean theorem, the square of this distance is $r^2 + x^2$. Thus, if we let K denote the constant of proportionality, we find that the contribution is

$$\approx K\frac{\Delta x}{r^2+x^2}.$$

Thus, the noise coming from the entire highway between a and b is found by summing up these contributions — or rather (after taking the limit as $\Delta x \longrightarrow 0$) taking the integral with Δx replaced by dx: Noise$=\int_a^b \frac{K\,dx}{r^2+x^2}$. Thus,

$$\begin{aligned}\text{Noise} &= K\int_a^b \frac{dx}{r^2+x^2} = \frac{K}{r^2}\int_a^b \frac{dx}{1+(x/r)^2} = \\ &= \frac{K}{r^2}\int_{a/r}^{b/r}\frac{r\,du}{1+u^2} = \frac{K}{r}\int_{a/r}^{b/r}\frac{du}{1+u^2} \\ &= \frac{K}{r}\arctan u\,\Big|_{a/r}^{b/r} = \frac{K}{r}\left(\arctan\big(\frac{b}{r}\big) - \arctan\big(\frac{a}{r}\big)\right).\end{aligned}$$

If the freeway had finite length and were contained between a and b, then this would give us a rather complicated expression for the noise from the freeway. It would not be easy to say how the noise drops off as our distance r increases. However, at this point we use simplifying assumption (3), which tells us to take the limit as $a \longrightarrow -\infty$ and $b \longrightarrow \infty$, i.e.,

$$\begin{aligned}\text{Noise} &= \frac{K}{r}\left(\lim_{b\longrightarrow\infty}\arctan\big(\frac{b}{r}\big) - \lim_{a\longrightarrow-\infty}\arctan\big(\frac{a}{r}\big)\right) \\ &= \frac{K}{r}\left(\frac{\pi}{2} - \big(-\frac{\pi}{2}\big)\right) = \frac{K\pi}{r}.\end{aligned}$$

Since K and π are constants, we end up with the simple conclusion: *noise from the freeway drops off proportionally to distance.* Thus, if you move twice as far away, the amount of noise drops only by 50%. (We say "only," because in the case of a point source the noise would drop by a factor of 4, i.e., by 75%.)

Even though the simplifying assumptions might seem to detract from the realism of this example, our conclusion roughly agrees with what one observes in practice. For example, real estate agents know that noise from Interstate 5 has a bad effect on property values for quite a distance — much more so than would a "point source" such as a noisy factory.

Notice the role of simplifying assumption (3). Although it might seem odd at first, improper integrals have many practical applications: it often simplifies the calculations to suppose that something extends out to infinity.

HOMEWORK 1.

1. Find the following improper integrals (some of them might be ∞):

$$\begin{array}{llllll}
\text{(a)} & \displaystyle\int_1^\infty dx/(x\sqrt{x}); & \text{(e)} & \displaystyle\int_0^\infty dx/(1+x^2); & \text{(i)} & \displaystyle\int_{-10}^{10} dx/\sqrt{100-x^2}; \\
\text{(b)} & \displaystyle\int_0^1 dx/(x\sqrt{x}); & \text{(f)} & \displaystyle\int_{-\infty}^\infty dx/(1+x^2); & \text{(j)} & \displaystyle\int_0^1 x\,dx/\sqrt{1-x^2}, \\
\text{(c)} & \displaystyle\int_1^\infty dx/\sqrt{x}; & \text{(g)} & \displaystyle\int_{-\infty}^\infty dx/(9+x^2); & \text{(k)} & \displaystyle\int_0^\infty x\,dx/(1+x^2); \\
\text{(d)} & \displaystyle\int_0^1 dx/\sqrt{x}; & \text{(h)} & \displaystyle\int_0^1 dx/\sqrt{1-x^2}; & \text{(l)} & \displaystyle\int_0^\infty x\,dx/(1+x^4).
\end{array}$$

2. Let n be a positive number. Find (in terms of n) the volume of the solid obtained by rotating around the y-axis the part of $y = -1/x^n$ below $y = -1$ in the fourth quadrant. For what values of n is this volume infinite?

3. Suppose that $n > 2$. Find (in terms of n) the balancing point of the uniform flat plate bounded by the graph of $y = 1/x^n$, the graph of $y = -1/x^n$, and the line $x = 1$. (The plate extends to the right all the way out to infinity.)

4. Show that the following uniform flat plates have finite mass but do not have center of mass at a finite distance:

(a) Problem 3 when $1 < n \leq 2$;

(b) the plate bounded by the graph of $y = 1/(1+x^2)$, the graph of $y = -1/(1+x^2)$, and the y-axis. This means that you could lift up the plate, but you could not balance it!