Vector Valued Functions and Motion in Space

$\widehat{\mathbf{T}}$, $\widehat{\mathbf{B}}$, and $\widehat{\mathbf{N}}$:

Let $\vec{\mathbf{r}}(t) = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}} + z(t)\widehat{\mathbf{k}}$, $a \le t \le b$, be the position vector describing the motion in the space of a particle. Then

$$\text{velocity } = \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}, \qquad s = \text{ arclength } = \int_a^t |\vec{\mathbf{v}}(u)|\,du, \qquad \text{speed } = \frac{ds}{dt} = |\vec{\mathbf{v}}|,$$

$$\text{acceleration } = \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \left(\frac{d^2s}{dt^2}\right)\widehat{\mathbf{T}} + \kappa\left(\frac{ds}{dt}\right)^2\widehat{\mathbf{N}} = \frac{d|\vec{\mathbf{v}}|}{dt}\widehat{\mathbf{T}} + \kappa|\vec{\mathbf{v}}|^2\widehat{\mathbf{N}},$$

$$\text{unit tangent vector } = \widehat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|},$$

$$\text{principal unit normal vector } = \widehat{\mathbf{N}} = \frac{1}{\kappa}\frac{d\widehat{\mathbf{T}}}{ds} = \frac{d\widehat{\mathbf{T}}/dt}{|d\widehat{\mathbf{T}}/dt|},$$

$$\text{binormal vector } = \widehat{\mathbf{B}} = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}},$$

$$\text{curvature } = \kappa = \left|\frac{d\widehat{\mathbf{T}}}{ds}\right| = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{|\vec{\mathbf{v}}|^3},$$

$$\text{torsion } = \tau = -\left(\frac{d\widehat{\mathbf{B}}}{ds}\right) \cdot \widehat{\mathbf{N}} = \frac{\det\begin{pmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{pmatrix}}{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|^2}.$$

Formulas for Integration in Vector Fields

Line Integrals:

The line integral of a continuous function $f(x,y,z)$ over a space curve C parametrized by $\vec{\mathbf{r}}(t) = g(t)\widehat{\mathbf{i}} + h(t)\widehat{\mathbf{j}} + k(t)\widehat{\mathbf{k}}$, $a \le t \le b$ is given by

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))|\vec{\mathbf{v}}|\,dt.$$

Green's Theorems:
(C bounds R)

If $\vec{\mathbf{F}} = M(x,y)\widehat{\mathbf{i}} + N(x,y)\widehat{\mathbf{j}}$ is a 2-dimensional vector field then:

$$\text{circulation around } C = \oint_C \vec{\mathbf{F}} \cdot \widehat{\mathbf{T}}\,ds = \oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dxdy,$$

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Functions of Two or More Variables and Their Derivatives

Gradient and Directional Derivative: Let $f(x,y,z)$ be a function differentiable throughout some region D in the space containing a point $P(x_0,y_0,z_0)$. The **gradient** of f at P is the vector

$$(\nabla f)_P = \frac{\partial f}{\partial x}(P)\widehat{\mathbf{i}} + \frac{\partial f}{\partial y}(P)\widehat{\mathbf{j}} + \frac{\partial f}{\partial z}(P)\widehat{\mathbf{k}}.$$

If $\widehat{\mathbf{u}}$ is a unit vector, then the **directional derivative** of f at P in the direction of $\widehat{\mathbf{u}}$ is

$$(D_{\widehat{\mathbf{u}}} f)_P = (\nabla f)_P \cdot \widehat{\mathbf{u}}.$$

Tangent Plane and Normal Line: The eq. of the tangent plane at $P(x_0,y_0,z_0)$ on the level surface $f(x,y,z) = c$ is:

$$\frac{\partial f}{\partial x}(P)(x-x_0) + \frac{\partial f}{\partial y}(P)(y-y_0) + \frac{\partial f}{\partial z}(P)(z-z_0) = 0.$$

The eq. of normal line of the surface at P is:

$$x(t) = x_0 + t\,\frac{\partial f}{\partial x}(P), \qquad y(t) = y_0 + t\,\frac{\partial f}{\partial y}(P), \qquad z(t) = z_0 + t\,\frac{\partial f}{\partial z}(P).$$

Critical Points: Let $f(x,y)$ be a continuous function of two independent variables. The points where

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \qquad \text{and the ones where} \qquad \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{fails to exist}$$

are called **critical points** of $f(x,y)$.

Local Max & Min Test: Let P be a critical point of $f(x,y)$, then

- f has a **local maximum** at P if

$$\frac{\partial^2 f}{\partial x^2}(P) < 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 > 0;$$

- f has a **local minimum** at P if

$$\frac{\partial^2 f}{\partial x^2}(P) > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 > 0;$$

- f has a **saddle point** at P if

$$\frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 < 0;$$

- the **test is inconclusive** if

$$\frac{\partial^2 f}{\partial x^2}(P)\frac{\partial^2 f}{\partial y^2}(P) - \left(\frac{\partial^2 f}{\partial x \partial y}(P)\right)^2 = 0.$$

Multiple Integrals

Masses, etc.: If $\delta = \delta(x,y,z)$ is the density function of an object occupying a region D in space, then the mass, the first moments, the center of mass, and the second moments are given by the following formulas:

$$M = \iiint_D \delta\, dV$$

$$M_{yz} = \iiint_D x\delta\, dV \qquad M_{xz} = \iiint_D y\delta\, dV \qquad M_{xy} = \iiint_D z\delta\, dV$$

$$\bar{x} = \frac{M_{yz}}{M} \qquad \bar{y} = \frac{M_{xz}}{M} \qquad \bar{z} = \frac{M_{xy}}{M}$$

$$I_x = \iiint_D (y^2+z^2)\delta\, dV \qquad I_y = \iiint_D (x^2+z^2)\delta\, dV \qquad I_z = \iiint_D (x^2+y^2)\delta\, dV$$

NOTE: similar (even though simpler) formulas hold in the case of an object in the xy-plane.

Jacobians: Suppose that the region G in the uvw-space is transformed one-to-one into the region D in the xyz-space by differentiable equations of the form

$$x = g(u,v,w), \qquad y = h(u,v,w), \qquad z = k(u,v,w)$$

under mild assumptions (always met in this exam) ones has that

$$\iiint_D F(x,y,z)\, dxdydz = \iiint_G F(g,h,k)|J|\, dudvdw$$

where

$$J = J(u,v,w) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \text{Jacobian determinant.}$$

NOTE: a similar (even though simpler) formula holds in the case of a one-to-one transformation between the uv-plane into the xy-plane.

Cylindrical Coordinates: The following are selected equations relating cartesian and cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z \qquad x^2+y^2 = r^2.$$

Here $0 \le \theta \le 2\pi$.
The Jacobian of this transformation is $J(r,\theta,z) = r$.

Spherical Coordinates: The following are selected equations relating cartesian and spherical coordinates:

$$x = \rho\sin\varphi\cos\theta, \qquad y = \rho\sin\varphi\sin\theta, \qquad z = \rho\cos\varphi,$$

$$x^2+y^2+z^2 = \rho^2, \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

Here $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$.
The Jacobian of this transformation is $J(\rho,\theta,\varphi) = \rho^2\sin\varphi$.
