FIBONACCI SUMSETS AND THE GONALITY OF STRIP GRAPHS

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Abstract. We provide a new perspective on the divisor theory of graphs, using additive combinatorics. As a test case for this perspective, we compute the gonality of certain families of outerplanar graphs, specifically the strip graphs. The Jacobians of such graphs are always cyclic of Fibonacci order. As a consequence, we obtain several results on the additive properties of Fibonacci numbers.

1. Introduction

1.1. **Divisors and Gonality.** The divisor theory of graphs was first introduced in [BN07, Bak08]. This theory, which mirrors and informs that of divisors on algebraic curves, has attracted a great deal of interest in the intervening 17 years. In [Bak08], Baker defines a new graph invariant, the *gonality*, which is the minimum degree of a divisor of positive rank. More generally, once can define the *r-gonality*, the minimum degree of a divisor of rank r. Computing the gonality of a graph is NP-hard [GSvdW20], and there is a wealth of literature on bounding this invariant [vDdBG20, HJJS22, vDdBSvdW22, DEM23].

In this paper, we recast the gonality problem in the language of additive combinatorics. The group of equivalence classes of degree-0 divisors on a graph G is a finite abelian group, known variously as the *Jacobian* $\mathrm{Jac}(G)$, the *sandpile group*, or the *critical group*. This group contains a subset $\mathcal{A}(G) \subset \mathrm{Jac}(G)$, whose elements are in bijection with equivalence classes of vertices in G. Additive combinatorics is largely concerned with the additive structure of subsets of abelian groups, known as *additive sets*. Of particular interest are the *iterated sumsets* $m\mathcal{A}(G)$, consisting of all sums of m elements of $\mathcal{A}(G)$. In Corollary 2.4, we provide an equivalent characterization of the r-gonality in terms of the iterated sumsets of $\mathcal{A}(G)$.

1.2. **Outerplanar Graphs.** As a test case for this perspective, we consider the gonality of *outerplanar graphs*, which are graphs that can be embedded in the plane so that all vertices belong to a single face. More specifically, we focus on two families of outerplanar graphs, the *fan graphs* $\mathcal{F}_n$ (see Figure 1) and the *strip graphs* G_n (see Figure 2).

These families of graphs form a good test case for two reasons. First, the lower bounds on gonality that can be found in the existing literature are insufficient for computing the gonality of these graphs. For example, it is shown in [vDdBG20] that the gonality is bounded below by a much-studied graph invariant known as the *treewidth*. In Lemma 3.3, however, we show that the treewidth of an outerplanar graph is at most two, which is as small as possible. Similarly, in [HJJS22], it is shown that the gonality is bounded below by the so-called *scramble number*, but in Lemma 5.6, we show that the scramble number of the strip graph G_n is three.

Second, the Jacobian of an outerplanar graph is relatively easy to understand. More specifically, if G is a maximal outerplanar graph with exactly two vertices of valence two, then $\mathrm{Jac}(G)$ is cyclic of Fibonacci order (see Corollary 3.2). Throughout, we let F_n denote the nth Fibonacci number, indexed so that $F_0 = 0$ and $F_1 = 1$. For the specific families of graphs $\mathcal{F}_n$ and G_n, the additive sets $\mathcal{A}(G) \subset \mathrm{Jac}(G)$ admit nice descriptions in terms of Fibonacci numbers. Specifically, the set $\mathcal{A}(\mathcal{F}_n)$ consists of 0 and all odd-index Fibonacci numbers between 1 and F_{2n}. In other words,

$$\mathcal{A}(\mathcal{F}_n) = \{0\} \cup \{F_{2k-1} \mid 1 \leq k \leq n\} \subset \mathbb{Z}/F_{2n}\mathbb{Z}.$$

The set $\mathcal{A}(G_n)$ also admits a nice description:

$$\mathcal{A}(G_n) = \{F_k F_{2n-k} \mid 0 \leq k \leq n\} \subset \mathbb{Z}/F_{2n}\mathbb{Z}.$$

By Catalan's identity, $\mathcal{A}(G_n)$ is a translate of the subset

$$\mathcal{B}(G_n) = \{(-1)^{k+1} F_k^2 \mid 0 \leq k \leq n\} \subset \mathbb{Z}/F_{2n}\mathbb{Z}.$$

The gonality of the fan graph $\mathcal{F}_n$ was computed in [Hen18]. The back half of this paper is primarily focused on computing the gonality of the strip graphs G_n.

Theorem 1.1. *We have*

$$\mathrm{gon}(G_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } n \leq 7 \\ 5 & \text{if } n \geq 8. \end{cases}$$

1.3. **Fibonacci Numbers.** While our focus in this paper is to use additive combinatorics to compute graph invariants like the gonality, one can also go the other way, using graph invariants to discover additive properties of Fibonacci numbers. We briefly mention a few results, which are purely statements about Fibonacci numbers, that follow from our graph-theoretic investigation. The first follows directly from [BN07, Theorem 1.7].

Theorem 1.2. *Every integer* $\pmod{F_{2n}}$ *can be expressed as a sum of* $n-1$ *elements of* $\mathcal{A}(\mathcal{F}_n)$, $\mathcal{A}(G_n)$, *or* $\mathcal{B}(G_n)$. *In other words,*

$$\begin{aligned} (n-1)\mathcal{A}(\mathcal{F}_n) &= \mathbb{Z}/F_{2n}\mathbb{Z} \\ (n-1)\mathcal{A}(G_n) &= \mathbb{Z}/F_{2n}\mathbb{Z} \\ (n-1)\mathcal{B}(G_n) &= \mathbb{Z}/F_{2n}\mathbb{Z}. \end{aligned}$$

Moreover, there exists an integer $\pmod{F_{2n}}$ *that cannot be expressed as a sum of* $n-2$ *elements of* $\mathcal{A}(\mathcal{F}_n)$, $\mathcal{A}(G_n)$, *or* $\mathcal{B}(G_n)$. *In other words,*

$$\begin{aligned} (n-2)\mathcal{A}(\mathcal{F}_n) &\subsetneq \mathbb{Z}/F_{2n}\mathbb{Z} \\ (n-2)\mathcal{A}(G_n) &\subsetneq \mathbb{Z}/F_{2n}\mathbb{Z} \\ (n-2)\mathcal{B}(G_n) &\subsetneq \mathbb{Z}/F_{2n}\mathbb{Z}. \end{aligned}$$

The next two results are derived from the fact that automorphisms of the graph G induce Freiman isomorphisms of the additive set $\mathcal{A}(G) \subset \mathrm{Jac}(G)$.

Theorem 1.3. *Let* $1 \leq k_1, \ldots, k_m, k_1', \ldots, k_m' \leq n$. *Then*

$$\sum_{i=1}^m F_{2k_i-1} = \sum_{i=1}^m F_{2k_i'-1} \pmod{F_{2n}} \iff \sum_{i=1}^m F_{2n-2k_i+1} = \sum_{i=1}^m F_{2n-2k_i'+1} \pmod{F_{2n}}.$$

Theorem 1.4. *Let* $0 \leq k_1, \ldots, k_m, k_1', \ldots, k_m' \leq n$. *Then*

$$\begin{aligned} \sum_{i=1}^m F_{k_i} F_{2n-k_i} &= \sum_{i=1}^m F_{k_i'} F_{2n-k_i'} \pmod{F_{2n}} \iff \\ \sum_{i=1}^m F_{n-k_i} F_{n+k_i} &= \sum_{i=1}^m F_{n-k_i'} F_{n+k_i'} \pmod{F_{2n}}. \end{aligned}$$

Finally, the gonalities of these graphs can be reinterpreted in the following way.

Theorem 1.5. *There exists an integer* $x \in \mathbb{Z}/F_{2n}\mathbb{Z}$ *such that:*

(1) x *can written as a sum of* $d-1$ *or fewer odd-index Fibonacci numbers between 1 and* F_{2n-1} *and*
(2) for all $k \leq n$, x *can be written as a sum of* d *or fewer such numbers, with* F_{2k-1} *as one of the summands,*

if and only if

$$d \geq \phi_n := \min\left\{\lfloor\sqrt{n+1}\rfloor - 1 + \left\lceil\frac{n+1-\lfloor\sqrt{n+1}\rfloor}{\lfloor\sqrt{n+1}\rfloor}\right\rceil, \lceil\sqrt{n+1}\rceil - 1 + \left\lceil\frac{n+1-\lceil\sqrt{n+1}\rceil}{\lceil\sqrt{n}\rceil}\right\rceil\right\}.$$

Theorem 1.6. *There exists an integer $x \in \mathbb{Z}/F_{2n}\mathbb{Z}$ such that, for all $j \leq n$, x can be written as a sum of d integers of the form F_kF_{2n-k}, with F_jF_{2n-j} as one of the summands, if and only if $d \geq \min\{\lceil\frac{n+1}{2}\rceil, 5\}$.*

1.4. **Outline of the Paper.** In Section 2, we introduce the basic theory of divisors on graphs, including the monodromy pairing, and its relation to counts of certain types of spanning forests. In Section 3, we begin our discussion of outerplanar graphs, proving in Corollary 3.2 that the Jacobian of certain outerplanar graphs is always cyclic of order F_{2n}. In Section 4, we discuss the fan graphs, describing the set $\mathcal{A}(\mathcal{F}_n)$ and proving Theorems 1.3 and 1.5. Then, in Section 5, we turn to the strip graphs, proving Theorems 1.2 and Theorem 1.4. We also prove, in Theorems 5.7 and 5.13, that the gonality of G_n is bounded below by 4 for $n \geq 6$. Finally, in Section 6, we improve this bound to 5 for $n \geq 8$, proving Theorem 1.1. This last section involves a significant amount of tedious computation, but the strategy of proof is the same as for the simpler lower bounds in Theorems 5.7 and 5.13. Readers who are uninterested in these technical calculations are encouraged to read the proofs of these simpler theorems instead.

Acknowledgements. This research was supported by NSF DMS-2054135.

2. Preliminaries

2.1. **Divisors on Graphs.** In this section, we describe the basic theory of divisors on graphs. For more detail, we refer the reader to [Bak08, BJ16].

Let G be a graph with $n+1$ vertices, no loops, and possibly with parallel edges. Throughout, we fix an ordering $v_0, \ldots, v_n$ of the vertices of G. The vertex v_0 will be referred to as the *base vertex.* A *divisor* D on G is a formal linear combination of vertices $D = \sum_{i=0}^{n} a_i \cdot v_i$. We may think of a divisor as an integer vector of length $n+1$.

The *graph Laplacian* of G is the $(n+1) \times (n+1)$ matrix with rows and columns indexed by the vertices of G, and whose (i,j)th entry is

$$\Delta_{i,j} = \begin{cases} \mathrm{val}(v_i) & \text{if } i = j \\ -\#\text{ of edges between } v_i \text{ and } v_j & \text{if } i \neq j. \end{cases}$$

That is, Δ is the difference of the valency matrix and the adjacency matrix.

Considering Δ as a map from $\mathbb{Z}^{n+1}$ to $\mathbb{Z}^{n+1}$, its image is the set of *principal divisors.* Two divisors D and D' are *equivalent* if their difference is principal. In other words, D is equivalent to D' if there exists $\vec{v} \in \mathbb{Z}^{n+1}$ such that $D - D' = \Delta\vec{v}$. The set of equivalence classes of divisors on G is a group under addition, known as the *Picard group*

$$\mathrm{Pic}(G) = \mathbb{Z}^{n+1}/\Delta\mathbb{Z}^{n+1}.$$

The *degree* of a divisor $D = \sum_{i=0}^{n} a_i \cdot v_i$ is the integer $D = \sum_{i=0}^{n} a_i$. Since all principal divisors have degree zero, the degree of a divisor is invariant under equivalence. The group of equivalence classes of divisors of degree zero is called the *Jacobian* $\mathrm{Jac}(G)$. It is a consequence of Kirchoff's matrix tree theorem that $|\mathrm{Jac}(G)|$ is equal to the number of spanning trees in G, often denoted $\kappa(G)$.

2.2. **The Monodromy Pairing.** Since every row of the graph Laplacian Δ sums to zero, Δ is not invertible. Every matrix Δ, however, has a *generalized inverse* – that is, a matrix L such that $\Delta L \Delta = \Delta$.

One way to construct a generalized inverse is as follows. Let $\widetilde{\Delta}$ be the $n \times n$ matrix obtained from Δ by deleting the first row and first column. Now, let L be the $(n+1) \times (n+1)$ matrix obtained by appending a zero row and zero column to the top and left of $\widetilde{\Delta}^{-1}$. Then L is a generalized inverse of Δ. For the remainder of the paper, we fix L to be this particular generalized inverse.

In [Sho10], Shokrieh defines the *monodromy pairing* on $\mathrm{Jac}(G)$. The pairing

$$\langle\cdot,\cdot,\rangle\colon\ \mathrm{Jac}(G)\times\mathrm{Jac}(G)\to\mathbb{Q}/\mathbb{Z}$$

is given by:

$$\langle D,D'\rangle=[D]^T L[D'] \pmod{\mathbb{Z}}.$$

The monodromy pairing is independent of the choice of generalized inverse L.

For specific generators of $\mathrm{Jac}(G)$, the monodromy pairing can be explicity described as follows. Let $\kappa_{i,j}(G)$ denote the number of 2-component spanning forests of G such that one component contains the base vertex v_0 and the other component contains both v_i and v_j.

Theorem 2.1. *We have*

$$\langle v_i-v_0,v_j-v_0\rangle=L_{i,j}=\frac{\kappa_{i,j}(G)}{\kappa(G)}.$$

Proof. The fact that $\langle v_i-v_0,v_j-v_0\rangle=L_{i,j}$ is immediate from the definition of the monodromy pairing. If either i or j is equal to 0, then $L_{i,j}=\kappa_{i,j}(G)=0$. By Cramer's rule, for $i,j\neq 0$, we have

$$L_{i,j}=\frac{C_{i,j}}{\det(\widetilde{\Delta})},$$

where $C_{i,j}$ is the (i,j)th cofactor of $\widetilde{\Delta}$. By the matrix tree theorem, we have

$$\det(\widetilde{\Delta})=|\,\mathrm{Jac}(G)|=\kappa(G),$$

and by the all minors matrix tree theorem from [Cha82], we have $C_{i,j}=\kappa_{i,j}(G)$. ☐

2.3. **The Rank of a Divisor.** A divisor $D=\sum_{i=0}^n a_i\cdot v_i$ is called *effective* if $a_i\geq 0$ for all i. If a divisor D is not equivalent to an effective divisor, we say that it has rank -1. Otherwise, we define the *rank* of a divisor D to be the maximum integer r such that $D-E$ is equivalent to an effective divisor for all effective divisors E of degree r.

In this section, we reinterpret the rank of a divisor in the language of additive combinatorics. Define the set

$$\mathcal{A}(G):=\{v_i-v_0\mid 0\leq i\leq n\}\subseteq\mathrm{Jac}(G).$$

For a positive integer m, we write

$$m\mathcal{A}(G):=\{a_1+\cdots+a_m\mid a_i\in\mathcal{A}(G)\}.$$

Since $0\in\mathcal{A}(G)$, we have $(m-1)\mathcal{A}(G)\subseteq m\mathcal{A}(G)$ for all m. Note that a divisor D of degree d is equivalent to an effective divisor if and only if $D-dv_0\in\mathcal{A}(G)$. This yields the following result.

Lemma 2.2. [BN07, Theorem 1.7] *For any graph G, the first Betti number $g=|E(G)|-|V(G)|+1$ is the smallest integer such that $g\mathcal{A}(G)=\mathrm{Jac}(G)$.*

Given a divisor D, we write

$$D-\mathcal{A}(G)=\{D-a\mid a\in\mathcal{A}(G)\}.$$

Proposition 2.3. *Let D be a divisor of degree d on a graph G. Then D has rank at least r if and only if*

$$(D-dv_0)-r\mathcal{A}(G)\subseteq(d-r)\mathcal{A}(G).$$

Proof. By definition, D has rank at least r if, for all effective divisors E of degree r, $D-E$ is equivalent to an effective divisor – that is, if $D-E-(d-r)v_0\in(d-r)\mathcal{A}(G)$. Since E is effective if and only if $E-rv_0\in r\mathcal{A}(G)$, we see that D has rank at least r if and only if $(D-dv_0)-r\mathcal{A}(G)\subseteq(d-r)\mathcal{A}(G)$. ☐

The *r-gonality* $\mathrm{gon}_r(G)$ of a graph G is the minimum degree of a divisor with rank at least r. The 1-gonality is typically just called the gonality. By Proposition 2.3, we have the following.

Corollary 2.4. *For a graph G, we have*

$$\mathrm{gon}_r(G) = \min\{d \mid \exists D \in \mathrm{Jac}(G) \text{ such that } D - r\mathcal{A}(G) \subseteq (d-r)\mathcal{A}(G)\}.$$

Note that, since $0 \in r\mathcal{A}(G)$, if $D - r\mathcal{A}(G) \subseteq (d-r)\mathcal{A}(G)$, then $D \in (d-r)\mathcal{A}(G)$. We will make frequent use of this simple observation in the proofs of Theorems 1.1, 5.7, and 5.13. We also have the following lower bound.

Corollary 2.5. *Let H be an abelian group and let $\varphi\colon \mathrm{Jac}(G) \to H$ be a homomorphism. Then*

$$\mathrm{gon}_r(G) \geq \min\{d \mid \exists D \in \mathrm{Jac}(G) \text{ such that } \varphi(D) - r\varphi(\mathcal{A}(G)) \subseteq (d-r)\varphi(\mathcal{A}(G))\}.$$

Corollary 2.5 is useful in applications because, for any vertex v_i in G, we have the homomorphism

$$\langle \cdot, v_i - v_0\rangle\colon \mathrm{Jac}(G) \to \mathbb{Q}/\mathbb{Z},$$

and we can use Theorem 2.1 to completely describe the set

$$\langle \mathcal{A}(G), v_i - v_0\rangle = \{\kappa_{i,j}(G) \mid 0 \leq j \leq n\} \subseteq \mathbb{Z}/\kappa(G)\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.$$

Since every finite subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic, this allows us to bound the gonality of a graph using techniques from additive combinatorics on cyclic groups.

2.4. **Freiman Isomorphisms.** A fundamental concept in additive combinatorics is that of the *Freiman isomorphism.* Let $\mathcal{A}, \mathcal{B}$ be subsets of abelian groups G and H, respectively. A *Freiman isomorphism* of order m from $\mathcal{A}$ to $\mathcal{B}$ is a bijection $\psi\colon \mathcal{A} \to \mathcal{B}$ such that

$$\sum_{i=1}^m a_i = \sum_{i=1}^m a_i' \iff \sum_{i=1}^m \psi(a_i) = \sum_{i=1}^m \psi(a_i')$$

for all $a_1, \ldots, a_m, a_1', \ldots, a_m' \in \mathcal{A}$. The following simple observation will be useful for identifying Freiman isomorphisms.

Proposition 2.6. *Let ψ be an automorphism of a graph G. Then the induced map $\psi_*\colon \mathcal{A}(G) \to \mathcal{A}(G)$ given by $\psi_*(v_i - v_0) = \psi(v_i) - v_0$ is a Freiman isomorphism of arbitrary order.*

Proof. We have

$$\sum_{i=1}^m (v_{j_i} - v_0) \sim \sum_{i=1}^m (v_{j_i'} - v_0) \iff \sum_{i=1}^m v_{j_i} \sim \sum_{i=1}^m v_{j_i'}.$$

Similarly,

$$\sum_{i=1}^m (\psi(v_{j_i}) - v_0) \sim \sum_{i=1}^m (\psi(v_{j_i'}) - v_0) \iff \sum_{i=1}^m \psi(v_{j_i}) \sim \sum_{i=1}^m \psi(v_{j_i'}).$$

Because ψ is an automorphism, we have

$$\sum_{i=1}^m v_{j_i} \sim \sum_{i=1}^m v_{j_i'} \iff \sum_{i=1}^m \psi(v_{j_i}) \sim \sum_{i=1}^m \psi(v_{j_i'})$$

and the result follows. $\square$

3. Outerplanar Graphs

3.1. **Outerplanar Graphs.** We now turn our attention to a specific family of graphs. A graph G is called *outerplanar* if it can be embedded in the plane in such a way that all vertices belong to a single face. An outerplanar graph is called *maximal* if it is simple, and adding an edge between any two non-adjacent vertices results in a non-outerplanar graph. Equivalently, a simple outerplanar graph on $n+1$ vertices is maximal if and only if it has $2n-1$ edges. Note that the first Betti number of a maximal outerplanar graph is $n-1$.

Two examples of maximal outerplanar graphs that we will discuss in this paper are the *fan graphs*, pictured in Figure 1 and the *strip graphs*, pictured in Figure 2. The *fan graph* $\mathcal{F}_n$ is the graph with $n+1$ vertices, and edges between v_0 and v_i for all $i \geq 1$, and between v_i and v_j if $|i-j| = 1$ for all $i, j \geq 1$. See Figure 1.

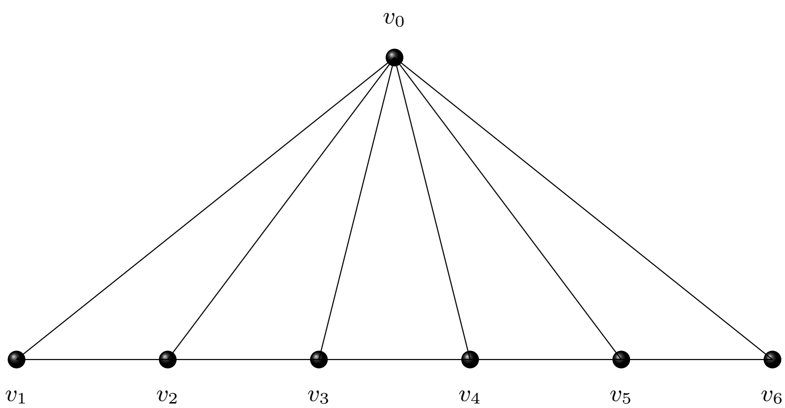

Figure 1. The fan graph $\mathcal{F}_6$.

The *strip graph* G_n is the graph with $n+1$ vertices and edges between v_i and v_j if $|i-j| \in \{1, 2\}$. See Figure 2.

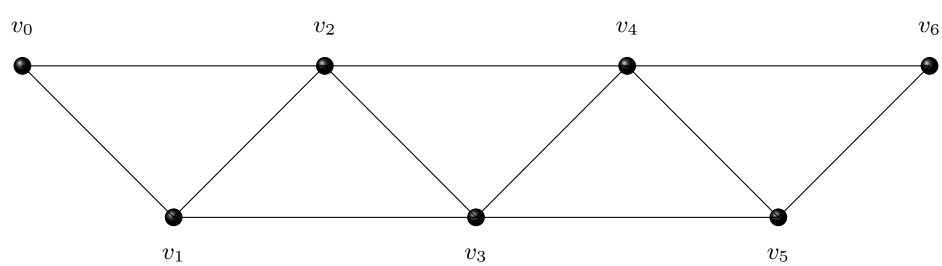

Figure 2. The strip graph G_6.

3.2. **The Jacobian of an Outerplanar Graph.** Throughout, we let F_n denote the nth Fibonacci number, indexed so that $F_0 = 0$ and $F_1 = 1$. In [Sla77], Slater shows that, if G is a maximal outerplanar graph with $n+1$ vertices, exactly 2 of which have valence 2, then $\kappa(G) = F_{2n}$. In addition, we will show that the Jacobian of any such graph is cyclic. We first need the following preliminary lemma.

Lemma 3.1. *Let G be a maximal outerplanar graph with $n+1$ vertices, exactly 2 of which have valence 2. Let v_0 be a vertex of valence 2 and v_1 a vertex adjacent to v_0. Then $\kappa_{1,1}(G) = F_{2n-1}$.*

Proof. The invariant $\kappa_{1,1}(G)$ counts the number of 2-component spanning forests such that v_0 is in one component and v_1 is in the other. Such forests are in bijection with spanning trees containing the edge from v_0 to v_1. Specifically, given a spanning tree containing this edge, remove it to obtain a 2-component spanning forest such that v_0 is in one component and v_1 is in the other. This operation is clearly invertible – given a 2-component spanning forest such that v_0 is in one component and v_1 is in the other, add the edge from v_0 to v_1 to obtain a spanning tree.

Now, let G' be the graph obtained by deleting the two edges adjacent to v_0. As shown in the proof of [Sla77, Proposition 1], G' is a maximal outerplanar graph on n vertices, exactly 2 of which have valence 2. By [Sla77, Proposition 1], we have $\kappa(G') = F_{2n-2}$. It follows that the number of spanning trees in G that do not contain the edge from v_0 to v_1 is F_{2n-2}, hence the number of spanning trees in G that do contain this edge is $F_{2n} - F_{2n-2} = F_{2n-1}$. □

Corollary 3.2. *Let G be a maximal outerplanar graph with $n+1$ vertices, exactly 2 of which have valence 2. Let v_0 be a vertex of valence 2 and v_1 a vertex adjacent to v_0. The map $\varphi\colon \mathrm{Jac}(G) \to \mathbb{Z}/F_{2n}\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ given by*

$$\varphi(D) = \langle D, v_1 - v_0\rangle \pmod{\mathbb{Z}}$$

is an isomorphism. In particular, $\mathrm{Jac}(G) \cong \mathbb{Z}/F_{2n}\mathbb{Z}$.

Proof. Combining Theorem 2.1 with Lemma 3.1, we have

$$\varphi(v_1 - v_0) = \frac{\kappa_{1,1}(G)}{\kappa(G)} = \frac{F_{2n-1}}{F_{2n}} \in \mathbb{Q}/\mathbb{Z}.$$

Since F_{2n-1} and F_{2n} are relatively prime, this element generates the cyclic subgroup of order F_{2n} in $\mathbb{Q}/\mathbb{Z}$, hence φ maps $\mathrm{Jac}(G)$ onto this cyclic subgroup. Since, by [Sla77, Proposition 1], we have $|\mathrm{Jac}(G)| = \kappa(G) = F_{2n}$, it follows that φ is an isomorphism onto this subgroup. □

3.3. **Bounds on the Gonality of Outerplanar Graphs.** In the next sections, we will discuss the gonality of certain families of outerplanar graphs. Here, we note that bounds in the existing literature are insufficient for computing the gonality of these graphs. In [vDdBG20], it is shown that a well-known graph invariant, the *treewidth*, is a lower bound on gonality. Outerplanar graphs, however, have treewidth at most 2, and as we shall see in the later sections, the gonality is often much higher.

Lemma 3.3. *Let G be an outerplanar graph. Then $\mathrm{tw}(G) \leq 2$.*

Proof. Both outerplanar graphs and graphs of treewidth at most 2 have forbidden minor characterizations. Specifically, a graph is outerplanar if and only if it has neither of the forbidden minors K_4 nor $K_{2,3}$ [Die18, Exercise 4.23]. Similarly, a graph has treewidth at most 2 if and only if it does not have the forbidden minor K_4 [Bod98]. It follows that the treewidth of an outerplanar graph is at most 2. □

A divisor $D = \sum_{i=0}^{n} a_i \cdot v_i$ on a graph is *multiplicity-free* if a_i is equal to either 0 or 1 for all i. In [DEM23], Dean, Everett, and Morrison define the *multiplicity-free gonality* $\mathrm{mfgon}(G)$ of a graph G to be the minimum degree of a multiplicity-free divisor of positive rank. Of course, the multiplicity-free gonality is an upper bound on the gonality. For maximal outerplanar graphs, however, the gonality is typically much smaller than the multiplicity-free gonality. Recall that an *independent set* in a graph G is a set of vertices, no two of which are adjacent. The *independence number* $\alpha(G)$ is the maximal size of an independent set.

Lemma 3.4. *Let G be a simple planar graph on $n+1$ vertices, with all of its faces triangles, except for possibly the outer face. Then $\mathrm{mfgon}(G) = n + 1 - \alpha(G)$. In particular, if G is maximal outerplanar, then $\mathrm{mfgon}(G) \geq \frac{n}{2}$.*

Proof. By [DEM23, Lemma 2.4], if $S \subset V(G)$ is an independent set, then

$$D = \sum_{v_i \notin S} v_i$$

is a multiplicity-free divisor of positive rank. It follows that $\mathrm{mfgon}(G) \leq n + 1 - \alpha(G)$.

Now, let D be a multiplicity-free divisor of degree less than $n + 1 - \alpha(G)$. Then there exists a pair of adjacent vertices v and w so that neither v nor w is in the support of D. Now, run Dhar's burning algorithm starting at v. By assumption, any pair of adjacent vertices is contained in a triangle, and since D has at most 1 chip on each vertex, if two vertices of the triangle burn, then so does the third. It follows by induction that every vertex burns, hence D is v-reduced. Since v is not in the support of D, it follows that D does not have positive rank.

For the final statement, note that a maximal outerplanar graph contains a Hamiltonian cycle, so the independence number $\alpha(G)$ is less than or equal to that of the cycle, which is $\lceil \frac{n}{2} \rceil$. □

4. Fan Graphs

The set $\mathcal{A}(\mathcal{F}_n)$ has a particularly nice description.

Lemma 4.1. *For all $1 \leq k \leq n$, we have $\kappa_{1,k}(\mathcal{F}_n) = F_{2n-2k+1}$.*

Proof. By Lemma 3.1, we have $\kappa_{1,1}(\mathcal{F}_n) = F_{2n-1}$. (Note that, in Lemma 3.1, it is v_0 rather than v_1 that has valence 2. However, the number $\kappa_{1,1}$, which counts the number of 2-component spanning forests such that v_0 is in one component and v_1 is in the other, is invariant under switching the labels of v_0 and v_1.)

Now assume that $k \geq 2$. We will show that $\kappa_{1,k}(\mathcal{F}_n) = \kappa_{1,1}(\mathcal{F}_{n-k+1})$. From the previous paragraph, it then follows that $\kappa_{1,k}(\mathcal{F}_n) = F_{2n-2k+1}$. Given a 2-component spanning forest, let T_0 denote the component containing v_0 and T_1 denote the component containing v_1 and v_k. Then T_1 must contain the unique path from v_1 to v_k that does not pass through v_0. By deleting this path, we obtain a 2-component spanning forest in $\mathcal{F}_n \smallsetminus \{v_1, \ldots, v_{k-1}\} \cong \mathcal{F}_{n-k+1}$ such that one component contains v_0, and the other contains v_k. This operation is clearly invertible, hence this yields a bijection between the two sets of 2-component spanning forests, and $\kappa_{1,k}(\mathcal{F}_n) = \kappa_{1,1}(\mathcal{F}_{n-k+1})$. □

Since, by Corollary 3.2, the map $\varphi\colon \mathrm{Jac}(\mathcal{F}_n) \to \mathbb{Z}/F_{2n}\mathbb{Z}$ is an isomorphism, we may identify $\mathrm{Jac}(\mathcal{F}_n)$ with its image under φ. By Lemma 5.2, under this identification, we have the following.

Corollary 4.2. *The set $\mathcal{A}(\mathcal{F}_n)$ consists of 0 and all odd-index Fibonacci numbers between 1 and F_{2n}. In other words,*

$$\mathcal{A}(\mathcal{F}_n) = \{0\} \cup \{F_{2k-1} \mid 1 \leq k \leq n\} \subset \mathbb{Z}/F_{2n}\mathbb{Z}.$$

Theorems 1.3 and 1.5 follow immediately.

Proof of Theorem 1.3. There is an involution of $\mathcal{F}_n$ that fixes v_0 and sends v_k to v_{n+1-k} for all $k \geq 1$. Thus, by Proposition 2.6 the involution $\iota\colon \mathcal{A}(\mathcal{F}_n) \to \mathcal{A}(\mathcal{F}_n)$ given by $\iota(0) = 0$ and $\iota(F_{2k-1}) = F_{2n-2k+1}$ is a Freiman isomorphism of arbitrary order. □

Proof of Theorem 1.5. By [Hen18, Theorem 11], $\mathrm{gon}(\mathcal{F}_n) = \phi_n$, and by Corollary 2.4, we have

$$\mathrm{gon}(\mathcal{F}_n) = \phi_n = \min\{d \mid \exists x \in \mathbb{Z}/F_{2n}\mathbb{Z} \text{ such that } x - \mathcal{A}(\mathcal{F}_n) \subseteq (d-1)\mathcal{A}(\mathcal{F}_n)\}.$$

Finally, by Corollary 4.2, we have $x - \mathcal{A}(\mathcal{F}_n) \subseteq (d-1)\mathcal{A}(\mathcal{F}_n)$ if and only if x satisfies the two conditions in the statement of the theorem. □

5. Strip Graphs

5.1. **The Set $\mathcal{A}(G_n)$.** We now turn to the strip graphs G_n. We aim to represent the set $\mathcal{A}(G_n)$ explicitly as a subset of $\mathbb{Z}/F_{2n}\mathbb{Z}$. To do this, we will use Theorem 2.1.

Lemma 5.1. *We have*

$$\begin{aligned}
\kappa_{1,k}(G_n) &= F_{2n-2} + \kappa_{1,k-2}(G_{n-2}) \text{ for } 2 \leq k \leq n\\
\kappa_{1,1}(G_n) &= F_{2n-1}\\
\kappa_{1,0}(G_n) &= 0.
\end{aligned}$$

Proof. To see that $\kappa_{1,0}(G_n) = 0$, note that if one component of a 2-component forest contains v_0, then the other does not. The fact that $\kappa_{1,1}(G_n) = F_{2n-1}$ is Lemma 3.1.

We now assume that $k \geq 2$. Given a 2-component spanning forest, let T_0 denote the component containing v_0 and T_1 denote the component containing v_1 and v_k. Note that the edge from v_0 to v_1 cannot appear in such a spanning forest. There are two cases:

(1) If the edge from v_0 to v_2 is not in T_0, then $T_0 = \{v_0\}$. As such, T_1 is a spanning tree of the graph $G_n \smallsetminus \{v_0\} \cong G_{n-1}$. By [Sla77, Proposition 1] (or [KG75, Lemma 1]), there are exactly F_{2n-2} such spanning trees.
(2) If the edge from v_0 to v_2 is in T_0, then the edge from v_1 to v_3 is in any path from v_1 to v_k. As such, this edge is contained in T_1. Thus, the restriction of our spanning forest to $G_n \smallsetminus \{v_0, v_1\} \cong G_{n-2}$ is a 2-component spanning forest, where one component contains v_2 and the other component contains both v_3 and v_k. By definition, the number of such spanning forests is equal to $\kappa_{1,k-2}(G_{n-2})$.

Combining the two cases, we obtain

$$\kappa_{1,k}(G_n) = F_{2n-2} + \kappa_{1,k-2}(G_{n-2}).$$

□

As in Section 4, since the map φ is an isomorphism, we may identify the set $\mathcal{A}(G_n)$ with its image under φ.

Lemma 5.2. *For all $0 \leq k \leq n$, we have $\kappa_{1,k}(G_n) = F_k F_{2n-k}$.*

Proof. We fix n and prove this by induction on k. Note that the base cases $k = 0, 1$ are done in Lemma 5.1. For $k \geq 1$, by Lemma 5.1, we have

$$\begin{aligned}
\kappa_{1,k+1}(G_n) &= F_{2n-2} + \kappa_{1,k-1}(G_{n-2})\\
&= F_{2n-2} + F_{k-1}F_{2n-k+1},
\end{aligned}$$

where the second equality holds by induction. Now, by the identity on the top of page 48 from [BE22], the above is equal to

$$\begin{aligned}
&F_{k-1}F_{2n-k-2} + F_kF_{2n-k-1} + F_{k-1}F_{2n-k-3}\\
&= F_kF_{2n-k-2} + F_{k-1}F_{2n-k-2} + F_kF_{2n-k-3} + F_{k-1}F_{2n-k-3}\\
&= F_{k+1}F_{2n-k-2} + F_{k+1}F_{2n-k-3}\\
&= F_{k+1}F_{2n-k-1}.
\end{aligned}$$

□

Corollary 5.3. *We have*

$$\mathcal{A}(G_n) = \{F_kF_{2n-k} \mid 0 \leq k \leq n\} \subset \mathbb{Z}/F_{2n}\mathbb{Z}.$$

Theorems 1.2 and 1.4 follow immediately.

Proof of Theorem 1.2. The statements about $\mathcal{A}(\mathcal{F}_n)$ and $\mathcal{A}(G_n)$ follow directly from Lemma 2.2, using the fact that the first Betti number of $\mathcal{F}_n$ is $n-1$. To see the statement about $\mathcal{B}(G_n)$, note that by the Catalan identity, one has $F_{n-k}F_{n+k} - F_n^2 = (-1)^{k+1}F_k^2$. Since translation by $-F_n^2$ is a Freiman isomorphism of arbitrary order, the result follows. $\square$

Proof of Theorem 1.4. There is an involution of G_n that sends v_k to v_{n-k} for all k. Thus, by Proposition 2.6, the involution $\iota\colon \mathcal{A}(G_n) \to \mathcal{A}(G_n)$ given by $\iota(F_kF_{2n-k}) = F_{n-k}F_{n+k}$ is a Freiman isomorphism of arbitrary order. $\square$

5.2. **The Zeckendorf Form.** Our goal for the remainder of the paper is to use Corollary 2.4 to compute the gonality of G_n. To do this, we need to describe the sets $m\mathcal{A}(G_n)$ for certain small values of m. In this section, we introduce a fundamental tool for describing these sets.

Every nonnegative integer can be written uniquely as a sum of non-consecutive Fibonacci numbers. This expression is called the *Zeckendorf form* of the number. When $x \in \mathbb{Z}/F_{2n}\mathbb{Z}$, we define the Zeckendorf form of x to be the Zeckendorf form of its unique representative in the range $0 \leq x < F_{2n}$. We will primarily be interested in the *leading terms* of a number written in Zeckendorf form, which are the largest Fibonacci numbers appearing in this sum. Equivalently, the leading term of a number x is the largest Fibonacci number smaller than x. Our next goal is to write every element of $\mathcal{A}(G_n)$ in Zeckendorf form.

Lemma 5.4. [FP98, Theorem 1] *For $m \geq n$, the Zeckendorf form of F_mF_n is:*

$$F_mF_n = \begin{cases} \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} F_{m+n+2-4r} & \text{if } n \text{ is even} \\ F_{m-n+1} + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} F_{m+n+2-4r} & \text{if } n \text{ is odd.} \end{cases}.$$

Because of this, we have the following corollary.

Corollary 5.5. *All elements of $\mathcal{A}(G_n)$ have the following Zeckendorf form:*

$$F_kF_{2n-k} = \begin{cases} \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} F_{2n+2-4r} & \text{if } k \text{ is even} \\ F_{2n-2k+1} + \left(\sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} F_{2n+2-4r}\right) & \text{if } k \text{ is odd.} \end{cases}$$

As such, all non-zero elements of $\mathcal{A}(G_n)$, except F_{2n-1}, have F_{2n-2} as the leading term of their Zeckendorf form.

Corollary 5.5 can also be seen by induction, using Lemma 5.1.

5.3. **A Lower Bound on the Gonality of Strip Graphs.** In Section 6, we will prove that $\mathrm{gon}(G_n) = 5$ when n is sufficiently large. To show that $\mathrm{gon}(G_n) \leq 5$, it suffices to exhibit a divisor of degree 5 and positive rank. The more difficult part of the argument is to establish a lower bound on the gonality. Here, we demonstrate a lower bound of 3 when $n \geq 4$. Later in the paper, we will improve this bound to 4 when $n \geq 6$ and then to 5 when $n \geq 8$. These arguments all follow a similar approach, but the latter bounds are much more involved, with many more cases. We structure the argument in this way is to highlight the technique in a simpler case before proving the main theorem.

The *scramble number* of a graph was first defined in [HJJS22], where it was shown that it is a lower bound on gonality. Here, we show that the scramble number of G_n is 3 for all $n \geq 4$.

Lemma 5.6. *If $n \geq 4$, then* $\mathrm{sn}(G_n) = 3$.

Proof. The graph G_n has a topological subgraph isomorphic to the graph $C_{3;2,2,1}$ from [EM23], where it is shown to have scramble number 3. Since the scramble number is topological subgraph monotone, we see that $\mathrm{sn}(G_n) \geq 3$.

To show that $\mathrm{sn}(G_n) \leq 3$, we use [CFG+22, Theorem 1], which shows that the scramble number is bounded above by a graph invariant known as the *screewidth.* For i an even number less than n, let $X_i = \{v_i, v_{i+1}\}$, and if n is even, let $X_n = \{v_n\}$. Let $\mathcal{T}$ be the path with nodes X_i where X_i is adjacent to X_{i+2} for all i. The adhesion of each link in $\mathcal{T}$ is either 2 or 3, and the adhesion of each node is either 1 or 2. Thus, the width of this tree-cut decomposition is 3, hence $\mathrm{scw}(G_n) \leq 3$. $\square$

Lemma 5.6 implies that, for $n \geq 4$, the gonality of G_n is at least 3. We can also prove this using our approach. We find it helpful to illustrate our approach first in this simple case.

Theorem 5.7. *If $n \geq 4$, then $\mathrm{gon}(G_n) \geq 3$.*

Proof. We show that for $n \geq 4$, there does not exist an element $D \in \mathcal{A}(G_n)$ such that $D - x \in \mathcal{A}(G_n)$ for all $x \in \mathcal{A}(G_n)$. It will then follow from Corollary 2.4 that $\mathrm{gon}(G_n) \geq 3$ for $n \geq 4$. We break this into cases.

(1) If $D = F_{2n-1}$, then $D - F_{2n-2} = F_{2n-3}$. Since $F_{2n-3} \notin \{0, F_{2n-1}\}$, and the leading term of its Zeckendorf form is not F_{2n-2}, we have $D - F_{2n-2} = F_{2n-3} \notin \mathcal{A}(G_n)$ by Corollary 5.5.
(2) If $D = 0$, then $D - 2F_{2n-3} = F_{2n-2} + F_{2n-4}$. By Corollary 5.5, the second largest even-index Fibonacci number of an element of $\mathcal{A}(G_n)$ is F_{2n-6}. As such, $D - 2F_{2n-3} \notin \mathcal{A}(G_n)$.
(3) Otherwise, by Corollary 5.5, D has leading term F_{2n-2} in its Zeckendorf form. Thus, by Corollary 5.5 again, we have $D - F_{2n-2} \in \mathcal{A}(G_n)$ if and only if either $D - F_{2n-2} = 0$ or $D - F_{2n-2} = F_{2n-1}$. In the first case, we see that D is equal to F_{2n-2}, and the second, we see that D is equal to 0. We have already considered the case where $D = 0$. If $D = F_{2n-2}$, then $D - F_{2n-1} = F_{2n-1} + F_{2n-4}$, which is not in $\mathcal{A}(G_n)$ by Corollary 5.5.

$\square$

In Theorem 5.13 below, we will prove that $\mathrm{gon}(G_n) \geq 4$ when $n \geq 6$. Because $\mathrm{sn}(G_n) = 3$, the scramble number is insufficient to compute this bound. Instead, we will argue in a similar way to the proof of Theorem 5.7.

5.4. **The Set $2\mathcal{A}(G_n)$.** In this section, we describe the Zeckendorf form of all elements of $2\mathcal{A}(G_n)$. Describing this set will require us to write down the Zeckendorf form of the sum of two numbers. This is a component of *Zeckendorf arithmetic*, as described in [FP98, Fen03]. A key idea, used implicitly in all the proofs of this section, is that if x and y are both smaller than F_k, then $x + y$ is smaller than F_{k+2}. Thus, if we know only the first few terms of the Zeckendorf forms of x and y, we can compute the leading terms of the Zeckendorf form of $x + y$.

By Corollary 5.5, most elements of $2\mathcal{A}(G_n)$ are of the following form.

Lemma 5.8. *Let $2 \leq a \leq b \leq n$. Then the leading term of the Zeckendorf form of $F_aF_{2n-a} + F_bF_{2n-b}$ is F_{2n-1}, followed by either F_{2n-3} or F_{2n-4}. Moreover:*

(1) if the leading terms are $F_{2n-1} + F_{2n-3} + F_{2n-5}$, then $a = b = 3$,
(2) if the leading terms are $F_{2n-1} + F_{2n-4}$, then either $a = b = 2$ or the next term is F_{2n-6}; if there is another term, it is at most F_{2n-9}, and
(3) if the leading terms are $F_{2n-1} + F_{2n-3} + F_{2n-6}$, then $a = 3, b > 3$ and next term is at most F_{2n-9}.

Proof. By Corollary 5.5 we have

$$F_aF_{2n-a} + F_bF_{2n-b} = \sum_{r=1}^{\lfloor \frac{a}{2} \rfloor} F_{2n-4r+2} + \sum_{r=1}^{\lfloor \frac{b}{2} \rfloor} F_{2n-4r+2} + (\varepsilon_a F_{2n-2a+1} + \varepsilon_b F_{2n-2b+1}),$$

where

$$\varepsilon_a = \begin{cases} 0 & \text{if } a \text{ is even} \\ 1 & \text{if } a \text{ is odd.} \end{cases}$$

Combining the like terms, the above is equal to

$$= \sum_{r=1}^{\lfloor \frac{a}{2} \rfloor} 2F_{2n-4r+2} + \sum_{r=\lfloor \frac{a}{2} \rfloor+1}^{\lfloor \frac{b}{2} \rfloor} F_{2n-4r+2} + (\varepsilon_a F_{2n-2a+1} + \varepsilon_b F_{2n-2b+1})$$

$$= \sum_{r=1}^{\lfloor \frac{a}{2} \rfloor} (F_{2n-4r+3} + F_{2n-4r}) + \sum_{r=\lfloor \frac{a}{2} \rfloor+1}^{\lfloor \frac{b}{2} \rfloor} F_{2n-4r+2} + (\varepsilon_a F_{2n-2a+1} + \varepsilon_b F_{2n-2b+1}).$$

If $a = 2$ and $b = 3$, then the expression above is $F_{2n-1} + F_{2n-4} + F_{2n-5} = F_{2n-1} + F_{2n-3}$. If $a = b = 3$, then it is equal to $F_{2n-1} + F_{2n-4} + 2F_{2n-5} = F_{2n-1} + F_{2n-3} + F_{2n-5}$. If $a = 2$ and $b \neq 3$, the leading terms of the above expression are $F_{2n-1} + F_{2n-4}$, and if $b > 3$ the next term is F_{2n-6}. Moreover, if the next possible largest term is F_{2n-9}. Similarly, if $a = 3$ and $b > 3$, then the leading terms of the above expression are $F_{2n-1} + F_{2n-4} + F_{2n-5} + F_{2n-6} = F_{2n-1} + F_{2n-3} + F_{2n-6}$. The next possible largest term is F_{2n-9}.

Otherwise, we have that the left hand sum has leading terms $F_{2n-1} + F_{2n-4} + F_{2n-5} + F_{2n-8}$, which becomes $F_{2n-1} + F_{2n-3} + F_{2n-8}$ in Zeckendorf form. Because the righthand sum has leading term at most F_{2n-9}, it does not affect the first two leading terms, so in this case, we have leading terms $F_{2n-1} + F_{2n-3}$. □

We now turn to the elements of $2\mathcal{A}(G_n)$ that are not described by Lemma 5.8.

Lemma 5.9. *If $a \geq 4$, the leading term of $F_{2n-1} + F_a F_{2n-a}$ in Zeckendorf form is F_{2n-6}, with the next possible leading term either F_{2n-9} or F_{2n-10}.*

Proof. Let $a \geq 4$. We note

$$\begin{aligned} F_{2n-1} + F_a F_{2n-a} &= F_{2n-1} + \left(\sum_{r=1}^{\lfloor \frac{a}{2} \rfloor} F_{2n-4r+2} \right) + \varepsilon_a F_{2n-2a+1} \\ &= \left(\sum_{r=2}^{\lfloor \frac{a}{2} \rfloor} F_{2n-4r+2} \right) + \varepsilon_a F_{2n-2a+1} \pmod{F_{2n}}. \end{aligned}$$

If $a = 4$, then $\varepsilon_a = 0$, and if $a \geq 5$, then $F_{2n-2a+1}$ is at most F_{2n-9}. As such, the leading term is unaffected by this term, so the leading term is F_{2n-6}. Moreover, if $a = 5$, the second leading term is F_{2n-9}. Otherwise, we have that the second leading term is F_{2n-10}, since we obtain the term with $r = 3$. □

Below we list the remaining elements of $2\mathcal{A}(G_n)$ not covered by one of the previous cases.

Lemma 5.10. *We have:*

- $2F_{2n-1} = F_{2n-3} \pmod{F_{2n}}$
- $F_{2n-1} + F_{2n-2} = 0 \pmod{F_{2n}}$
- $F_{2n-1} + 2F_{2n-3} = F_{2n-5} \pmod{F_{2n}}$.

We summarize the results of this subsection in the following corollary.

Corollary 5.11. *Let $D \in 2\mathcal{A}(G_n)$. Then either:*

(1) $D \in \mathcal{A}(G_n)$,
(2) $D = F_{2n-3}$,
(3) $D = F_{2n-5}$,
(4) D has leading term F_{2n-6}, followed by either F_{2n-9} or F_{2n-10}.
(5) D has leading term F_{2n-1}, followed by either F_{2n-3} or F_{2n-4}. Moreover:
   (a) if the leading terms are $F_{2n-1} + F_{2n-3} + F_{2n-5}$, then $D = F_{2n-1} + F_{2n-3} + F_{2n-5}$, and

(b) *if the leading terms are* $F_{2n-1} + F_{2n-4}$*, then either* $D = F_{2n-1} + F_{2n-4}$ *or the next term is* F_{2n-6}*; if there is another term, it is at most* F_{2n-9}*.*
(c) *if the leading terms are* $F_{2n-1} + F_{2n-3} + F_{2n-6}$*, then the next term is at most* F_{2n-9}*.*

The following lemma will not be used in this section, but we will need it in Section 6.

Lemma 5.12. *If* $D \in 2\mathcal{A}(G_n)$ *has leading terms* $F_{2n-1} + F_{2n-3} + F_{2n-8}$*, then the next possible leading term is* F_{2n-10}*.*

Proof. From the proof of Lemma 5.8, we have that elements in $D \in 2\mathcal{A}(G_n)$ have possible leading terms $F_{2n-1} + F_{2n-3} + F_{2n-8}$ if they are the sum of two elements $D_1 + D_2$ with leading terms $F_{2n-2} + F_{2n-6}$. If either has next term F_{2n-9}, then we have that the leading terms of D would be $F_{2n-1} + F_{2n-3} + F_{2n-7}$ or $F_{2n-1} + F_{2n-3} + F_{2n-6}$. In a similar manner, if both have next term F_{2n-10}, then D would have leading terms $F_{2n-1} + F_{2n-3} + F_{2n-7}$. As such, the only remaining case is $D_1 = F_{2n-2} + F_{2n-6}$ and, $D_2 = D_1$ or D_2 has leading terms $F_{2n-2} + F_{2n-6} + F_{2n-10}$. Thus, if D has leading terms $F_{2n-1} + F_{2n-3} + F_{2n-8}$, then the next possible leading term is F_{2n-10}. □

5.5. **A Stronger Lower Bound on the Gonality of Strip Graphs.** To compute the gonality of G_n, it will be helpful to use the Zeckendorf representation of some elements of $-\mathcal{A}(G_n) \pmod{F_{2n}}$. Below we leave a table:

D	$-D$
F_{2n-1}	F_{2n-2}
F_{2n-2}	F_{2n-1}
$2F_{2n-3}$	$F_{2n-2} + F_{2n-4}$
$3F_{2n-4}$	$F_{2n-2} + F_{2n-4} + F_{2n-7}$
$5F_{2n-5}$	$F_{2n-2} + F_{2n-4} + F_{2n-8}$
$8F_{2n-6}$	$F_{2n-2} + F_{2n-4} + F_{2n-8} + F_{2n-11}$
$13F_{2n-7}$	$F_{2n-2} + F_{2n-4} + F_{2n-8} + F_{2n-12}$

We now show that $\text{gon}(G_n) \geq 4$ when $n \geq 6$. Our approach will be similar to that of Theorem 5.13, using our description of the set $2\mathcal{A}(G_n)$.

Theorem 5.13. *If* $n \geq 6$*, then* $\text{gon}(G_n) \geq 4$*.*

Proof. We show that, for $n \geq 6$, there does not exist an element $D \in 2\mathcal{A}(G_n)$ such that $D - x \in 2\mathcal{A}(G_n)$ for all $x \in \mathcal{A}(G_n)$. It will then follow from Corollary 2.4 that $\text{gon}(G_n) \geq 4$ for $n \geq 6$. We again break this into cases. The first 5 cases cover elements of $\mathcal{A}(G_n)$, the next 2 cover elements of $F_{2n-1} + \mathcal{A}(G_n)$, and the last case covers all the remaining elements of $2\mathcal{A}(G_n)$.

(1) If $D = 0$, then $D - 2F_{2n-3} = F_{2n-2} + F_{2n-4}$. By Corollary 5.11, if $D \in 2\mathcal{A}(G_n)$ has leading term F_{2n-2}, then $D \in \mathcal{A}(G_n)$. By Corollary 5.5, if an element has leading term F_{2n-2}, then the next largest even-index term is F_{2n-6}. As such, no element in $2\mathcal{A}(G_n)$ has leading terms $F_{2n-2} + F_{2n-4}$. Hence $D - 2F_{2n-3} \notin 2\mathcal{A}(G_n)$.
(2) If $D = F_{2n-1}$, then $D - 2F_{2n-3} = D + F_{2n-2} + F_{2n-4} = F_{2n-4}$. By Corollary 5.11, no element in $2\mathcal{A}(G_n)$ has leading term F_{2n-4}, so $D - 2F_{2n-3} \notin 2\mathcal{A}(G_n)$.
(3) If $D = F_{2n-2}$, then $D - 5F_{2n-5}$ has leading terms $F_{2n-1} + F_{2n-3} + F_{2n-6} + F_{2n-8}$. By Corollary 5.11, if an element has these first 3 leading terms in $2\mathcal{A}(G_n)$, then the next term is at most F_{2n-9}. Since $F_{2n-8} > F_{2n-9}$, we see that $D - 5F_{2n-5}$ is not in $2\mathcal{A}(G_n)$.
(4) If $D = 2F_{2n-3}$, then $D - 3F_{2n-4} = F_{2n-7}$, which is not in $2\mathcal{A}(G_n)$ by Corollary 5.11.
(5) If $D \in \mathcal{A}(G_n) \setminus \{0, F_{2n-1}, F_{2n-2}, 2F_{2n-3}\}$, then $D = F_k F_{2n-k}$ with $k \geq 4$. By Corollary 5.5, its leading terms are $F_{2n-2} + F_{2n-6}$. As such, $D - 2F_{2n-3} = D + F_{2n-2} + F_{2n-4}$ has leading terms $F_{2n-1} + F_{2n-3} + F_{2n-5}$. By Corollary 5.11, the only element with these leading terms in $2\mathcal{A}(G_n)$ is $4F_{2n-3}$. If this is the case, then $D = 6F_{2n-3}$. However, $6F_{2n-3} = F_{2n-2} + F_{2n-7}$, which is not in $2\mathcal{A}(G_n)$ by Corollary 5.11.

(6) If $D = 2F_{2n-1} = F_{2n-3}$, then $D - 3F_{2n-4} = D + F_{2n-2} + F_{2n-4} + F_{2n-7} = F_{2n-1} + F_{2n-4} + F_{2n-7}$. By Corollary 5.11, if an element has leading terms $F_{2n-1} + F_{2n-4}$, then the next term is F_{2n-6}. Thus, $D - 3F_{2n-4} \notin 2\mathcal{A}(G_n)$.

(7) If $D = F_{2n-1} + F_k F_{2n-k}$ with $k \geq 3$, then by Lemma 5.9, $D - F_{2n-2}$ has leading terms $F_{2n-1} + F_{2n-5}$ (if $k = 3$) or $F_{2n-1} + F_{2n-6}$ (if $k \geq 4$). By Corollary 5.11, there is no element in $2\mathcal{A}(G_n)$ with these leading terms.

(8) Finally, assume that D is a sum of two elements of $\mathcal{A}(G_n)$, neither of which is 0 or F_{2n-1}. By Lemma 5.8, D has leading terms $F_{2n-1} + F_{2n-3}$ or $F_{2n-1} + F_{2n-4}$. Then $D - F_{2n-1}$ has leading term F_{2n-3} or F_{2n-4}. By Corollary 5.11, there is no element of $2\mathcal{A}(G_n)$ with leading term F_{2n-4}, and the only element of $2\mathcal{A}(G_n)$ with leading term F_{2n-3} is F_{2n-3} itself. It follows that $D = F_{2n-1} + F_{2n-3}$. However, $D - 3F_{2n-4} = F_{2n-2} + F_{2n-7}$. Hence $D - 3F_{2n-4} \notin 2\mathcal{A}(G_n)$ by Corollary 5.11.

□

6. The Gonality of Strip Graphs

In this section, we prove Theorem 1.1. Our proof follows the same strategy as that of Theorems 5.7 and 5.13, though there are many more cases. We start by showing that $\text{gon}(G_n) \leq 5$ for all n.

6.1. **A Divisor of Rank 1.** We now find a divisor of degree 5 and rank at least 1 on the strip graph G_n. This shows that the gonality of G_n is at most 5.

Lemma 6.1. *For $3 \leq k \leq n$, we have*

$$F_{2n} + 2F_{2n-1} - F_k F_{2n-k} = F_{k-2} F_{2n-k+2} + 3F_{k-1} F_{2n-k+1}.$$

Proof. We fix n and prove this by induction on k. For the base case $k = 3$, we have

$$F_{2n} + 2F_{2n-1} - 2F_{2n-3} = 2F_{2n-1} + 2F_{2n-2} - F_{2n-3} = F_{2n-1} + 3F_{2n-2}.$$

Now, assume the equation holds for k. We will prove it for $k+1$ by the following tedious calculation.

$$\begin{aligned}
&F_{2n} + 2F_{2n-1} - F_{k+1} F_{2n-k-1} \\
&= F_{2n} + 2F_{2n-1} - F_k F_{2n-k-1} - F_{k-1} F_{2n-k-1} \\
&= F_{2n} + 2F_{2n-1} + F_k F_{2n-k-2} - F_k F_{2n-k} - F_{k-1} F_{2n-k-1} \\
&= F_{k-2} F_{2n-k+2} + 3F_{k-1} F_{2n-k+1} + F_k F_{2n-k-2} - F_{k-1} F_{2n-k-1}
\end{aligned}$$

where the last equality holds by inductive hypothesis. Now, the above is equal to

$$\begin{aligned}
&3F_{k-1} F_{2n-k+1} + F_{k-2} F_{2n-k+1} + F_{k-2} F_{2n-k} + F_k F_{2n-k-2} - F_{k-1} F_{2n-k-1} \\
&= 2F_{k-1} F_{2n-k+1} + F_k F_{2n-k+1} + F_{k-2} F_{2n-k} + F_k F_{2n-k-2} - F_{k-1} F_{2n-k-1} \\
&= 2F_{k-1} F_{2n-k+1} + F_k F_{2n-k} + F_k F_{2n-k-1} + F_{k-2} F_{2n-k} + F_k F_{2n-k-2} - F_{k-1} F_{2n-k-1} \\
&= 2F_{k-1} F_{2n-k+1} + 2F_k F_{2n-k} + F_{k-2} F_{2n-k} - F_{k-1} F_{2n-k-1} \\
&= F_{k-1} F_{2n-k+1} + 2F_k F_{2n-k} + F_{k-1} F_{2n-k} + F_{k-1} F_{2n-k-1} + F_{k-2} F_{2n-k} - F_{k-1} F_{2n-k-1} \\
&= F_{k-1} F_{2n-k+1} + 2F_k F_{2n-k} + F_{k-1} F_{2n-k} + F_{k-2} F_{2n-k} \\
&= F_{k-1} F_{2n-k+1} + 3F_k F_{2n-k}
\end{aligned}$$

□

This yields the following result.

Lemma 6.2. *If $D = 3v_0 + 2v_1$, then D has rank at least 1.*

Proof. By Proposition 2.3, it suffices to show that $D - 5v_0 - \mathcal{A}(G_n) \subseteq 4\mathcal{A}(G_n)$. Note that $D - 5v_0 = 2v_1 - 2v_0$, so by Lemma 5.2 we have $\varphi(D - 5v_0) = 2F_{2n-1}$. By Lemma 6.1, we see that $2F_{2n-1} - F_k F_{2n-k} \in 4\mathcal{A}(G_n)$ for all $k \geq 3$. It therefore suffices to check the cases where $k \leq 2$. For these cases, we have

$$2F_{2n-1} - 0 = 2F_{2n-1} \in 2\mathcal{A}(G_n) \subseteq 4\mathcal{A}(G_n)$$
$$2F_{2n-1} - F_{2n-1} = F_{2n-1} \in \mathcal{A}(G_n) \subseteq 4\mathcal{A}(G_n) \text{ and}$$
$$2F_{2n-1} - F_{2n-2} = F_{2n-1} - F_{2n-3} = F_{2n-2} + 2F_{2n-3} \in 3\mathcal{A}(G_n) \subseteq 4\mathcal{A}(G_n).$$

□

Lemma 6.2 demonstrates that the gonality of G_n is at most 5. One can also check that the divisor $D = 3v_0 + 2v_1$ has positive rank "by hand", using Dhar's burning algorithm to compute the v_i-reduced divisor equivalent to D for all i. We have chosen to avoid this in order to emphasize our approach.

6.2. **The Set $3A$.** In this section, we describe the Zeckendorf form of all elements of $3\mathcal{A}(G_n)$. The approach here is similar to that of Section 5.4. In particular, we use Corollary 5.5 to describe the Zeckendorf form of every element of $\mathcal{A}(G_n)$, and Corollary 5.11 to describe that of every element of $2\mathcal{A}(G_n)$, and then use Zeckendorf arithmetic to describe their sum.

Lemma 6.3. *If $D \in 2\mathcal{A}(G_n)$ has leading terms $F_{2n-1} + F_{2n-4}$, then $D + F_{2n-1} \in \mathcal{A}(G_n)$.*

Proof. By the proof of Lemma 5.8, $D \in F_{2n-2} + \mathcal{A}(G_n)$, hence $D + F_{2n-1} \in \mathcal{A}(G_n)$. □

Lemma 6.4. *If $D \in 2\mathcal{A}(G_n)$ has leading terms $F_{2n-1} + F_{2n-4}$ and $D' \in \mathcal{A}(G_n)$ has leading term F_{2n-2}, then either:*

(1) $D + D' = F_{2n-4}$,
(2) $D + D'$ has leading terms $F_{2n-4} + F_{2n-6}$, with next possible term F_{2n-9} or F_{2n-10},
(3) $D + D'$ has leading terms $F_{2n-3} + F_{2n-6}$, with next possible term F_{2n-9} or F_{2n-10},
(4) $D + D'$ has leading terms $F_{2n-3} + F_{2n-7}$, or
(5) $D + D'$ has leading terms $F_{2n-3} + F_{2n-8}$.

Proof. By Lemma 5.8, either $D = F_{2n-1} + F_{2n-4}$, or the next leading term is F_{2n-6}. By Corollary 5.5, either $D' = F_{2n-2}$, or the next leading term is F_{2n-5}, or F_{2n-6}. The proof then follows by case analysis, for each possible combination of leading terms of D and D'. □

The remaining lemmas in this section follow from a straightforward case analysis, similar to that of Lemma 6.4. We omit the details.

Lemma 6.5. *If $D \in 2\mathcal{A}(G_n)$ has leading terms $F_{2n-1} + F_{2n-3}$, then either*

(1) $D + F_{2n-1} = F_{2n-3} + F_{2n-4} + F_{2n-7}$,
(2) $D + F_{2n-1}$ has leading terms $F_{2n-2} + F_{2n-4}$, with next possible term smaller than F_{2n-9},
(3) $D + F_{2n-1}$ has leading terms $F_{2n-2} + F_{2n-5} + F_{2n-7}$, or
(4) $D + F_{2n-1}$ has leading terms $F_{2n-2} + F_{2n-5} + F_{2n-8}$, with next possible term F_{2n-10}.

Lemma 6.6. *If $D \in 2\mathcal{A}(G_n)$ has leading terms $F_{2n-1} + F_{2n-3}$, then either:*

(1) $D + F_{2n-2} = F_{2n-3} + F_{2n-5}$,
(2) $D + F_{2n-2}$ has leading terms $F_{2n-3} + F_{2n-6}$, with next possible term F_{2n-9} or F_{2n-10},
(3) $D + F_{2n-2}$ has leading terms $F_{2n-3} + F_{2n-7}$, or
(4) $D + F_{2n-2}$ has leading terms $F_{2n-3} + F_{2n-8}$.

Lemma 6.7. *If $D \in 2\mathcal{A}(G_n)$ has leading terms $F_{2n-1} + F_{2n-3}$, then either:*

(1) $D + F_{2n-2} + F_{2n-5} = F_{2n-2} + F_{2n-7}$,
(2) $D + F_{2n-2} + F_{2n-5}$ has leading terms $F_{2n-2} + F_{2n-10}$,
(3) $D + F_{2n-2} + F_{2n-5}$ has leading terms $F_{2n-3} + F_{2n-5} + F_{2n-7}$, or

(4) $D + F_{2n-2} + F_{2n-5}$ *has leading terms* $F_{2n-3} + F_{2n-5} + F_{2n-8}$.

Lemma 6.8. *If* $D \in 2\mathcal{A}(G_n)$ *has leading terms* $F_{2n-1} + F_{2n-3}$ *and* $D' \in \mathcal{A}(G_n)$ *has leading terms* $F_{2n-2} + F_{2n-6}$, *then either:*

(1) $D + D' = F_{2n-2} + F_{2n-9}$,
(2) $D + D'$ *has leading terms* $F_{2n-2} + F_{2n-10}$,
(3) $D + D'$ *has leading terms* $F_{2n-3} + F_{2n-5}$,
(4) $D + D'$ *has leading terms* $F_{2n-3} + F_{2n-5} + F_{2n-8}$, *or*
(5) $D + D'$ *has leading terms* $F_{2n-3} + F_{2n-6} + F_{2n-8}$.

Lemma 6.9. *If* $D' \in \mathcal{A}(G_n)$, *then either:*

(1) $F_{2n-3} + D' = F_{2n-1} + F_{2n-5}$, *or*
(2) $F_{2n-3} + D'$ *has leading terms* $F_{2n-1} + F_{2n-6}$, *with next possible term* F_{2n-9} *or* F_{2n-10}.

Lemma 6.10. *If* $D' \in \mathcal{A}(G_n)$, *then either:*

(1) $F_{2n-5} + D' = F_{2n-2} + F_{2n-5}$, *or*
(2) $F_{2n-5} + D'$ *has leading terms* $F_{2n-2} + F_{2n-4}$, *with next possible term* F_{2n-7}, F_{2n-9} *or* F_{2n-10}.

Lemma 6.11. *If* $D \in 2\mathcal{A}(G_n)$ *has leading term* F_{2n-6} *and* $D' \in \mathcal{A}(G_n)$, *then either:*

(1) $D + D' = F_{2n-1} + F_{2n-6}$,
(2) $D + D' = F_{2n-2} + F_{2n-4}$, *or*
(3) $D+D'$ *has leading terms* $F_{2n-2}+F_{2n-5}$, *with next possible term* F_{2n-7} *or* $(F_{2n-8}+F_{2n-10})$.

We summarize the results of Lemmas 6.4-6.11 as follows:

Corollary 6.12. *Let* $D \in 3\mathcal{A}(G_n) \smallsetminus 2\mathcal{A}(G_n)$.

(1) If D *has leading term* F_{2n-4}, *then either:*
  (a) $D = F_{2n-4}$, *or*
  (b) D *has leading terms* $F_{2n-4} + F_{2n-6}$, *followed by either* F_{2n-9} *or* F_{2n-10}.
(2) If D *has leading term* F_{2n-3}, *then either:*
  (a) D *has leading terms* $F_{2n-3} + F_{2n-8}$,
  (b) D *has leading terms* $F_{2n-3} + F_{2n-7}$,
  (c) D *has leading terms* $F_{2n-3} + F_{2n-6}$,
  (d) $D = F_{2n-3} + F_{2n-5}$,
  (e) D *has leading terms* $F_{2n-3} + F_{2n-5} + F_{2n-8}$, *or*
  (f) D *has leading terms* $F_{2n-3} + F_{2n-5} + F_{2n-7}$.
(3) If D *has leading term* F_{2n-2}, *then either:*
  (a) D *has leading terms* $F_{2n-2} + F_{2n-10}$,
  (b) $D = F_{2n-2} + F_{2n-9}$,
  (c) $D = F_{2n-2} + F_{2n-7}$,
  (d) $D = F_{2n-2} + F_{2n-5}$,
  (e) D *has leading terms* $F_{2n-2} + F_{2n-5} + F_{2n-8}$ *with next possible term* F_{2n-10},
  (f) D *has leading terms* $F_{2n-2} + F_{2n-5} + F_{2n-7}$,
  (g) $D = F_{2n-2} + F_{2n-4}$, *or*
  (h) D *has leading terms* $F_{2n-2} + F_{2n-4}$, *followed by either* F_{2n-7} *or a term smaller than* F_{2n-8}.
(4) If D *has leading term* F_{2n-1}, *then either:*
  (a) D *has leading terms* $F_{2n-1} + F_{2n-6}$, *followed by either* F_{2n-9} *or* F_{2n-10},
  (b) $D = F_{2n-1} + F_{2n-6}$, *or*
  (c) $D = F_{2n-1} + F_{2n-5}$.

6.3. **Proof of Theorem 1.1.** We now complete our proof of the gonality of G_n.

Proof of Theorem 1.1. By Lemma 6.2, we have $\text{gon}(G_n) \leq 5$. By [AR18], the Brill-Noether existence conjecture holds for all graphs of genus at most 5. As a consequence, $\text{gon}(G_n) \leq \lceil \frac{n}{2} \rceil$ for $n \leq 6$. When $n = 7$, we note that the divisor $2v_4 + 2v_5$ has positive rank, hence $\text{gon}(G_7) \leq 4$.

For $n \leq 7$, it therefore suffices to show that $\text{gon}(G_n) \geq \lceil \frac{n+1}{2} \rceil$. If $n = 0$ or 1, then G_n is a tree, and hence has gonality 1. If $n = 2$ or 3, then then G_n is not a tree, hence it has gonality at least 2. If $n = 4$ or 5, then $\text{gon}(G_n) \geq 3$ by Theorem 5.7, and if $n = 6$ or 7, then $\text{gon}(G_n) \geq 4$ by Theorem 5.13.

For the rest of the proof, assume that $n \geq 8$. We will show that there does not exist a $D \in 3\mathcal{A}(G_n)$ such that $D - x \in 3\mathcal{A}(G_n)$ for all $x \in A$. It will then follow from Corollary 2.4 that $\text{gon}(G_n) \geq 5$ for $n \geq 8$. As in the proofs of Theorem 5.7 and Theorem 5.13, we shall proceed by examining the different possible leading terms of D. Although there are many cases, nearly all are proven by straightforward calculations.

(1) Let D have leading term F_{2n-6}. Then $D - 2F_{2n-3} = D + F_{2n-2} + F_{2n-4}$ has leading terms $F_{2n-2} + F_{2n-4} + F_{2n-6}$. By Corollary 6.12 there is no such case, so $D - 2F_{2n-3} \notin 3\mathcal{A}(G_n)$.
(2) Let D have leading term F_{2n-5}. By Corollary 6.12, D must be in $2\mathcal{A}(G_n)$, and by Corollary 5.11, we see that $D = F_{2n-5}$. Note that $D - 3F_{2n-4} = F_{2n-1} + F_{2n-7}$. By Corollary 6.12, no element in $3\mathcal{A}(G_n)$ has such leading terms, so $D - 3F_{2n-4} \notin 3\mathcal{A}(G_n)$.
(3) Let D have leading term F_{2n-4}. We break this into subcases:
   (a) Let D have leading terms $F_{2n-4} + F_{2n-6}$. Then $D - 2F_{2n-3}$ has leading terms $F_{2n-1} + F_{2n-5} + F_{2n-8}$. By Corollary 6.12, the only element in $3\mathcal{A}(G_n)$ with leading terms $F_{2n-1} + F_{2n-5}$ is $F_{2n-1} + F_{2n-5}$ itself. As such, $D - 2F_{2n-3} \notin 3\mathcal{A}(G_n)$.
   (b) Let $D = F_{2n-4}$. Then $D - 5F_{2n-5}$ has leading terms $F_{2n-1} + F_{2n-6} + F_{2n-8}$. By Corollary 6.12, we have that $D - 5F_{2n-5} \notin 3\mathcal{A}(G_n)$.
(4) Let D has leading term F_{2n-3}. We again break this into subcases.
   (a) Let D have leading terms $F_{2n-3} + F_{2n-8}$. Then $D - F_{2n-1}$ has leading terms $F_{2n-1} + F_{2n-8}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.
   (b) Let D have leading terms $F_{2n-3} + F_{2n-7}$. Then $D - F_{2n-1}$ has leading terms $F_{2n-1} + F_{2n-7}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.
   (c) Let D have leading terms $F_{2n-3} + F_{2n-5}$. Then $D - F_{2n-2}$ has leading terms $F_{2n-1} + F_{2n-3} + F_{2n-5}$. By Corollary 6.12, if $D - F_{2n-2} \in 3\mathcal{A}(G_n)$, we see that it must in fact be in $2\mathcal{A}(G_n)$. By Lemma 5.8, it follows that $D - F_{2n-2} = F_{2n-1} + F_{2n-3} + F_{2n-5}$. In particular, $D = F_{2n-3} + F_{2n-5}$. Then $D - 8F_{2n-6} = F_{2n-1} + F_{2n-3} + F_{2n-8} + F_{2n-11}$. If this element is in $3\mathcal{A}(G_n)$, then it is in $2\mathcal{A}(G_n)$ by Corollary 6.12, but this is not the case by Lemma 5.12.
   (d) Let D have leading terms $F_{2n-3} + F_{2n-6}$. We must consider further subcases.
      (i) If $D = F_{2n-3} + F_{2n-6}$, then $D - 5F_{2n-5} = F_{2n-1} + F_{2n-4} + F_{2n-6} + F_{2n-8}$, which is not in $3\mathcal{A}(G_n) \setminus 2\mathcal{A}(G_n)$ by Corollary 6.12. Moreover, by Corollary 5.11 since $F_{2n-8} > F_{2n-9}$, $D - 5F_{2n-5} \notin 2\mathcal{A}(G_n)$.
      (ii) If D has leading terms $F_{2n-3} + F_{2n-6} + F_{2n-8}$, then $D - F_{2n-2}$ has leading terms $F_{2n-1} + F_{2n-6} + F_{2n-8}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.
      (iii) If D has leading terms $F_{2n-3} + F_{2n-6} + F_{2n-9}$, then $D - 3F_{2n-4}$ has leading terms $F_{2n-1} + F_{2n-3} + F_{2n-9}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.
      (iv) If D has leading terms $F_{2n-3} + F_{2n-6} + F_{2n-k}$ with $k \geq 10$. Then $D - 5F_{2n-5}$ has leading terms $F_{2n-1} + F_{2n-4} + F_{2n-6} + F_{2n-8} + F_{2n-k}$. By the same argument as with $F_{2n-3} + F_{2n-6}$, $D - 5F_{2n-5} \notin 3\mathcal{A}(G_n)$.
(5) Let D have leading term F_{2n-2}.
   (a) Let $D = F_{2n-2}$. Then $D - 5F_{2n-5} = F_{2n-1} + F_{2n-3} + F_{2n-6} + F_{2n-8}$, which is not in $3\mathcal{A}(G_n) \setminus 2\mathcal{A}(G_n)$ by Corollary 6.12. Moreover, by Corollary 5.11, $D - 5F_{2n-5} \notin 2\mathcal{A}(G_n)$. Altogether, $D - 5F_{2n-5} \notin 3\mathcal{A}(G_n)$.

(b) Let D have leading terms $F_{2n-2}+F_{2n-k}$ with $k\geq 7$. By Corollary 6.12, this implies that $k\in\{7,8,9,10\}$. Then $D-8F_{2n-6}$ has leading terms $F_{2n-1}+F_{2n-3}+F_{2n-5}$. If $D\neq F_{2n-2}+F_{2n-10}$, then there are additional terms. By Corollary 6.12, the only element of $3\mathcal{A}(G_n)$ with leading terms $F_{2n-1}+F_{2n-3}+F_{2n-5}$ is itself, so $D-8F_{2n-6}\notin 3\mathcal{A}(G_n)$. If $D=F_{2n-2}+F_{2n-10}$, then $D-3F_{2n-4}=F_{2n-1}+F_{2n-3}+F_{2n-5}+F_{2n-10}$. Again, the only element of $3\mathcal{A}(G_n)$ with leading terms $F_{2n-1}+F_{2n-3}+F_{2n-5}$ is itself, so $D-3F_{2n-4}\notin 3\mathcal{A}(G_n)$.

(c) Let D have leading terms $F_{2n-2}+F_{2n-6}$. Then by Corollaries 6.12 and 5.11, we have $D\in\mathcal{A}(G_n)$. By Corollary 5.5, either $D=F_{2n-2}+F_{2n-6}$, or the next term is F_{2n-k} with $k\geq 9$. We break this into further cases.

(i) If $D=F_{2n-2}+F_{2n-6}$, we have $D-8F_{2n-6}=F_{2n-1}+F_{2n-3}+F_{2n-5}+F_{2n-7}+F_{2n-9}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(ii) If $k=9$, then $D-8F_{2n-6}$ has leading term F_{2n-0}, hence $D-8F_{2n-6}\notin 3\mathcal{A}(G_n)$.

(iii) If $k=10$, then either $D=F_{2n-2}+F_{2n-6}+F_{2n-10}$, or there are more terms. If there are more terms, then the leading term of $D-8F_{2n-6}$ is smaller than F_{2n-6}, so $D-8F_{2n-6}\notin 3\mathcal{A}(G_n)$. If $D=F_{2n-2}+F_{2n-6}+F_{2n-10}$, then $D-2F_{2n-3}=F_{2n-1}+F_{2n-3}+F_{2n-5}+F_{2n-8}+F_{2n-10}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(iv) If $k>10$, then the leading terms of $D-8F_{2n-6}$ are $F_{2n-1}+F_{2n-3}+F_{2n-5}+F_{2n-7}+F_{2n-9}+F_{2n-k}$, so $D-8F_{2n-6}\notin 3\mathcal{A}(G_n)$ by Corollary 6.12.

(d) Let D have leading terms $F_{2n-2}+F_{2n-5}+F_{2n-k}$ with $k\geq 7$. Then $D-2F_{2n-3}$ has leading term F_{2n-k}. By Corollary 6.12, the smallest leading term possible for an element in $3\mathcal{A}(G_n)$ is F_{2n-6}, so $D-2F_{2n-3}\notin 3\mathcal{A}(G_n)$

(e) Let $D=F_{2n-2}+F_{2n-5}$. Then $D-8F_{2n-6}=F_{2n-8}+F_{2n-11}$. Again, the smallest leading term possible for an element in $3\mathcal{A}(G_n)$ is F_{2n-6}, so $D-2F_{2n-3}\notin 3\mathcal{A}(G_n)$

(f) Let D have leading terms $F_{2n-2}+F_{2n-4}$. By Corollary 6.12, if there is another term, then it is either F_{2n-7} or smaller than F_{2n-8}. If D has no other term, then $D-5F_{2n-5}$ has leading terms $F_{2n-6}+F_{2n-8}$, which is not in $3\mathcal{A}(G_n)\setminus\mathcal{A}(G_n)$ by Corollary 6.12. Moreover, by Corollary 5.11, $D-5F_{2n-5}\notin\mathcal{A}(G_n)$.

If the next term is of the form F_{2n-k} with $k\geq 7$, then $D-F_{2n-2}$ has leading terms $F_{2n-4}+F_{2n-k}$. Hence, $D-F_{2n-2}\notin 3\mathcal{A}(G_n)$ by Corollary 6.12.

(6) Let D have leading term F_{2n-1}.

(a) Let D have leading terms $F_{2n-1}+F_{2n-6}$. If there are no other terms, then $D-8F_{2n-6}=F_{2n-4}+F_{2n-6}+F_{2n-8}+F_{2n-11}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12. Otherwise, the next term is either F_{2n-9} or F_{2n-10}. Then $D-3F_{2n-4}$ has leading term F_{2n-3}, followed by either F_{2n-9} or F_{2n-10}. In either case, by Corollary 6.12, we see that $D-3F_{2n-4}\notin 3\mathcal{A}(G_n)$.

(b) Let $D=F_{2n-1}$. Then $D-3F_{2n-4}=F_{2n-4}+F_{2n-7}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(c) Let $D=F_{2n-1}+F_{2n-5}$. Then $D-8F_{2n-6}=F_{2n-3}+F_{2n-8}+F_{2n-11}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(d) Let D have leading terms $F_{2n-1}+F_{2n-3}+F_{2n-8}$. Then $D-2F_{2n-3}$ has leading terms $F_{2n-2}+F_{2n-8}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(e) Let D have leading terms $F_{2n-1}+F_{2n-3}+F_{2n-7}$. Then $D-2F_{2n-3}$ has leading terms $F_{2n-2}+F_{2n-7}$. By Corollary 6.12, the only element in $3\mathcal{A}(G_n)$ with these leading terms is $F_{2n-2}+F_{2n-7}$ itself. Hence, if there are more terms, then $D-2F_{2n-3}\notin 3\mathcal{A}(G_n)$. Otherwise, if $D=F_{2n-1}+F_{2n-3}+F_{2n-7}$, then $D-8F_{2n-6}=F_{2n-2}+F_{2n-6}+F_{2n-11}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

(f) Let D have leading terms $F_{2n-1}+F_{2n-3}+F_{2n-6}$. If $D=F_{2n-1}+F_{2n-3}+F_{2n-6}$, then $D-8F_{2n-6}=F_{2n-2}+F_{2n-6}+F_{2n-8}+F_{2n-11}$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12. Otherwise, the next term is either F_{2n-9} or F_{2n-10}. Then $D-3F_{2n-4}$

has leading terms $F_{2n-2}+F_{2n-5}$, followed by either F_{2n-9} or F_{2n-10}. By Corollary 6.12, this is not in $3\mathcal{A}(G_n)$.

(g) Let D have leading terms $F_{2n-1}+F_{2n-3}+F_{2n-5}$. Then $D = F_{2n-1}+F_{2n-3}+F_{2n-5}$ and $D-8F_{2n-6} = F_{2n-2}+F_{2n-5}+F_{2n-8}+F_{2n-11}$. Again, by Corollary 6.12, this is not an element of $3\mathcal{A}(G_n)$.

(h) Finally, let D have leading terms $F_{2n-1}+F_{2n-4}$. If there is a next term, then it is F_{2n-6}. We again consider further subcases.

  (i) If D has leading terms $F_{2n-1}+F_{2n-4}+F_{2n-6}+F_{2n-9}$, then $D-8F_{2n-6}$ has leading term $F_{2n-2}+F_{2n-11}$. Hence $D-8F_{2n-6} \notin 3\mathcal{A}(G_n)$.

  (ii) If D has leading terms $F_{2n-1}+F_{2n-4}+F_{2n-6}+F_{2n-10}$, then $D-8F_{2n-6}$ has leading term F_{2n-2}. If there is a next term, then $D-8F_{2n-6}$ has leading terms $F_{2n-2}+F_{2n-k}$ with $k \geq 12$, which is not in $3\mathcal{A}(G_n)$ by Corollary 6.12.

  The cases above reduce the problem to three specific divisors D. Specifically, $D = F_{2n-1}+F_{2n-4}$, $D = F_{2n-1}+F_{2n-4}+F_{2n-6}$, and $D = F_{2n-1}+F_{2n-4}+F_{2n-6}+F_{2n-10}$. We will use Dhar's burning algorithm starting at v_8 to show that the v_8-reduced divisors equivalent to these do not contain v_8 in their support. As a consequence, none of these divisors have positive rank. Equivalently, this shows that $D-21F_{2n-8} \notin 3\mathcal{A}(G_n)$ for these three divisors D.

  (iii) Let $D = F_{2n-1}+F_{2n-4}$. Identifying $\mathrm{Jac}(G_n)$ with $\mathbb{Z}/F_{2n-2}\mathbb{Z}$ by the isomorphism φ, we have $D = 2v_0+2v_2$. We shall run Dhar's burning algorithm starting at the vertex v_8. Initially, everything burns except for v_0. As such, we fire v_0 to obtain v_1+3v_2. At the next step, everything burns except v_0, v_1 and v_2. As such, we fire these vertices to obtain $v_2+2v_3+v_4$. At this point, everything burns except for v_0, v_1, v_2 and v_3. Firing these vertices, we obtain $3v_4+v_5$, at which point the entire graph burns. It follows that $3v_4+v_5$ is v_8-reduced. Since it does not contain v_8 in its support, the divisor D does not have positive rank.

  (iv) Let $D = F_{2n-1}+F_{2n-4}+F_{2n-6}$. Equivalently, $D = 2v_0+v_2+v_4$. We again run Dhar's burning algorithm starting at v_8. Again, initially, everything burns except for v_0. As such, we fire v_0 to obtain $v_1+2v_2+v_4$. At the next step, everything burns except v_0, v_1 and v_2. Firing these vertices, we obtain $2v_3+2v_4$. Now, everything burns except for v_0, v_1, v_2, v_3 and v_4. Firing these vertices, we obtain $v_3+2v_5+v_6$, at which point the entire graph burns.

  (v) Let $D = F_{2n-1}+F_{2n-4}+F_{2n-6}+F_{2n-10}$. Equivalently, $D = 2v_0+v_2+v_6$. We once again run Dhar's burning algorithm starting at v_8. Initially, everything burns except for v_0. As such, we fire v_0 to obtain $v_1+2v_2+v_6$. At the next step, everything burns except v_0, v_1 and v_2. As such, we fire v_0, v_1 and v_2 to obtain $2v_3+v_4+v_6$, at which point the entire graph burns.

□

Proof of Theorem 1.6. By Theorem 1.1, $\mathrm{gon}(G_n) = \min\{\lceil \frac{n}{2} \rceil, 5\}$, and by Corollary 2.4, we have

$$\mathrm{gon}(G_n) = \min\{d \mid \exists x \in \mathbb{Z}/F_{2n}\mathbb{Z} \text{ such that } x - \mathcal{A}(G_n) \subseteq (d-1)\mathcal{A}(G_n)\}.$$

Finally, by Corollary 5.3, we have $x - \mathcal{A}(G_n) \subseteq (d-1)\mathcal{A}(G_n)$ if and only if x satisfies the condition in the statement of the theorem. □

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