4.2 Exponential Functions

Unlike the case of the polynomial functions that we have studied earlier, exponential functions are of the type

$$f(x) = (\text{costant base})^{\text{variable power in } x}$$

These type of functions are important in applications: carbon dating, economic, biological sciences, etc...

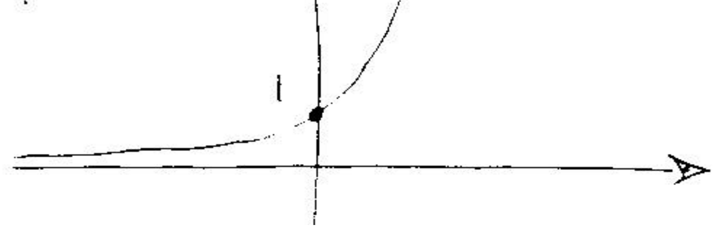
Ex: let's consider $f(x) = 2^x$. We can write a table of values for $x \in \mathbb{Z}$

x	3	2	1	0	-1	-2	-3	$\ldots$
2^x	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$	$2^{-1} = 1/2$	$2^{-2} = 1/4$	$2^{-3} = 1/8$	

For rational values $2^{m/n} = \left(2^{1/n}\right)^m = \left(\sqrt[n]{2}\right)^m$ etc... whereas if x is a real number we can use the fact that any real is in between two rational numbers and approximate the value of 2^x in that w

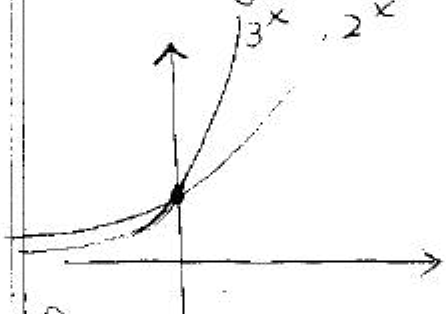
The graph looks like:

growth function



Ex: Let's compare the graphs of

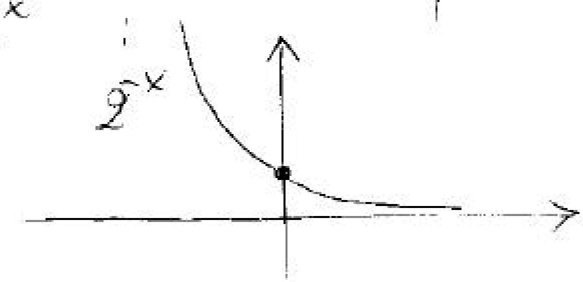
$y = 2^x, \; 3^x, \; 2^{-x}$



2^x and 3^x they meet at $(0,1)$. 3^x grows faster than 2^x for positive values of x.

For negative values of x then 3^x approaches the horizontal asymptote $y=0$ much faster than 2^x.

Note that the graph of 2^{-x} is symmetric with respect to the y-axis of 2^x



decay function

$$y = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$

This is the typical graph of an exponential $y = a^x$ with $0<a<1$

Observe that exponential function are one-one.

Hence $a^{x_1} = a^{x_2} \leadsto x_1 = x_2$

This observation helps in solving equations involving exponentials:

Solve $9^{x^2} = 3^{3x+2}$

$$(3^2)^{x^2} = 3^{3x+2} \rightsquigarrow 3^{2x^2} = 3^{3x+2}$$

$\rightsquigarrow$ exponents are the same $\rightsquigarrow 2x^2 = 3x+2$

$2x^2 - 3x - 2 = 0 \qquad (2x+1)(x-2) = 0$

$\therefore$ $x = -\frac{1}{2}$ or $x = 2$.

Solve $9^{2x} \cdot \left(\frac{1}{3}\right)^{x+2} = 27(3^x)^{-2}$

$$(3^2)^{2x} \cdot (3^{-1})^{x+2} = 3^3 \cdot 3^{-2x}$$

$$3^{4x-x-2} = 3^{3-2x} \qquad 3^{3x-2} = 3^{3-2x}$$

$$\rightsquigarrow 3x-2 = 3-2x \rightsquigarrow 5x = 5 \rightsquigarrow x = 1$$

The half-life of radium is 1,600 years. If the initial amount is q_0 milligrams then the quantity remaining after t year is

$$q(t) = q_0 2^{kt}$$

Find k.

We need to solve an equation!!

$\frac{1}{2} q_0 =$ is what is left after 1,600 year

$= q(1{,}600) = q_0\, 2^{k \cdot 1{,}600}$

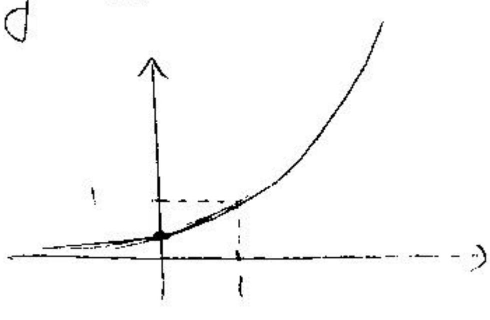
$\therefore \quad \frac{1}{2} \cancel{q_0} = \cancel{q_0}\, 2^{k\, 1{,}600} \rightsquigarrow \frac{1}{2} = 2^{k \cdot 1{,}600}$

$\rightsquigarrow \quad 2^{-1} = 2^{k \cdot 1{,}600} \rightsquigarrow -1 = k \cdot 1{,}600$

$\therefore \quad k = -\frac{1}{1{,}600}$

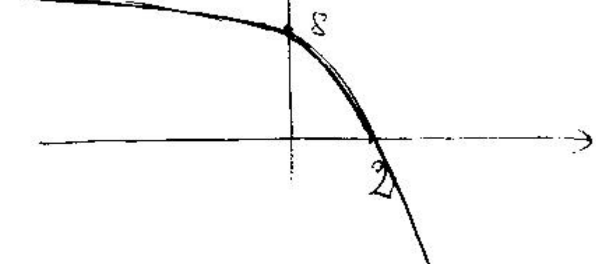
* We can also use shifting and stretching techniques to graph more complex exponential functions.

Ex: $y = 2^{x-1}$ we shif 2^x to the righ of 1 unit



when $x = 0$ then $y = 2^{0-1} = \frac{1}{2}$.

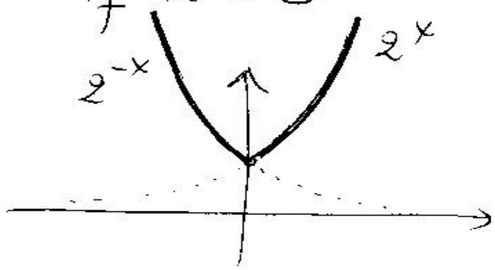
Ex: $y = -3^x + 9$



$y = 9$

Ex: $y = 2^{|x|} = \begin{cases} 2^{x} & \text{if } x \geq 0 \\ 2^{-x} & \text{if } x < 0 \end{cases}$

thus the graph looks like



* Find an exponential function of the form $f(x) = b\,a^{x}$ such that it has y-intercept 6 and passes through $P(2, 3/32)$.

$$6 = f(0) = b\,a^{0} = b \quad \therefore \quad b = 6$$

$$3/32 = 6 \cdot a^{2} \rightsquigarrow \frac{1}{64} = a^{2} \quad \therefore$$

$$a = \frac{1}{8} \longrightarrow f(x) = 6 \cdot \left(\frac{1}{8}\right)^{x}$$

* As we mentioned earlier, exponential functions are useful to compute the future value of a certain amount of money invested in a bank at a fix interest rate.

P_0 = principal invested at a simple interest rate r (say 12% or 0.12)

after 1 year $\quad P(1) = \underbrace{P_0}_{\text{amount in bank}} + \underbrace{r P_0}_{\text{interest}} = P_0(1+r)$

after 2 years $P(2) = \underbrace{P_0(1+r)}_{\text{amount in bank}} + \underbrace{r\,P_0(1+r)}_{\text{interest}}$

$$= P_0(1+r)\left[1+r\right] = P_0(1+r)^2$$

In general : $P(t) = P_0\,(1+r)^t$

$1+r$ is a real number greater than 1.

In general, if the interest is compounded n times per year the formula is

$$P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$$

* If \$1,000 is invested at a rate of 12% per year compounded monthly, find the principal after 6 months, after 20 years :

* after 6 months $P(0.5) = 1{,}000\left(1+\frac{0.12}{12}\right)^{12\cdot(0.5)}$

$$= 1{,}000\,(1.01)^6 = \$1{,}061.52$$

* after 20 years $P(20) = 1{,}000\left(1+\frac{0.12}{12}\right)^{12\cdot 20}$

$$= 1{,}000\,(1.01)^{240} = \$10{,}892.55$$