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Veronese Quotient Models of $\overline{\mathrm{M}}_{0,n}$ and Conformal Blocks

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Introduction

The moduli space of Deligne–Mumford stable n-pointed rational curves, $\overline{\mathrm{M}}_{0,n}$, is a natural compactification of the moduli space of smooth pointed genus 0 curves and has figured prominently in the literature. A central motivating question is to describe other compactifications of $\mathrm{M}_{0,n}$ that receive morphisms from $\overline{\mathrm{M}}_{0,n}$. From the perspective of Mori theory, this is tantamount to describing semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$. This work is concerned with two recent constructions that each yield an abundance of such semi-ample divisors on $\overline{\mathrm{M}}_{0,n}$ and with the relationship between them. The first construction comes from geometric invariant theory (GIT) and the second from conformal field theory.

There are birational models of $\overline{\mathrm{M}}_{0,n}$ obtained via GIT that are moduli spaces of pointed rational normal curves of fixed degree d, where the curves and the marked points are weighted by nonnegative rational numbers $(\gamma, A) = (\gamma, (a_1, \dots, a_n))$ [Gi; GiJM; GiSi]. These *Veronese quotients* $V^d_{\gamma,A}$ are remarkable in that they specialize to nearly every known compactification of $\mathrm{M}_{0,n}$ [GiJM]. There are birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to these GIT quotients, and their natural polarization can be pulled back along this morphism to yield semi-ample divisors $\mathcal{D}_{\gamma,A}$ on $\overline{\mathrm{M}}_{0,n}$.

A second recent development in the birational geometry of $\overline{\mathrm{M}}_{0,n}$ involves divisors that arise from conformal field theory. These divisors are first Chern classes of vector bundles of conformal blocks $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ on the moduli stack $\overline{\mathcal{M}}_{g,n}$. Constructed using the representation theory of affine Lie algebras [Fa; TUY], these vector bundles depend on the choice of a simple Lie algebra $\mathfrak{g}$, a nonnegative integer ℓ, and an n-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant integral weights in the Weyl alcove for $\mathfrak{g}$ of level ℓ. For the definition of vector bundles of conformal blocks and related representation-theoretic notations, see Section 4.1. Vector bundles of conformal blocks are globally generated when $g = 0$ [Fa, Lemma 2.5], and their first Chern classes $c_1(\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})) = \mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$, the conformal block divisors, are semi-ample.

For $\gamma = 0$, it was shown in [Gi; GiG] that the divisors $\mathcal{D}_{0,A}$ coincide with conformal block divisors for $\mathfrak{sl}_r$ and level 1. Our guiding philosophy is that there is a

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general correspondence between Veronese quotients and conformal block divisors. After first giving background information about Veronese quotients in Section 1, in support of this we:

2. derive intersection numbers for all $\mathcal{D}_{\gamma,A}$ with curves on $\overline{\mathrm{M}}_{0,n}$ (Theorem 2.1);
3. give a new modular interpretation for a particular family of Veronese quotients (Section 3);
4. show that the models described in Section 3 are given by conformal block divisors (Theorem 4.6);
5. provide several conjectures (and supporting evidence) generalizing these results (Section 5).

In order to further motivate and put this work in context, we next say a bit more about items 2–5.

SECTION 2. THE CLASSES OF ALL VERONESE QUOTIENT DIVISORS $\mathcal{D}_{\gamma,A}$. For each allowable (γ, A), there exists a morphism $\varphi_{\gamma,\mathcal{A}} \colon \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$. In Section 2 we study the divisors $\mathcal{D}_{\gamma,A} = \varphi^*_{\gamma,\mathcal{A}}(L_{\gamma,\mathcal{A}})$, where $L_{\gamma,\mathcal{A}}$ is the canonical ample polarization on $V^d_{\gamma,A}$. In Theorem 2.1 we give a formula for the intersection of $\mathcal{D}_{\gamma,A}$ with F-curves (Definition 1.5), a collection of curves that span the vector space of numerical equivalence classes of 1-cycles on $\overline{\mathrm{M}}_{0,n}$. Theorem 2.1 is a vast generalization of formulas that have appeared for $d \in \{1,2\}$ and for $\gamma = 0$ (see [AS; Gi; GiG; GiSi]) and captures a great deal of information about the nef cone $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. For example, since adjacent chambers in the GIT cone correspond to adjacent faces of the nef cone, by combining Theorem 2.1 with the results of [GiJM] we could describe many faces of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. Moreover, Theorem 2.1 is equivalent to giving the class of $\mathcal{D}_{\gamma,A}$ in the Néron–Severi space. To illustrate this claim, we give the classes of the conformal block divisors with S_n-invariant weights (Corollary 2.12) and the particularly simple formula for the divisors that give rise to the maps to the Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ (Example 2.13).

SECTION 3. A NEW MODULAR INTERPRETATION FOR A PARTICULAR FAMILY OF VERONESE QUOTIENTS. Much work has focused on alternative compactifications of $\mathrm{M}_{0,n}$ [B; Fe; Gi; GiJM; GiSi; Ha; Ka1; Ka2; LMa; Si; Sm]. As shown in [GiJM], every choice of allowable weight data for Veronese quotients (Definition 1.1) yields such a compactification and nearly every previously known compactification arises as such a Veronese quotient. In Section 3, we study the particular Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ for $1 \le \ell \le g$. In Theorem 3.5 we provide a new modular interpretation for these spaces and note that, prior to [GiJM], this moduli space had not appeared in the literature (see Remark 3.1). Our main application is to show that the nontrivial conformal block divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ are pullbacks of ample classes from Veronese quotients (Theorem 4.6). For this, we prove several results concerning morphisms between these Veronese quotients (see Corollary 3.7 and Proposition 3.8).

SECTION 4. A PARTICULAR FAMILY OF CONFORMAL BLOCK DIVISORS. In [Gi; GiG] it was shown that the divisors $\mathcal{D}_{0,A}$ coincide with conformal block divisors of $\mathfrak{sl}_r$ and level 1, and in [AGS] it is shown that the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2}}$ are proportional for the two special cases $\ell = 1$ and g. In [AGS] the authors ask whether there is a more general correspondence between $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^n)$ and Veronese quotient divisors. Theorem 4.6 gives a complete, affirmative answer to their question (cf. Remark 4.7). One of the main insights in this work is that, while not proportional for the remaining levels $\ell \in \{2, \dots, g-1\}$, the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2}}$ lie on the same face of the nef cone of $\overline{\mathrm{M}}_{0,n}$. In other words, the two semi-ample divisors define maps to isomorphic birational models of $\overline{\mathrm{M}}_{0,n}$. The corresponding birational models are precisely the spaces described in Section 3.

Section 5. Generalizations. Evidence suggests that $\mathfrak{sl}_r$ conformal block divisors with nonzero weights give rise to compactifications of $\mathrm{M}_{0,n}$ and that these compactifications coincide with Veronese quotients. This is certainly true for $\ell = 1$ and for the family of higher-level $\mathfrak{sl}_2$ divisors considered in this paper as well as for a large number of cases found using [S], software written for Macaulay 2 by D. Swinarski. In Section 5.1 we provide evidence in support of these ideas in the $\mathfrak{sl}_2$ cases. In Section 5.2 we describe consequences of and evidence for Conjecture 5.6, which asserts that conformal block divisors (with strictly positive weights) separate points on $\mathrm{M}_{0,n}$.

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1. Background on Veronese Quotients

We begin by reviewing general facts about Veronese quotients, including a description of them as moduli spaces of weighted pointed (generalized) Veronese curves (Section 1.1) and the morphisms $\varphi_{\gamma,A} \colon \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$ (Section 1.2), from [GiJM].

1.1. The Spaces $V^d_{\gamma,A}$

Following [GiJM], we write $\mathrm{Chow}(1, d, \mathbb{P}^d)$ for the irreducible component of the Chow variety parameterizing curves of degree d in $\mathbb{P}^d$ and their limit cycles, and we consider the incidence correspondence

$$U_{d,n} := \{(X, p_1, \dots, p_n) \in \mathrm{Chow}(1, d, \mathbb{P}^d) \times (\mathbb{P}^d)^n : p_i \in X \ \forall i\}.$$

There is a natural action of $\mathrm{SL}(d+1)$ on $U_{d,n}$, and one can form the GIT quotients $U_{d,n} /\!/_L \mathrm{SL}(d+1)$, where L is an $\mathrm{SL}(d+1)$-linearized ample line bundle. The Chow variety and each copy of $\mathbb{P}^d$ have a tautological ample line bundle $\mathcal{O}_{\mathrm{Chow}}(1)$ and $\mathcal{O}_{\mathbb{P}^d}(1)$, respectively. By taking external tensor products of them, for each sequence of positive rational numbers $(\gamma, (a_1, \dots, a_n))$ we obtain a $\mathbb{Q}$-linearized ample line bundle $L = \mathcal{O}(\gamma) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_n)$.

DEFINITION 1.1. We say that a linearization L is *allowable* if it is an element of the set Δ^0, where

$$\Delta^0 = \Big\{(\gamma, A) = (\gamma, (a_1, \dots, a_n)) \in \mathbb{Q}^{n+1}_{\geq 0} : \\ \gamma < 1,\ 0 < a_i < 1, \text{ and } (d-1)\gamma + \textstyle\sum_i a_i = d+1\Big\}.$$

If $a_1 = \cdots = a_n = a$, then we write a^n for $A = (a_1, \dots, a_n)$.

Let

$$V^d_{\gamma,A} := U_{d,n} /\!/_{\gamma,A} \mathrm{SL}(d+1).$$

We call GIT quotients of this form *Veronese quotients* because they are quotients of a space parameterizing pointed Veronese curves.

Given $(X, p_1, \dots, p_n) \in U_{d,n}$, we may think of a choice $L \in \Delta^0$ as an assignment of a rational weight γ to the curve X and another weight a_i to each of the marked points p_i. The conditions $\gamma < 1$ and $0 < a_i < 1$ for all i imply that the quotient $U_{d,n} /\!/_{\gamma,A} \mathrm{SL}(d+1)$ is a compactification of $\mathrm{M}_{0,n}$ [GiJM, Prop. 2.10]. As reflected in Lemma 1.2 to follow, the quotients have a modular interpretation parameterizing pointed degenerations of Veronese curves.

By taking $d = 1$ and $\gamma = 0$, one obtains the GIT quotients $(\mathbb{P}^1)^n /\!/_A \mathrm{SL}(2)$ with various weight data A. This quotient, which appears in [MuFoK, Chap. 3] under the heading "An Elementary Example", has been studied by many authors. It was generalized first to $d = 2$ by Simpson in [Si] and later by Giansiracusa and Simpson in [GiSi]; it was then generalized for arbitrary d and $\gamma = 0$ by Giansiracusa in [Gi]. More generally, the quotients for arbitrary d and $\gamma \geq 0$ are defined and studied by Giansiracusa, Jensen, and Moon in [GiJM].

The semistable points of $U_{d,n}$ with respect to the linearization (γ, A) have the following nice geometric properties.

LEMMA 1.2 [GiJM, Cor. 2.4, Prop. 2.5, Cor. 2.6, Cor. 2.7]. *For an allowable choice (γ, A) as in Definition 1.1, a semistable point $(X, p_1, \dots, p_n)$ of $U_{d,n}$ has the following properties.*

(i) *X is an arithmetic genus 0 curve having at worst multinodal singularities.*

(ii) *Given a subset $J \subset \{1, \dots, n\}$, the marked points $\{p_j : j \in J\}$ can coincide at a point of multiplicity m on X provided*

$$(m-1)\gamma + \sum_{j \in J} a_j \leq 1.$$

In particular, a collection of marked points can coincide at a smooth point of X as long as their total weight is at most 1. Also, a semistable curve cannot have a singularity of multiplicity m unless $\gamma \leq 1/(m-1)$.

(iii) *X is nondegenerate; that is, it is not contained in a hyperplane.*

1.2. The Morphisms $\varphi_{\gamma,A} : \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$

In [GiJM] the authors prove the existence and several properties of birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to Veronese quotients.

Proposition 1.3 [GiJM, Thm. 1.2, Prop. 4.7]. *For an allowable choice* (γ, A), *there exists a regular birational map* $\varphi_{\gamma,A}\colon \overline{\mathrm{M}}_{0,n} \to V^d_{\gamma,A}$ *preserving the interior* $\mathrm{M}_{0,n}$. *Moreover,* $\varphi_{\gamma,A}$ *factors through the contraction maps* ρ_A *to Hassett's moduli spaces* $\overline{\mathrm{M}}_{0,A}$:

$$\begin{tikzcd} \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_A"'] \arrow[dr, "\varphi_{\gamma,A}"] & \\ \overline{\mathrm{M}}_{0,A} \arrow[r, "\phi_\gamma"] & V^d_{\gamma,A}. \end{tikzcd}$$

By the definition of the projective GIT quotient, there is a natural choice of an ample line bundle on each GIT quotient. By pulling it back to $\overline{\mathrm{M}}_{0,n}$, we obtain a semi-ample divisor.

Definition 1.4. Let $L = (\gamma, A)$ be an allowable linearization on $U_{d,n}$, and let $\bar{L} = L /\!/_L \mathrm{SL}(d+1)$ be the natural $\mathbb{Q}$-ample line bundle on $V^d_{\gamma,A}$. Then $\mathcal{D}_{\gamma,A}$ is defined as the semi-ample line bundle $\varphi^*_{\gamma,A}(\bar{L})$.

Next, following [Ha] and [GiJM], we describe the F-curves contracted by $\varphi_{\gamma,A}$ and ρ_A. Toward this end we first define F-curves, which together span the vector space $N_1(\overline{\mathrm{M}}_{0,n})$ of numerical equivalence classes of 1-cycles on $\overline{\mathrm{M}}_{0,n}$.

Definition 1.5. Let $A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4 = [n] = \{1, \dots, n\}$ be a partition into nonempty subsets, and put $n_i = |A_i|$. There is an embedding

$$f_{A_1,A_2,A_3,A_4}\colon \overline{\mathrm{M}}_{0,4} \to \overline{\mathrm{M}}_{0,n}$$

given by attaching four legs $L(A_i)$ to $(X, (p_1, \dots, p_4)) \in \overline{\mathrm{M}}_{0,4}$ at the marked points. More specifically, to each p_i we attach a stable $(n_i + 1)$-pointed fixed rational curve $L(A_i) = (X_i, (p^i_1, \dots, p^i_{n_i}, p^i_a))$ by identifying p^i_a and p_i; if $n_i = 1$ for some i, we just keep p_i as is. The image is a curve in $\overline{\mathrm{M}}_{0,n}$ whose equivalence class is the *F-curve* denoted $F(A_1, A_2, A_3, A_4)$. Each member of the F-curve consists of a (varying) *spine* and four (fixed) *legs*.

In many parts of this paper we will focus on symmetric divisors and F-curves, in which case the equivalence class is determined by the number of marked points on each leg. We shall write F_{n_1,n_2,n_3,n_4} for *any* F-curve class $F(A_1, A_2, A_3, A_4)$ with $|A_i| = n_i$.

The F-curves $F(A_1, A_2, A_3, A_4)$ contracted by the Hassett morphism ρ_A are precisely those for which one of the legs, say $L(A_i)$, has weight $\sum_{j\in A_i} a_j \geq \sum_{j\in[n]} a_j - 1$. We can always order the cells of the partition so that A_4 is the heaviest—that is, $\sum_{j\in A_4} a_j \geq \sum_{j\in A_i} a_j$ for all i. Because the morphism $\varphi_{\gamma,A}$ factors through ρ_A, these curves are also contracted by $\varphi_{\gamma,A}$. This morphism may contract additional F-curves as well, which we describe here.

As proved in [GiJM], the map $\varphi_{\gamma,A}$ contracts those curves $F(A_1, A_2, A_3, A_4)$ for which the sum of the degrees of the four legs is equal to d. We next define two functions ϕ and σ that are useful for computing the degree of the legs of an F-curve.

DEFINITION 1.6 [GiJM, Sec. 3.1]. Consider the function $\phi\colon 2^{[n]} \times \Delta^0 \to \mathbb{Q}$ given by

$$\phi(J,\gamma,A) = \frac{a_J - 1}{1-\gamma}, \quad \text{where } a_J = \sum_{j\in J} a_j \text{ for } J \in 2^{[n]}.$$

For a fixed allowable linearization $(\gamma, A) = (\gamma, (a_1,\dots,a_n))$ (cf. Definition 1.1), let

$$\sigma(J) = \begin{cases} \lceil \phi(J,\gamma,A)\rceil & \text{if } 1 < a_J < a_{[n]} - 1, \\ 0 & \text{if } a_J < 1, \\ d & \text{if } a_J > a_{[n]} - 1. \end{cases}$$

Finally, for $(X, p_1,\dots,p_n) \in U_{d,n}$ and $E \subset X$ a subcurve, define $\sigma(E) = \sigma(\{j \in [n] \mid p_j \in E\})$.

PROPOSITION 1.7 [GiJM, Prop. 3.5]. *For an allowable choice of* (γ, A), *suppose that* $\phi(J,\gamma,A) \notin \mathbb{Z}$ *for any nonempty* $J \subset [n]$. *If* X *is a GIT-semistable curve and* $E \subset X$ *is a tail* (*a subcurve such that* $E \cap \overline{X - E}$ *is one point*), *then* $\deg(E) = \sigma(E)$.

COROLLARY 1.8 [GiJM, Cor. 3.7]. *Suppose that* $\phi(J,\gamma,A) \notin \mathbb{Z}$ *for any* $\emptyset \neq J \subset [n]$, *and let* $E \subseteq X$ *be a connected subcurve of* $(X, p_1,\dots,p_n) \in U^{ss}_{d,n}$. *Then*

$$\deg(E) = d - \sum \sigma(Y),$$

where the sum is over all connected components Y *of* $\overline{X\backslash E}$.

Given an F-curve $F(A_1, A_2, A_3, A_4)$ as described perviously, Proposition 1.7 states that $\deg(L(A_i)) = \sigma(A_i)$ if $\phi(A_i,\gamma,A)$ is not an integer. It follows that the degree of the spine is 0, and hence that the F-curve is contracted, if and only if $\sum_{i=1}^4 \sigma(A_i) = d$.

REMARK 1.9. When $U^{ss}_{d,n}$ has a strictly semistable point, it is possible that $\phi(A_i,\gamma,A) \in \mathbb{Z}$. If $\phi(A_i,\gamma,A) = k$ is an integer then $\deg(L(A_i))$ may be either k or $k+1$. In this case, both curves are identified in the GIT quotient and it suffices to consider the case where the legs have the maximum possible total degree.

2. The Veronese Quotient Divisors $\mathcal{D}_{\gamma,A}$

The Veronese quotient divisors $\mathcal{D}_{\gamma,A}$ are semi-ample divisors that give rise to morphisms from $\overline{\mathrm{M}}_{0,n}$ to the Veronese quotients $V^d_{\gamma,A}$. One of the main results of this paper is to give the combinatorial tools necessary to study these divisors as elements of the cone of nef divisors on $\overline{\mathrm{M}}_{0,n}$.

In Theorem 2.1, we give a formula for the intersection of the $\mathcal{D}_{\gamma,A}$ (as given in Definition 1.4) with F-curves on $\overline{\mathrm{M}}_{0,n}$ (as described in Definition 1.5). As a first application of Theorem 2.1, in Section 2.5 we give a simple formula for the intersection of the particular divisors $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ with a basis of F-curves. As a second application of Theorem 2.1, in Corollary 2.12 we specify the class of $\mathcal{D}_{\gamma,A}$ when A is S_n-invariant. We have already described (at the end of Section 1) a criterion for determining when these numbers are 0, but computing them in the nonzero case is substantially more complicated.

We first state Theorem 2.1 and Corollary 2.2; the latter exhibits the intersection numbers in a particular case. Before proving Theorem 2.1, in Section 2.1 we give an overview of our approach.

Notation. Let $[n] = A_1 \sqcup A_2 \sqcup A_3 \sqcup A_4$ be a partition and let $F(A_1, A_2, A_3, A_4)$ be the corresponding F-curve (cf. Definition 1.5). Recall that $\sigma(A_i)$ is the degree of the leg $L(A_i)$ (Definition 1.6). In this section, we establish the following explicit formula for the intersection of $\mathcal{D}_{\gamma,A}$ and $F(A_1, A_2, A_3, A_4)$.

Theorem 2.1. *Suppose we have an allowable linearization* (γ, A) *with* $d \geq 2$ *and also an* F*-curve* $F(A_1, A_2, A_3, A_4)$*. Then*

$$\begin{aligned}
&F(A_1, A_2, A_3, A_4) \cdot \mathcal{D}_{\gamma,A}\\
&\quad= \left(\sum_{i=1}^{3} c_{i4}^2\right)\frac{w}{2d} + \left(w_{A_4} - \frac{w}{d}\sigma(A_4)\right)b\\
&\qquad+ \sum_{i=1}^{3}\left(\frac{w}{d}(\sigma(A_i)+\sigma(A_4)) - w_{A_i} - w_{A_4}\right)c_{i4}\\
&\qquad- \frac{1+\gamma}{2d}\left(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i)) - \sum_{i=1}^{3}\sigma(A_i\cup A_4)(d-\sigma(A_i\cup A_4))\right),
\end{aligned}$$

where

$$\begin{aligned}
c_{ij} &:= d - \sigma(A_i) - \sigma(A_j) - \sigma([n]\backslash(A_i\cup A_j))\\
&= \sigma(A_i\cup A_j) - \sigma(A_i) - \sigma(A_j),\\
b &= d - \sum_{i=1}^{4}\sigma(A_i),\\
w &= \sum_{i=1}^{n} a_i,\\
w_{A_j} &= \sum_{i\in A_j} a_i.
\end{aligned}$$

Note that the case $d = 1$ was studied previously in [AS, Sec. 2]. If there is an A_i such that $\phi(A_i, \gamma, A)$ is an integer, then the σ-function does not give a unique degree for each leg (Remark 1.9). Nonetheless, the result of Theorem 2.1 is independent of the choice of semistable degree distribution.

As an example for how simple this formula can be, consider the following.

COROLLARY 2.2. *For* $n = 2g + 2$ *and* $1 \le \ell \le g$,

$$F_{n-i-2,i,1,1} \cdot \mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2}} = \begin{cases} \frac{1}{\ell+1} & \text{if } i \equiv \ell \pmod 2 \text{ and } i \ge \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Before delving into the proof of Theorem 2.1 (in Section 2.4), we first explain our approach (in Section 2.1) and develop a useful tool (in Section 2.2) that is a rational lift to $U_{d,n}$ of the image C in $V^d_{\gamma,A}$ of a given F-curve.

2.1. Approach to the Proof of Theorem 2.1

Let C be the image of $F(A_1, A_2, A_3, A_4)$ in $V^d_{\gamma,A}$ under the map $\varphi_{\gamma,A}$. Let $L = \mathcal{O}(\gamma, A)$ be an allowable polarization on $U_{d,n}$, and let $\bar{L} = L /\!/_{\gamma,A} \mathrm{SL}(d+1)$ be the associated ample line bundle on $V^d_{\gamma,A}$. By the projection formula, we have

$$F(A_1, A_2, A_3, A_4) \cdot \mathcal{D}_{\gamma,A} = C \cdot \bar{L}.$$

Therefore, proving Theorem 2.1 requires that we compute $C \cdot \bar{L}$. To do this, we will lift C to an appropriate curve $\tilde{C}$ on $U_{d,n}$ and perform the intersection there.

DEFINITION 2.3. Let C be a curve in $V^d_{\gamma,A}$, and let $\pi : U^{\mathrm{ss}}_{d,n} \to V^d_{\gamma,A}$ be the quotient map. A *rational lift* of C to $U_{d,n}$ is a curve $\tilde{C}$ in $U_{d,n}$ such that:

- a general point of $\tilde{C}$ lies in $U^{\mathrm{ss}}_{d,n}$; and
- $\overline{\pi(\tilde{C})} = C$ and $\pi|_{\tilde{C}} : \tilde{C} \dashrightarrow C$ is of degree 1.

A section of $\bar{L}$ can be pulled back to a section of L that vanishes on the unstable locus. It follows that if we have a rational lifting $\tilde{C}$ then, by the projection formula,

$$C \cdot \bar{L} = \tilde{C} \cdot \left(L - \sum t_i E_i\right)$$

for some rational numbers $t_i > 0$; here the sum is taken over all irreducible unstable divisors. By the proof of [GiJM, Prop. 4.6], there are two types of unstable divisors. One is a divisor of curves with unstable degree distribution and the other is D_{deg}, the divisor of curves contained in a hyperplane. If $\tilde{C}$ intersects D_{deg} only among unstable divisors, then $C \cdot \bar{L} = \tilde{C} \cdot (L - tD_{\mathrm{deg}})$ for some $t > 0$.

2.2 An Explicit Rational Lift

In this section we will construct a rational lift $\tilde{C}$ to $U_{d,n}$ of the image C in $V^d_{\gamma,A}$ of an F-curve $F(A_1, A_2, A_3, A_4)$ in $\overline{\mathrm{M}}_{0,n}$. This lift $\tilde{C}$ will be used to prove Theorem 2.1 in Section 2.4.

An F-curve is isomorphic to $\overline{\mathrm{M}}_{0,4} \cong \mathbb{P}^1$. Thus the total space of an F-curve is a family of curves over $\mathbb{P}^1$, which is a reducible surface for $n \ge 5$ that consists of five components. One component corresponds to a varying spine, which is isomorphic to the universal curve over $\overline{\mathrm{M}}_{0,4}$ and hence to $\overline{\mathrm{M}}_{0,5}$; the other four components are constant families over $\mathbb{P}^1$, which correspond to four fixed legs. We will think of the total space $X \cong \overline{\mathrm{M}}_{0,5}$ of spines as the blowup of three points on the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The points of attachment to the legs $L(A_i)$ that are labeled by A_1, A_2, and A_3 will correspond to the three sections of X through the exceptional divisors, while the point of attachment to the leg $L(A_4)$ will correspond to the diagonal. We denote the classes of the total transforms of two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ by F (for fiber) and S (for section), and we denote the exceptional divisors by E_i. Then, on $X \cong \overline{\mathrm{M}}_{0,5}$, the ten boundary classes are given by

$$\begin{gathered} D_{15} = S - E_1,\ D_{25} = S - E_2,\ D_{35} = S - E_3, \\ D_{45} = F + S - E_1 - E_2 - E_3,\ D_{14} = F - E_1,\ D_{24} = F - E_2, \\ D_{34} = F - E_3,\ D_{23} = E_1,\ D_{13} = E_2,\ D_{12} = E_3. \end{gathered} \tag{1}$$

Here is an outline of the construction of an explicit rational lift of a curve on $V^d_{\gamma,A}$. For a curve isomorphic to $\mathbb{P}^1$ on $V^d_{\gamma,A}$, we need to construct a family of rational curves in $\mathbb{P}^d$ of degree d over $\mathbb{P}^1$ and with n sections. To begin, we construct a map from X to $\mathbb{P}^d$ by constructing a base point free sublinear system V of a certain divisor class on X. We then attach four fixed legs to make a family of degree d rational curves. To make a family of curves whose general member is in $U^{\mathrm{ss}}_{d,n}$, the general member must satisfy certain degree conditions on each irreducible component and must also be nondegenerate. Let $\sigma(A_i)$ be the degree of the leg containing marked points in A_i. (If $U^{\mathrm{ss}}_{d,n} = U^{\mathrm{s}}_{d,n}$ then the degree of the leg is uniquely determined by the σ-function in [GiJM], but if there are strictly semistable points then the degree is not determined uniquely; in the latter case, we can take any degree distribution that gives semistable points—see Remark 1.9.) Then the general fiber must have degree

$$b := d - \sum_{i=1}^{4} \sigma(A_i).$$

As the cross ratio of the four points on the spine varies, there are three points at which the spine breaks into two components. The degree of one of these components where A_i and A_j come together is exactly

$$c_{ij} := d - \sigma(A_i) - \sigma(A_j) - \sigma([n]\backslash(A_i \sqcup A_j)) = \sigma(A_i \sqcup A_j) - \sigma(A_i) - \sigma(A_j).$$

Hence we consider the following divisor class on X (which depends on an integer $a \geq 0$):

$$H(a) := aF + bS - \sum_{i=1}^{3} c_{i4} E_i.$$

When $a \gg 0$, this divisor class is base point free (Lemma 2.4) and so defines a map to $\mathbb{P}^d$. Moreover, for $a \gg 0$ we can take a subspace $V \subset H^0(X, H(a))$ of dimension $b + 2$ such that its restriction $V|_F$ to every fiber defines a rational normal curve of degree b; thus the image of X is nondegenerate (Lemma 2.6). In Proposition 2.7, we establish that the general point of the family obtained via attaching four fixed tails is semistable by showing that it satisfies certain degree conditions.

LEMMA 2.4. *For $a \gg 0$, $H(a)$ is base point free.*

Proof. Since X is a del Pezzo surface, it is well known that if $H(a)$ is nef then $H(a)$ is base point free. On $X \cong \overline{\mathrm{M}}_{0,5}$, the cone of curves is generated by the classes D_{ij}. Thus, by using the explicit descriptions of the divisors D_{ij} given in (1), it is straightforward to check that $H(a)$ is nef if and only if

$$a \geq c_{i4}, \quad b \geq c_{i4}, \quad \text{and} \quad a+b \geq \sum_{i=1}^{3} c_{i4}.$$

The second inequality is immediate because $b = c_{12}+c_{34} = c_{13}+c_{24} = c_{14}+c_{23}$. So if a is sufficiently large, then $H(a)$ is nef and base point free. $\square$

LEMMA 2.5. *For $a \gg 0$, the map $H^0(X, H(a)) \to H^0(F, H(a)|_F)$ is surjective.*

Proof. By the exact sequence

$$\begin{aligned} 0 &\to H^0(X, H(a)-F) \\ &\to H^0(X, H(a)) \to H^0(F, H(a)|_F) \to H^1(X, H(a)-F), \end{aligned}$$

it suffices to show that $h^1(X, H(a)-F) = 0$. Since X is a del Pezzo surface, $-K_X$ is ample. Thus $H(a) - K_X$ is ample for $a \gg 0$ by Lemma 2.4 and $h^i(X, H(a)) = h^i(X, H(a) - K_X + K_X) = 0$ for $i > 0$ by the Kodaira vanishing theorem. Since $H(a) - F = H(a-1)$ by definition, $h^1(X, H(a)-F) = 0$ for large a as well. $\square$

By a Riemann–Roch calculation, if $a \gg 0$ then

$$h^0(X, H(a)) = 3ab - \sum_{i=1}^{3} \binom{c_{i4}+1}{2} + 1.$$

So for sufficiently large a, $h^0(X, H(a))$ is greater than $d+1$; hence we cannot use the complete linear system $|H(a)|$ to construct a map to $\mathbb{P}^d$. To deal with this problem, we use the following lemma.

LEMMA 2.6. *Let $V \subset H^0(X, H(a))$ be a general linear subspace of dimension $h^0(F, H(a)|_F)+1 = b+2$. For $a \gg 0$, the map $V \to H^0(F, H(a)|_F)$ is surjective for every fiber F.*

Proof. For a given fiber F, write K_F for the kernel of the map $H^0(X, H(a)) \to H^0(F, H(a)|_F)$. From Lemma 2.5 it follows that K_F is a linear space of dimension $h^0(X, H(a)-F)$. We will show that $\dim V \cap K_F = 1$ for every fiber F. In particular, denote the fiber over a point $y \in \overline{\mathrm{M}}_{0,4} \cong \mathbb{P}^1$ by F_y, and consider the variety

$$Z = \{(y, V) \in \mathbb{P}^1 \times \mathrm{Gr}(b+2, H^0(X, H(a))) \mid \dim V \cap K_{F_y} \geq 2\}.$$

The fibers of Z over $\mathbb{P}^1$ are Schubert varieties, which are known to be irreducible of codimension 2 in the Grassmannian. It follows that $\dim Z < \dim \mathrm{Gr}(b+2, H^0(X, H(a))$ and so Z does not map onto the Grassmannian. Consequently, for the general $V \in \mathrm{Gr}(b+2, H^0(X, H(a)))$, $\dim V \cap K_{F_y} < 2$ for every $y \in \mathbb{P}^1$; however, $\dim V \cap K_{F_y} \geq 1$ trivially for dimension reasons. It follows that the map $V \to H^0(F, H(a)|_F)$ is surjective for every fiber F. $\square$

By Lemma 2.6, if we consider the map $X \to \mathbb{P}^{b+1}$ corresponding to the linear series V then it is clear that each individual fiber is mapped to $\mathbb{P}^{b+1}$ via a complete linear series. Hence the general fiber maps to a smooth rational normal curve of degree b, and the three special fibers map to nodal curves whose two components have the appropriate degrees. Then, provided $b < d$, one can embed this $\mathbb{P}^{b+1}$ in $\mathbb{P}^d$ and obtain a family of curves in that projective space.

Now consider the case $b = d$. Since X is a surface and $d \geq 2$, we can take a point $p \in \mathbb{P}^{b+1} \setminus X$. Considering a projection from p, we obtain a family of curves in $\mathbb{P}^d$ with the same degree distribution. We must choose the point p so that a general member of such a family of curves is semistable. Because that member has the correct degree distribution, it suffices to check that a general member of the family is not contained in a hyperplane. Yet the image of a curve under projection is degenerate only if the original curve is degenerate.

To each of the four sections we attach a family of curves that does not vary in moduli. Using the same trick as before, we may take four copies of $\mathbb{P}^1 \times \mathbb{P}^1$, mapped into $\mathbb{P}^d$ via a linear series $V_i \subset |\mathcal{O}(x_i, y_i)|$, where

$$x_i = \begin{cases} H(a) \cdot (S - E_i) = a - c_{i4}, & i \neq 4, \\ H(a) \cdot \left(F + S - \sum_{j=1}^{3} E_j\right) = a + b - \sum_{j=1}^{3} c_{j4}, & i = 4, \end{cases}$$

and where $y_i = \sigma(A_i)$ is the degree of the leg. Note that if $b = d$ then $\sigma(A_i) = 0$; in that case, we need not worry about the construction of extra components.

PROPOSITION 2.7. *The family that we have constructed is a rational lift of $\varphi_{\gamma,A}(F(A_1, A_2, A_3, A_4))$. Furthermore, it does not intersect any GIT-unstable divisor other than D_{deg}.*

Proof. We claim that all of the members of this family satisfy the degree conditions required by semistability. Indeed, the general member is a nodal curve with four components labeled by the A_i. The degree of the leg labeled by A_i is $\mathcal{O}(x_i, y_i) \cdot \mathcal{O}(1,0) = \sigma(A_i)$, and the degree of the spine is $H(a) \cdot F = b = d - \sum_{i=1}^{4} \sigma(A_i)$. As one varies the cross ratio of the four points on the spine, there are three points where the spine breaks into two components. The degree of these components are, for instance, $H(a) \cdot E_1 = c_{14} = d - \sigma(A_4) - \sigma(A_1) - \sigma([n] \backslash (A_4 \cup A_1))$ and $H(a) \cdot (F - E_1) = b - c_{14} = d - \sigma(A_2) - \sigma(A_3) - \sigma([n] \backslash (A_2 \cup A_3))$. $\square$

2.3. Divisor Classes on $U_{d,n}$

The main result of this section is Lemma 2.9, which gives a numerical relation between several divisor classes on $U_{d,n}$.

DEFINITION 2.8. Let H be the divisor class on $U_{d,n}$ parameterizing curves that meet a fixed codimension 2 linear subspace in $\mathbb{P}^d$. Let D_k be the divisor class on $U_{d,n}$ that is the closure of the locus parameterizing curves with two irreducible components with respective degrees k and $d - k$. Finally, let D_{deg} be the divisor of curves contained in a hyperplane.

Lemma 2.9. *The following numerical relation holds in* $\mathrm{N}^1(U_{d,n})$:

$$D_{\mathrm{deg}} = \frac{1}{2d}\left((d+1)H - \sum_{k=1}^{\lfloor d/2\rfloor} k(d-k)D_k\right).$$

To prove this result, we will use a result of [CHSta] about the moduli space of stable maps. A map $f\colon (C, p_1, \dots, p_n) \to \mathbb{P}^r$ from an arithmetic genus 0 curve C with n marked points to $\mathbb{P}^r$ is called *stable* if

- C has at worst nodal singularities,
- p_i are distinct smooth points on C, and
- $\omega_C + \sum p_i + f^*\mathcal{O}(3)$ is ample.

We say that f has degree d if $f^*\mathcal{O}(1)$ has degree d on C. A moduli space of stable maps $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is the moduli space of degree d stable maps from genus 0 n-pointed curves to $\mathbb{P}^r$. For more information about moduli space of stable maps, see [FP].

Here is a list of properties of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ that we will use in the paper.

(i) There is a forgetful map $f\colon \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ that forgets the n marked points and stabilizes the map.

(ii) There are several functorial morphisms. A cycle morphism $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \mathrm{Chow}(1, d, \mathbb{P}^d)$ maps a stable map to its image of the fundamental cycle of the domain. There are n evaluation maps $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \mathbb{P}^d$ that send a stable map to the image of ith marked points on $\mathbb{P}^d$. Taking the product of these maps yields a cycle map

$$g\colon \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \mathrm{Chow}(1, d, \mathbb{P}^d) \times (\mathbb{P}^d)^n$$

that clearly factors through $U_{d,n}$.

(iii) We can define divisor classes H, D_k, and D_{deg} on $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$ using the descriptions given in Definition 2.8.

Proof of Lemma 2.9. By [CHSta, Lemma 2.1], on the moduli space of stable maps $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ we have

$$D_{\mathrm{deg}} = \frac{1}{2d}\left((d+1)H - \sum_{k=1}^{\lfloor d/2\rfloor} k(d-k)D_k\right).$$

If we pull back D_{deg} by the forgetful map $f\colon \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ then we obtain the same formula for D_{deg} on $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d)$. Now, for the cycle map $g\colon \overline{\mathrm{M}}_{0,n}(\mathbb{P}^d, d) \to U_{d,n}$, we have $g_*(H) = H = \mathcal{O}_{\mathrm{Chow}}(1)$, $g_*(D_k) = D_k$, and $g_*(D_{\mathrm{deg}}) = D_{\mathrm{deg}}$. As a result, the same formula holds for $U_{d,n}$. □

2.4. Proof of Theorem 2.1

In this section we prove Theorem 2.1, which relies on the curve constructed in Section 2.2.

Proof. As explained in Section 2.1, to prove Theorem 2.1 we shall compute the intersection of C, the image of $F(A_1, A_2, A_3, A_4)$ in $V^d_{\gamma,A}$, with the natural ample line bundle $\bar{L}$. For this it suffices to find a rational lift $\tilde{C}$ of this curve to $U_{d,n}$ such that a general element of $\tilde{C}$ is semistable and then to compute the intersection in $U_{d,n}$.

By Proposition 2.7, the family constructed in Section 2.2 has this property; we can therefore use it to carry out these computations. To compute the intersection of $\tilde{C}$ with $\mathcal{O}_{\mathrm{Chow}}(1)$, fix a general codimension 2 linear space in $\mathbb{P}^d$. The intersection number is precisely the number of curves in the family that intersect this linear space. In other words, it is the total degree of our five surfaces. Hence

$$\begin{aligned}
&\tilde{C} \cdot \mathcal{O}_{\mathrm{Chow}}(1) \\
&\qquad = H(a)^2 + \sum_{i=1}^{4} \mathcal{O}(x_i, y_i)^2 \\
&\qquad = 2ab - \sum_{i=1}^{3} c_{i4}^2 + \sum_{i=1}^{3} 2(a - c_{i4})\sigma(A_i) + 2\left(a + b - \sum_{j=1}^{3} c_{j4}\right)\sigma(A_4) \\
&\qquad = 2ad + 2\sigma(A_4)b - \sum_{i=1}^{3} c_{i4}^2 - \sum_{i=1}^{3} 2(\sigma(A_i) + \sigma(A_4))c_{i4}.
\end{aligned}$$

Similarly, to compute the intersection of $\tilde{C}$ with $\mathcal{O}_{\mathbb{P}^d_j}(1)$, fix a general hyperplane in $\mathbb{P}^d$. The intersection number is precisely the number of points at which the jth section meets this hyperplane; in other words, it is the degree of the jth section. If A_i is the part of the partition containing j, then

$$\tilde{C} \cdot \mathcal{O}_{\mathbb{P}^d_j}(1) = \mathcal{O}(x_i, y_i) \cdot \mathcal{O}(0,1) = x_i = \begin{cases} a - c_{i4}, & i \neq 4, \\ a + b - \sum_{k=1}^{3} c_{k4}, & i = 4. \end{cases}$$

One can then easily compute the intersection with $L = \bigotimes_{j=1}^{n} \mathcal{O}_{\mathbb{P}^d_j}(a_j) \otimes \mathcal{O}_{\mathrm{Chow}}(\gamma)$ by linearity. If we let $w_{A_j} = \sum_{i \in A_j} a_i$ and $w = \sum_{i=1}^{n} a_i$, then

$$\begin{aligned}
\tilde{C} \cdot L &= \gamma\left(2ad + 2\sigma(A_4)b - \sum_{i=1}^{3} c_{i4}^2 - \sum_{i=1}^{3} 2(\sigma(A_i + A_4))c_{i4}\right) \\
&\qquad + \sum_{i=1}^{3} w_{A_i}(a - c_{i4}) + w_{A_4}\left(a + b - \sum_{i=1}^{3} c_{i4}\right) \\
&= (2d\gamma + w)a - \sum_{i=1}^{3} c_{i4}^2\gamma + (2\sigma(A_4)\gamma + w_{A_4})b \\
&\qquad - \sum_{i=1}^{3} \big(2\gamma(\sigma(A_1) + \sigma(A_4)) + w_{A_i} + w_{A_4}\big)c_{i4}.
\end{aligned}$$

Recall that $C \cdot \bar{L} = \tilde{C} \cdot (L - tD_{\mathrm{deg}})$ for some positive rational number t (Section 2.1). It remains to determine the value of t. By Lemma 2.9,

$$\begin{aligned}
&\tilde{C} \cdot D_{\text{deg}} \\
&\quad = \frac{d+1}{2d}\left(2ad + 2\sigma(A_4)b - \sum_{i=1}^{3} c_{i4}^2 - \sum_{i=1}^{3} 2(\sigma(A_i)+\sigma(A_4))c_{i4}\right) \\
&\qquad + \frac{1}{2d}\left(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i)) - \sum_{i=1}^{3}(\sigma(A_i \cup A_4))(d-\sigma(A_i \cup A_4))\right).
\end{aligned}$$

Note that the rational lift depends on the choice of a. To obtain an intersection number $C \cdot \bar{L} = \tilde{C} \cdot (L - tD_{\text{deg}})$ that is independent of our choice of a, the coefficient of a must be 0. Thus

$$2d\gamma + w - t\frac{(d+1)}{2d}2d = 0$$

and $t = \frac{2d\gamma + w}{1+d} = 1 + \gamma$. Therefore,

$$\begin{aligned}
&C \cdot \bar{L} \\
&\quad = \tilde{C} \cdot (L - (1+\gamma)D_{\text{deg}}) \\
&\quad = \left(\sum_{i=1}^{3} c_{i4}^2\right)\frac{w}{2d} + \left(w_{A_4} - \frac{w}{d}\sigma(A_4)\right)b \\
&\qquad + \sum_{i=1}^{3}\left(\frac{w}{d}(\sigma(A_i)+\sigma(A_4)) - w_{A_i} - w_{A_4}\right)c_{i4} \\
&\qquad - \frac{1+\gamma}{2d}\left(\sum_{i=1}^{4}\sigma(A_i)(d-\sigma(A_i)) - \sum_{i=1}^{3}\sigma(A_i \cup A_4)(d-\sigma(A_i \cup A_4))\right). \qquad \square
\end{aligned}$$

2.5. *Example and Application of Theorem 2.1*

Because the F-curves span the vector space of 1-cycles, Theorem 2.1 gives (in principal) the class of $\mathcal{D}_{\gamma,A}$ in the Nerón–Severi space. Using a particular basis (described in Definition 2.10), we explicitly write down the class of $\mathcal{D}_{\gamma,A}$ for S_n-invariant weights A. The classes depend on the intersection numbers, which—as shown in Example 2.13 to follow—are especially simple for $\mathcal{D}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$.

Definition 2.10 [AGStS, Sec. 2.2.2, Prop. 4.1]. For $1 \le j \le g := \lfloor n/2 - 1 \rfloor$, let F_j be the S_n-invariant F-curve $F_{1,1,j,n-j-2}$. The set $\{\mathrm{F}_j : 1 \le j \le g\}$ forms a basis for the group of 1-cycles $\mathrm{N}_1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$.

Definition 2.11 [KeMc, Sec. 3]. For $2 \le j \le \lfloor n/2 \rfloor$, let B_j be the S_n-invariant divisor given by the sum of boundary divisors indexed by sets of size j:

$$B_j = \sum_{J \subset [n], |J| = j} \delta_J.$$

The set $\{\mathrm{B}_j : 2 \le j \le g+1\}$ forms a basis for the group of cycles $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$ of codimension 1.

COROLLARY 2.12. *Fix* $n = 2g+2$ (*or* $n = 2g+3$) *and* $j \in \{1,\dots,g\}$, *and write* $a(\gamma, A)_j = \mathcal{D}_{\gamma,A} \cdot F_j$. *If* A *is an* S_n*-invariant choice of weights then* $\mathcal{D}_{\gamma,A} \equiv \sum_{r=1}^{g} b(\gamma,A)_r B_{r+1}$, *where*

$$b(\gamma,A)_r = \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j + \frac{r(r+1)}{n-1}\sum_{j=r}^{g} a(\gamma,A)_j$$

for $n = 2g+3$ *odd and*

$$\begin{aligned} b(\gamma,A)_r = {} & \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j \\ & + \frac{r(r+1)}{n-1}\sum_{j=r}^{g-1} a(\gamma,A)_j + \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g \end{aligned}$$

for $n = 2g+2$ *even.*

Proof. This claim follows from the formula given in [AGStS, Thm. 5.1]. □

EXAMPLE 2.13.

$$\begin{aligned} & \mathcal{D}_{\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}} \\ & = \frac{1}{\ell+1}\sum_{r=1}^{g}\left(\frac{r(r+1)}{n-1}\left(\frac{g-\ell+1}{2}\right) - \left\lceil\frac{r-\ell+1}{2}\right\rceil_+ \left\lfloor\frac{r-\ell+1}{2}\right\rfloor_+\right)B_{r+1}, \end{aligned}$$

where

$$\lceil x\rceil_+ = \max\{\lceil x\rceil, 0\}, \qquad \lfloor x\rfloor_+ = \max\{\lfloor x\rfloor, 0\}.$$

Proof. Indeed, by the previous results we have

$$\begin{aligned} b(\gamma,A)_r = {} & \sum_{j=1}^{r-1}\left(\frac{r(r+1)}{n-1} - (r-j)\right)a(\gamma,A)_j \\ & + \frac{r(r+1)}{n-1}\sum_{j=r}^{g-1} a(\gamma,A)_j + \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g \\ = {} & \frac{r(r+1)}{n-1}\sum_{j=1}^{g} a(\gamma,A)_j - \frac{r(r+1)}{2(n-1)}a(\gamma,A)_g - \sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j. \end{aligned}$$

By Corollary 2.2,

$$\sum_{j=1}^{g} a(\gamma,A)_j = \begin{cases} \frac{1}{\ell+1}\left(\frac{g-\ell}{2}+1\right) & \text{if } g \equiv \ell \pmod 2, \\ \frac{1}{\ell+1}\left(\frac{g-\ell+1}{2}\right) & \text{if } g \not\equiv \ell \pmod 2. \end{cases}$$

Also, $a(\gamma,A)_g = \frac{1}{\ell+1}$ if $g \equiv \ell \pmod 2$ and $a(\gamma,A)_g = 0$ if $g \not\equiv \ell \pmod 2$, so we can write

$$b(\gamma,A)_r = \frac{1}{\ell+1}\,\frac{r(r+1)}{n-1}\,\frac{g-\ell+1}{2} - \sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j.$$

A similar case-by-case computation yields

$$\sum_{j=1}^{r-1}(r-j)a(\gamma,A)_j = \begin{cases} \left(\frac{r-\ell+1}{2}\right)^2 & \text{if } r \not\equiv \ell \pmod 2 \text{ and } \ell \le r-1, \\ \frac{(r-\ell)(r-\ell+2)}{4} & \text{if } r \equiv \ell \pmod 2 \text{ and } \ell \le r-1, \\ 0 & \text{if } \ell > r-1 \end{cases}$$

$$= \left\lceil \frac{r-\ell+1}{2} \right\rceil_+ \left\lfloor \frac{r-\ell+1}{2} \right\rfloor_+ . \qquad \square$$

3. A New Modular Interpretation for a Particular Family of Veronese Quotients

In this section, we study the family $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ of birational models for $\overline{\mathrm{M}}_{0,n}$, where $n = 2(g+1)$ and $1 \le \ell \le g$. In Theorem 3.5 we give a new modular interpretation of them as certain contractions of Hassett spaces,

$$\tau_\ell : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\varepsilon)^n} \to V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}},$$

where *even chains* (described in Definition 3.4) are replaced by particular curves. In order to establish the existence of morphisms τ_ℓ we first prove Proposition 3.2, which identifies the Veronese quotient associated to a nearby linearization with the Hassett space $\overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\varepsilon)^n}$. The results in this section allow us to prove in Section 4 that nontrivial conformal block divisors $\mathbb{D}(\mathfrak{sl}_2,\ell,\omega_1^{2g+2})$ are pullbacks of ample classes from $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$.

REMARK 3.1. Because their defining linearizations $\left(\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}\right)$ lie on GIT walls, the Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$ admit strictly semistable points and so their corresponding moduli functors are not actually separated. The quotient described in Theorem 3.5 is not isomorphic to a modular compactification in the sense of [Sm] (cf. Remark 3.6). Indeed, the only known method for constructing this compactification is via GIT. This fact highlights the strength of the Veronese quotient construction, since it can be used (as we show here) to construct "new" compactifications of $\mathrm{M}_{0,n}$—in other words, compactifications that have not been described and cannot be described through any of the previously developed techniques.

3.1. Defining the Maps τ_ℓ

In this section we define the morphism

$$\tau_\ell : \overline{\mathrm{M}}_{0,(\frac{1}{\ell+1}-\varepsilon)^{2g+2}} \to V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},(\frac{1}{\ell+1})^{2g+2}}$$

obtained by variation of GIT. As mentioned previously, each of the linearizations $\left(\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}\right)$ lies on a wall. To show that τ_ℓ exists, we will use the general "variation of GIT" fact that any quotient corresponding to a GIT chamber admits a morphism to a quotient corresponding to a wall of that chamber. Namely, in Proposition 3.2 we shall identify the Veronese quotient corresponding to a GIT chamber bordering the GIT wall that contains the linearization $\left(\frac{\ell-1}{\ell+1}, \left(\frac{1}{\ell+1}\right)^{2g+2}\right)$. We then use this identification to describe the morphism to the Veronese quotient $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}}$. We remark that the Veronese quotient discussed in Proposition 3.2 is not, in general, a modular compactification of $\mathrm{M}_{0,n}$ but is (unlike the quotients described in Theorem 3.5) isomorphic to one.

PROPOSITION 3.2. *For $2 \leq \ell \leq g-1$ and $\varepsilon > 0$ sufficiently small, the Hassett space $\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$ is isomorphic to the normalization of the Veronese quotient*

$$V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\varepsilon',\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}.$$

Here ε' is a positive number that is uniquely determined by the data $d = g+1-\ell$ and $A = \left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}$ (cf. Definition 1.1).

Proof. By Proposition 1.3, there is a morphism

$$\phi_\gamma : \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \to V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\varepsilon',\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$$

that fits into the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}"'] \arrow[dr, "\varphi_{\gamma,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}"] & \\
\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \arrow[r, "\phi_\gamma"] & V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\varepsilon',\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}.
\end{tikzcd}$$

So to prove the result, it suffices to show that ϕ_γ is bijective. Since $(g-\ell)\gamma + (2g+2)\left(\frac{1}{\ell+1}-\varepsilon\right) = g+2-\ell$, it follows that

$$\gamma = 1 - \frac{2}{\ell+1} + \frac{2(g+1)}{g-\ell}\varepsilon.$$

If $\ell \geq 3$, then $\gamma > \frac{1}{2}$ and a curve in the semistable locus $U^{\mathrm{ss}}_{g+1-\ell,2g+2}$ does not have multinodal singularities by Lemma 1.2. Similarly, the sum of the weights at a node cannot exceed $1-\gamma = \frac{2}{\ell+1} - \frac{2(g+1)}{g-\ell}\varepsilon < 2\left(\frac{1}{\ell+1}-\varepsilon\right)$. At a node, then, there is at most one marked point.

If $\ell = 2$ then $\gamma > \frac{1}{3}$, so a curve in $U^{\mathrm{ss}}_{g+1-\ell,2g+2}$ has at worst a multinodal point of multiplicity 3. Note that $1-(3-1)\gamma = \frac{1}{3} - \frac{4(g+1)}{g-2}\varepsilon < \frac{1}{3}-\varepsilon$; hence, by Lemma 1.2, there can be no marked point at a triple point. Similarly, since $1-(2-1)\gamma = \frac{2}{3} - \frac{2(g+1)}{g-2}\varepsilon < 2\left(\frac{1}{3}-\varepsilon\right)$, there can be at most one marked point at a node.

In short: There is no positive-dimensional moduli of curves contracted to the same curve; that is, ϕ_γ is an injective map. The surjectivity follows directly from the properness of both sides. □

REMARK 3.3. 1. In [GiJM, Thm. 7.1, Cor. 7.2] the authors show that, for certain values of γ and A, the corresponding Veronese quotient is $\overline{\mathrm{M}}_{0,A}$. Proposition 3.2 indicates precise values of γ and A when the latter is symmetric.

2. The normalization map of a Veronese quotient is always bijective [GiJM, Rem. 6.2]. So at least on the level of topological spaces, the normalization is equal to the Veronese quotient itself.

3. If $g \equiv \ell \bmod 2$, then there are strictly semistable points on $U_{g+1-\ell,2g+2}$ for the linearization $\left(\frac{\ell-1}{\ell+1}+\varepsilon', \left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}\right)$. Indeed, for a set J of $g+1$ marked points, the weight function

$$\begin{aligned}\phi(J,\gamma,A) &= \frac{(g+1)\left(\frac{1}{\ell+1}-\varepsilon\right)-1}{\frac{2}{\ell+1}-\frac{2(g+1)}{g-\ell}\varepsilon} \\ &= \frac{(g-\ell)(g-\ell)-(g-\ell)(g+1)(\ell+1)\varepsilon}{2(g-\ell)-2(g+1)(\ell+1)\varepsilon} = \frac{g-\ell}{2}\end{aligned}$$

is an integer. Therefore, the quotient stack $[U^{\mathrm{ss}}_{g+1-\ell,2g+2}/\mathrm{SL}(g+2-\ell)]$ is not modular in the sense of [Sm].

4. Even if the GIT quotient is modular, the moduli-theoretic meaning of

$$\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \quad \text{and} \quad V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\varepsilon',\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$$

may be different in general because multinodal singularities and a marked point on a node are both allowed on the GIT quotient. Even so, the moduli spaces are isomorphic.

5. The condition on ℓ is necessary. Indeed, if $\ell = 1$ or g then the GIT quotient is not isomorphic to a Hassett space.

3.2. The New Modular Interpretation

In this section we will prove Theorem 3.5. This theorem describes the Veronese quotients $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}}$ as images of contractions where the *even chains* in the Hassett spaces $\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{n}}$ are replaced by other curves that we describe next.

DEFINITION 3.4. A curve $(C, x_1, \dots, x_{2g+2}) \in \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$ is an *odd chain* (resp., *even chain*) if C contains a connected chain $C_1 \cup \dots \cup C_k$ of rational curves such that the following statements hold.

(i) Each C_i contains exactly two marked points.

(ii) Each interior component C_i for $2 \le i \le k-1$ contains exactly two nodes: $C_i \cap C_{i-1}$ and $C_i \cap C_{i+1}$.

(iii) Aside from the two marked points, each of the two end components C_1 and C_k contains two "special" points, where a *special point* is either a node or a point at which $\ell+1$ marked points coincide. In the first case, we refer to the connected components of $\overline{C \setminus \bigcup_{i=1}^{k} C_i}$ as "tails"; we regard the second type of special point as a tail of degree 0.

(iv) The number of marked points on each of the tails is odd (resp., even).

Figure 1 shows two examples of odd chains when ℓ is even.

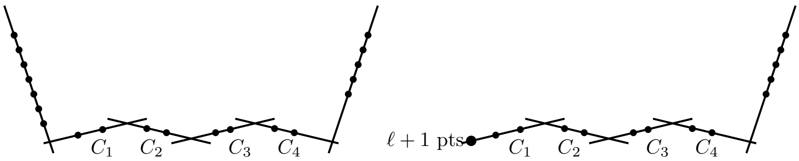

Figure 1 Examples of odd chains

THEOREM 3.5. *If* $3 \le \ell \le g-1$ *and* ℓ *is even* (*resp., odd*), *then the map* τ_ℓ *restricts to an isomorphism away from the locus of odd chains* (*resp., even chains*). *If* $(C, x_1, \ldots, x_{2g+2})$ *is an odd chain* (*resp., even chain*), *then* $\tau_\ell(C, x_1, \ldots, x_{2g+2})$ *is strictly semistable and its orbit closure contains a curve in which the chain* $C_1 \cup \cdots \cup C_k$ *has been replaced by a chain* $D_1 \cup \cdots \cup D_{k+1}$ *with two marked points at each node* $D_i \cap D_{i+1}$ (*see Figure 2*).

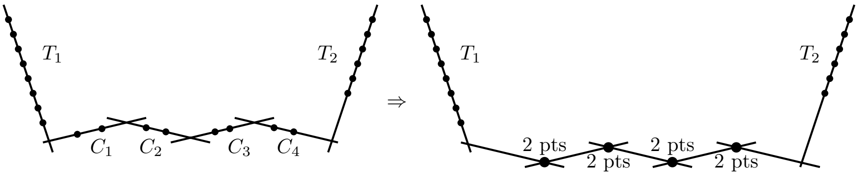

Figure 2 The contraction

Proof. Note that both the Hassett space and the GIT quotient are stratified by the topological types of parameterized curves. Furthermore, τ_ℓ is compatible with these stratifications. Hence τ_ℓ contracts a curve B if and only if

- B is in the closure of a stratum,
- a general point $(C, x_1, \ldots, x_{2g+2})$ of B has irreducible components $C_1, C_2, \ldots, C_k$ with positive-dimensional moduli,
- C_i is contracted by the map from C to $\tau_\ell(C, x_1, \ldots, x_{2g+2})$, and
- the configurations of points on the irreducible components other than the C_i are fixed.

A component $C_i \subset C \in \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$ has positive-dimensional moduli if it has four or more distinct special points. If a tail with k points is contracted, then $k\left(\frac{1}{\ell+1}\right) \le 1$. But then $k\left(\frac{1}{\ell+1} - \varepsilon\right) < 1$, so such a tail is impossible on $\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$. Thus no tail is contracted. Now $\gamma = \frac{\ell-1}{\ell+1} \ge \frac{1}{2}$. By Lemma 1.2, a curve $(D, y_1, \ldots, y_{2g+2}) \in U^{\mathrm{ss}}_{g+1-\ell, 2g+2}$ has at worst triple nodes if $\ell = 3$ and nodes if $\ell \ge 4$. Moreover, the sum of the weights on triple nodes cannot exceed $1 - 2\gamma = \frac{3-\ell}{\ell+1} \le 0$, so there are no marked points at a triple node. Since $1 - \gamma = \frac{2}{\ell+1}$, there are at most two marked points at a node. Therefore, the only possible contracted component is an interior component C_i with two points of attachment and two marked points.

Now let C_i be such a component. Connected to C_i are two tails T_1 and T_2 (not necessarily irreducible) with i and $2g-i$ marked points, respectively. (Here we will regard a point with $\ell+1$ marked points or, equivalently, of total weight $1-(\ell+1)\varepsilon$ as a tail of degree 0.) On the one hand, if $i \equiv \ell \bmod 2$ then $\phi(T_1) = \frac{i-\ell-1}{2}$ and $\phi(T_2) = \frac{2g-i-\ell-1}{2}$ (see Definition 1.6), so neither is an integer. Hence the degree of the component C_i is

$$d - (\sigma(T_1)+\sigma(T_2)) = g+1-\ell - \left(\left\lceil \frac{i-\ell-1}{2} \right\rceil + \left\lceil \frac{2g-i-\ell-1}{2} \right\rceil\right) = 1,$$

from which it follows that C_i is not contracted by the map to $\tau_\ell(C, x_1, \dots, x_{2g+2})$. On the other hand, if $i \equiv \ell+1 \bmod 2$, then both $\phi(T_1) = \frac{i-\ell-1}{2}$ and $\phi(T_2) = \frac{2g-i-\ell-1}{2}$ are integers. Therefore, this curve lies in the strictly semistable locus and the image $\tau_\ell(C, x_1, \dots, x_{2g+2})$ can be represented by several possible topological types. By [GiJM, Prop. 6.7], the orbit closure of $\tau_\ell(C, x_1, \dots, x_{2g+2})$ contains a curve in which C_i is replaced by the union of two lines $D_1 \cup D_2$ with two marked points at the node $D_1 \cap D_2$. □

REMARK 3.6. Note that τ_ℓ restricts to an isomorphism on the (nonclosed) locus of curves consisting of two tails connected by an irreducible bridge with four marked points. However, on the locus of curves consisting of two tails connected by a chain of two bridges with two marked points each, τ_ℓ forgets the data of the chain. Hence the map τ_ℓ fails to satisfy axiom (3) of [Sm, Def. 1.5]. In particular, the Veronese quotient described in Theorem 3.5 is not isomorphic to a modular compactification in the sense of [Sm].

3.3. Morphisms between the Moduli Spaces We Have Described

COROLLARY 3.7. *If $1 \le \ell \le g-2$, then there is a morphism*

$$\psi_{\ell,\ell+2} \colon \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},\left(\frac{1}{\ell+3}\right)^{2g+2}}$$

that preserves the interior.

Proof. For $1 \le \ell \le g-3$ we consider the composition

$$\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \to \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+3}-\varepsilon\right)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},\left(\frac{1}{\ell+3}\right)^{2g+2}},$$

where the first morphism is Hassett's reduction morphism [Ha, Thm. 4.1] and the last morphism is $\tau_{\ell+2}$.

If $\ell = g-2$, then $V^{g-1-\ell}_{\frac{\ell+1}{\ell+3}+\varepsilon',\left(\frac{1}{\ell+3}-\varepsilon\right)^{2g+2}} = (\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2)$ with symmetric weight datum. Because there is a morphism $\overline{\mathrm{M}}_{0,A} \to (\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2)$ for any symmetric weight datum A [Ha, Thm. 8.3], we obtain $\psi_{g-2,g}$. □

To derive morphisms between the moduli spaces described in Theorem 3.5, we consider the following diagram:

$$\begin{tikzcd}
 & \overline{\mathrm{M}}_{0,n} \arrow[dl] \arrow[dr] & \\
\overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \arrow[rr] \arrow[d, "\cong"] \arrow[ddrr, "\psi_{\ell,\ell+2}"] \arrow[dd, bend right=60, "\tau_\ell"'] & & \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+3}-\varepsilon\right)^{2g+2}} \arrow[d, "\cong"'] \arrow[dd, bend left=60, "\tau_{\ell+2}"] \\
V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}+\varepsilon',\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \arrow[d] & & V^{g-1-\ell}_{\frac{\ell+1}{\ell+3}+\varepsilon',\left(\frac{1}{\ell+3}-\varepsilon\right)^{2g+2}} \arrow[d] \\
V^{g+1-\ell}_{\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}} \arrow[rr, dashed] & & V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},\left(\frac{1}{\ell+3}\right)^{2g+2}}
\end{tikzcd}$$

PROPOSITION 3.8. *If* $3 \le \ell \le g-2$, *then the morphism* $\psi_{\ell,\ell+2}$ *factors through* τ_ℓ.

Proof. By the rigidity lemma [Ke, Def.-Lemma 1.0] it suffices to show that, for any curve $B \subset \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}}$ contracted by τ_ℓ, the morphism $\psi_{\ell,\ell+2}$ is constant. We have already described the curves contracted by τ_ℓ in the proof of Theorem 3.5, so it suffices to show that the same curves B are contracted by $\psi_{\ell,\ell+2}$.

When $\ell < g-2$, we have $i \equiv \ell+1 \bmod 2$ if and only if $i \equiv (\ell+2)+1 \bmod 2$; hence the image of B is contracted by

$$\tau_{\ell+2}\colon \overline{\mathrm{M}}_{0,\left(\frac{1}{\ell+1}-\varepsilon\right)^{2g+2}} \to V^{g-1-\ell}_{\frac{\ell+1}{\ell+3},\left(\frac{1}{\ell+3}\right)^{2g+2}}.$$

If $\ell = g-2$, then $\psi_{g-2,g}$ is Hassett's reduction morphism

$$\overline{\mathrm{M}}_{0,\left(\frac{1}{g-1}-\varepsilon\right)^{2g+2}} \to (\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2).$$

In this case there are two types of odd/even chains (of length 1 or 2). It is straightforward to check that these curves are contracted to an isolated singular point of $(\mathbb{P}^1)^{2g+2} /\!/ \mathrm{SL}(2)$ parameterizing strictly semistable curves. $\square$

4. Higher-Level Conformal Block Divisors and Veronese Quotients

The main goal of this section is to prove Theorem 4.6, which states that if $n = 2g+2$ then the divisors $\mathcal{D}_{\gamma,A} = \mathcal{D}_{\frac{\ell-1}{\ell+1},\left(\frac{1}{\ell+1}\right)^{2g+2}}$ and $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ determine the same birational models. We shall prove this claim by showing that $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ and $\mathcal{D}_{\gamma,A}$ lie on the same face of the semi-ample cone.

Toward this end, we use a set of S_n-invariant F-curves (given in Definition 2.10) that were shown in [AGStS, Prop. 4.1] to form a basis for $\mathrm{Pic}(\overline{\mathrm{M}}_{0,n})^{\mathrm{S}_n}$. Recall that we used Theorem 2.1 to obtain, in Corollary 2.2, a simple formula for the intersection of these curves with $\mathcal{D}_{\gamma,A}$. Here we will show (in Corollary 4.5) that $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is equivalent to a nonnegative combination of the divisors $\{\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1}, (\frac{1}{\ell+2k+1})^{2g+2}} : k \in \mathbb{Z}_{\geq 0}, \ell + 2k \leq g\}$. This claim follows from Proposition 4.4, which shows that the nonzero intersection numbers $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i$ are nondecreasing.

4.1. Definition of Vector Bundles of Conformal Blocks and Related Notation

In this section we briefly give the definition of conformal block divisors and explain our notation. The reader can find the details in [U, Chaps. 3, 4]. For representation-theoretic terminologies and definitions, consult [Hu].

Let $\mathfrak{g}$ be a simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and positive roots Δ^+. Let $\theta \in \Delta^+$ be the highest root, and let $(\cdot,\cdot)$ be the Killing form normalized so that $(\theta,\theta) = 2$. For a nonnegative integer ℓ, the *Weyl alcove* P_ℓ is the set of dominant integral weights λ satisfying $(\lambda,\theta) \leq \ell$.

For a collection of data $(\mathfrak{g}, \ell, \vec{\lambda} = (\lambda_1, \dots, \lambda_n))$—where $\mathfrak{g}$ is a simple Lie algebra, ℓ is a nonnegative integer, and $\vec{\lambda}$ is a collection of weights in P_ℓ—we can construct a conformal block vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ as follows.

For a simple Lie algebra $\mathfrak{g}$, we can construct an affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c,$$

where c is a central element, via the bracket operation

$$[X \otimes f, Y \otimes g] = [X,Y] \otimes fg + c(X,Y)\operatorname{Res}_{z=0} g\,df.$$

For each ℓ and $\lambda \in P_\ell$ there exists a unique integrable highest-weight $\hat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$, where c acts as multiplication by ℓ. Let $\mathcal{H}_{\vec{\lambda}} = \bigotimes_{i=1}^n \mathcal{H}_{\lambda_i}$. Then there is a natural $\hat{\mathfrak{g}}_n$-action on $\mathcal{H}_{\vec{\lambda}}$, where

$$\hat{\mathfrak{g}}_n = \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((z_i)) \oplus \mathbb{C}c.$$

Now we fix a stable curve $X = (C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$ and also set $U = C - \{p_1, \dots, p_n\}$. There is a natural map $\mathcal{O}_C(U) \hookrightarrow \bigoplus_{i=1}^n \mathbb{C}((z_i))$. Thus we have a map (indeed, it is a Lie algebra homomorphism)

$$\mathfrak{g}(X) = \mathfrak{g} \otimes \mathcal{O}_C(U) \hookrightarrow \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathbb{C}((z_i)) \oplus \mathbb{C}c = \hat{\mathfrak{g}}_n.$$

The vector space $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})|_X$ of conformal blocks is defined by $\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(X)\mathcal{H}_{\vec{\lambda}}$. By [U, Thm. 4.4], this construction can be sheafified; by [U, Thm. 4.19], these vector spaces form a vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$ of finite rank over the moduli stack $\overline{\mathcal{M}}_{g,n}$. Finally, a *conformal block divisor* $\mathbb{D}(\mathfrak{g}, \ell, \vec{\lambda})$ is the first Chern class of $\mathbb{V}(\mathfrak{g}, \ell, \vec{\lambda})$.

In this paper, we focus on the $\mathfrak{g} = \mathfrak{sl}_2$ cases.

4.2. *Intersections of F_i with $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ Are Nondecreasing*

In this section we prove that the nonzero intersection numbers of $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ with the S_n invariant F-curves F_i are nondecreasing. We recall some notation from [AGS].

DEFINITION 4.1. Define

$$r_\ell(a_1, \dots, a_n) := \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, (a_1\omega_1, \dots, a_n\omega_1))$$

and, as a special case,

$$r_\ell(k^j, t) := \operatorname{rank} \mathbb{V}(\mathfrak{sl}_2, \ell, (\underbrace{k\omega_1, \dots, k\omega_1}_{j \text{ times}}, t\omega_1)).$$

For the basic numerical properties of $r_\ell(a_1, \dots, a_n)$, see [AGS, Sec. 3].

PROPOSITION 4.2. *The ranks $r_\ell(1^j, t)$ are determined by the system of recurrences*

$$r_\ell(1^j, t) = r_\ell(1^{j-1}, t-1) + r_\ell(1^{j-1}, t+1), \quad t = 1, \dots, \ell, \tag{2}$$

together with seeds

$$r_\ell(1^j, j) = 1 \text{ if } j \le \ell \quad \text{and} \quad r_\ell(1^j, j) = 0 \text{ if } j > \ell.$$

REMARK. The recurrence (2) is reminiscent of the recurrence for Pascal's triangle.

Proof of Proposition 4.2. Partition the weight vector $(1, \dots, 1, t) = 1^j t$ as $1^{j-1} \cup (1, t)$. If $j + t$ is odd then, by the odd sum rule [AGS, Prop. 3.5], $r_\ell(1^j, t) = 0$. So assume that $j + t$ is even. Then the factorization formula [AGS, Prop. 3.3] states that

$$r_\ell(1^j, t) = \sum_{\mu=0}^{\ell} r_\ell(1^{j-1}, \mu) r_\ell(1, t, \mu). \tag{3}$$

We can simplify this expression. Recall that, by the $\mathfrak{sl}_2$ fusion rules [AGS, Prop. 3.4], $r_\ell(1, t, \mu)$ is 0 if $\mu > t + 1$ or if $\mu < t - 1$. Thus the only possibly nonzero summands in (3) are when $\mu = t - 1, t$, or $t + 1$. But when $\mu = t$, by the odd sum rule [AGS, Prop. 3.5] we have $r_\ell(1, t, t) = 0$. Thus (3) simplifies as follows:

$$r_\ell(1^j, t) = r_\ell(1^{j-1}, t-1) + r_\ell(1^{j-1}, t+1), \quad t = 1, \dots, \ell - 1;$$
$$r_\ell(1^j, \ell) = r_\ell(1^{j-1}, \ell - 1).$$

Since $r_\ell(1^{j-1}, \ell + 1) = 0$, we can unify these two lines and thereby obtain (2). $\square$

LEMMA 4.3. *Let $i_1 < i_2$ and $j_1 < j_2$. Suppose $i_1 \equiv i_2 \equiv j_1 \equiv j_2 \pmod 2$. Then $r_\ell(1^{i_1}, j_1) r_\ell(1^{i_2}, j_2) - r_\ell(1^{i_1}, j_2) r_\ell(1^{i_2}, j_1) \ge 0$.*

Proof. We prove the result by induction on i_2. For the base case, we can check that $(i_1, i_2) = (0, 2)$ and $(i_1, i_2) = (1, 3)$. If $(i_1, i_2) = (0, 2)$ then the result is true because $r_\ell(t) = 0$ if $t > 0$. Similarly, if $(i_1, i_2) = (1, 3)$ then the result is true because $r_\ell(1, t) = 0$ if $t > 1$.

So suppose the result has been established for all quadruples (i_1, i_2, j_1, j_2) with $i_2 \leq k - 1$. We now apply the recurrence (2):

$$\begin{aligned}
&r_\ell(1^{i_1}, j_1)r_\ell(1^{i_2}, j_2) - r_\ell(1^{i_1}, j_2)r_\ell(1^{i_2}, j_1)\\
&\quad = \big(r_\ell(1^{i_1-1}, j_1 - 1) + r_\ell(1^{i_1-1}, j_1 + 1)\big)\big(r_\ell(1^{i_2-1}, j_2 - 1) + r_\ell(1^{i_2-1}, j_2 + 1)\big)\\
&\qquad - \big(r_\ell(1^{i_1-1}, j_2 - 1) + r_\ell(1^{i_1-1}, j_2 + 1)\big)\big(r_\ell(1^{i_2-1}, j_1 - 1) + r_\ell(1^{i_2-1}, j_1 + 1)\big)\\
&\quad = r_\ell(1^{i_1-1}, j_1 - 1)r_\ell(1^{i_2-1}, j_2 - 1) - r_\ell(1^{i_1-1}, j_2 - 1)r_\ell(1^{i_2-1}, j_1 - 1)\\
&\qquad + r_\ell(1^{i_1-1}, j_1 - 1)r_\ell(1^{i_2-1}, j_2 + 1) - r_\ell(1^{i_1-1}, j_2 + 1)r_\ell(1^{i_2-1}, j_1 - 1)\\
&\qquad + r_\ell(1^{i_1-1}, j_1 + 1)r_\ell(1^{i_2-1}, j_2 - 1) - r_\ell(1^{i_1-1}, j_2 - 1)r_\ell(1^{i_2-1}, j_1 + 1)\\
&\qquad + r_\ell(1^{i_1-1}, j_1 + 1)r_\ell(1^{i_2-1}, j_2 + 1) - r_\ell(1^{i_1-1}, j_2 + 1)r_\ell(1^{i_2-1}, j_1 + 1).
\end{aligned}$$

By the induction hypothesis, each of the last four lines is nonnegative. □

PROPOSITION 4.4. *Suppose $\ell \leq i \leq g - 2$ and $i \equiv \ell \pmod 2$. Then*

$$\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i \leq \mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_{i+2}.$$

Yet if $i \equiv \ell + 1 \pmod 2$, then $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i = 0$.

Proof. By [AGS, Thm. 4.2] we have $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2}) \cdot F_i = r_\ell(1^i, \ell)r_\ell(1^{n-i-2}, \ell)$. By the odd sum rule [AGS, Prop. 3.5] we see that $r_\ell(1^i, \ell) = 0$ if $i \equiv \ell + 1 \pmod 2$.

In the remaining cases, we wish to show that

$$r_\ell(1^{i+2}, \ell)r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)r_\ell(1^{n-i-2}, \ell) \geq 0.$$

We apply the recurrence (2) and use $r_\ell(1^j, t) = 0$ if $t > \ell$ to obtain

$$\begin{aligned}
&r_\ell(1^{i+2}, \ell)r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)r_\ell(1^{n-i-2}, \ell)\\
&\quad = \big(r_\ell(1^{i+1}, \ell - 1) + r_\ell(1^{i+1}, \ell + 1)\big)r_\ell(1^{n-i-4}, \ell)\\
&\qquad - r_\ell(1^i, \ell)\big(r_\ell(1^{n-i-3}, \ell - 1) + r_\ell(1^{n-i-3}, \ell + 1)\big)\\
&\quad = r_\ell(1^{i+1}, \ell - 1)r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)r_\ell(1^{n-i-3}, \ell - 1)\\
&\quad = \big(r_\ell(1^i, \ell - 2) + r_\ell(1^i, \ell)\big)r_\ell(1^{n-i-4}, \ell)\\
&\qquad - r_\ell(1^i, \ell)\big(r_\ell(1^{n-i-4}, \ell - 2) + r_\ell(1^{n-i-2}, \ell)\big)\\
&\quad = r_\ell(1^i, \ell - 2)r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)r_\ell(1^{n-i-4}, \ell - 2).
\end{aligned}$$

By Lemma 4.3, we have

$$r_\ell(1^i, \ell - 2)r_\ell(1^{n-i-4}, \ell) - r_\ell(1^i, \ell)r_\ell(1^{n-i-4}, \ell - 2) \geq 0.$$ □

COROLLARY 4.5. *The divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is a nonnegative linear combination of the divisors $\big\{\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1}, (\frac{1}{\ell+2k+1})^{2g+2}} : k \in \mathbb{Z}_{\geq 0}, \ell + 2k \leq g\big\}$. Moreover, the coefficient of $\mathcal{D}_{\frac{\ell-1}{\ell+1}, (\frac{1}{\ell+1})^{2g+2}}$ in this expression is strictly positive.*

Proof. The statement follows from Proposition 4.4 and the intersection numbers computed in Corollary 2.2. $\square$

4.3. Morphisms Associated to Conformal Block Divisors

We are now in a position to prove that the divisors $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ give maps to Veronese quotients.

THEOREM 4.6. *The conformal block divisor* $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ *on* $\overline{\mathrm{M}}_{0,2g+2}$ *for* $1 \leq \ell \leq g$ *is the pullback of an ample class via the morphism*

$$\varphi_{\frac{\ell-1}{\ell+1}, A} = \phi_{\frac{\ell-1}{\ell+1}} \circ \rho_A : \overline{\mathrm{M}}_{0,n} \xrightarrow{\rho_A} \overline{\mathrm{M}}_{0,A} \xrightarrow{\phi_{(\ell-1)/(\ell+1)}} V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}, A}.$$

Here $A = \left(\frac{1}{\ell+1}\right)^{2g+2}$ *and* ρ_A *is the contraction to Hassett's moduli space* $\overline{\mathrm{M}}_{0,A}$ *of stable weighted pointed rational curves.*

Proof. By [AGS, Cors. 4.7, 4.9] and Corollary 2.2,

$$\mathcal{D}_{\frac{\ell-1}{\ell+1}, \left(\frac{1}{\ell+1}\right)^{2g+2}} \equiv \mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$$

if $\ell = 1, 2$. If $\ell \geq 3$ then, by Corollary 4.5, $\mathbb{D}(\mathfrak{sl}_2, \ell, \omega_1^{2g+2})$ is a nonnegative linear combination of $\mathcal{D}_{\frac{\ell+2k-1}{\ell+2k+1}, \left(\frac{1}{\ell+2k+1}\right)^{2g+2}}$ for $k \in \mathbb{Z}_{\geq 0}$ and $\ell + 2k \leq g$. In the latter case, by Proposition 3.8 we see that all of the divisors in this nonnegative linear combination are pullbacks of semi-ample divisors from $V^{g+1-\ell}_{\frac{\ell-1}{\ell+1}, A}$. Moreover, one of them is ample and appears with strictly positive coefficient. The result follows. $\square$

REMARK 4.7. If n is odd, then all of $\mathbb{D}(\mathfrak{sl}_2, \ell, (\omega_1, \dots, \omega_1))$ is trivial [Fa, Lemma 4.1]. It therefore suffices to consider $n = 2g + 2$ cases.

We note that, for a sequence of dominant integral weights $(k_1\omega_1, \dots, k_n\omega_1)$ of $\mathfrak{sl}_2$, the integer $\left(\sum_{i=1}^{n} \frac{k_i}{2}\right) - 1$ is called the *critical level* $c\ell$. By [Fa, Lemma 4.1], if ℓ is strictly greater than the critical level then $\mathbb{D}(\mathfrak{sl}_2, \ell, (k_1\omega_1, \dots, k_n\omega_1)) \equiv 0$, in which case the corresponding morphism is a constant map.

If $k_1 = \cdots = k_n = 1$ then the critical level is equal to g, so it is enough to study the cases $1 \leq \ell \leq g$. Hence Theorem 4.6 is a complete answer for the cases of Lie algebra $\mathfrak{sl}_2$ and weight data ω_1^n.

5. Conjectural Generalizations

Numerical evidence suggests that the connection between Veronese quotients and $\mathfrak{sl}_r$ conformal block divisors holds in a more general setting. In this section, we describe some of this evidence and make a few conjectures.

5.1. $\mathfrak{sl}_2$ *Cases*

We start by considering $\mathfrak{sl}_2$ symmetric weight cases—that is, $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ for $1 \leq k \leq \ell$. Theorem 4.6 tells us that when $k = 1$, the associated birational models are Veronese quotients. Before we can predict the birational models associated to other conformal block divisors, we need the following useful lemma.

LEMMA 5.1 [RW, (17)]. *The rank* $r_\ell(a_1,\dots,a_n) \neq 0$ *if and only if* (a) $\Lambda = \sum_{i=1}^n a_i$ *is even and* (b) *for any subset* $I \subset \{1,\dots,n\}$ *with* $n-|I|$ *odd,*

$$\Lambda - (n-1)\ell \le \sum_{i\in I}(2a_i - \ell).$$

Observe that, for a given weight datum, the left-hand side of this expression is fixed and the right-hand side is minimized by summing over all weights such that $2a_i < \ell$.

The next result shows that, when $k=\ell$, we get the same birational model as in the case $k=1$.

PROPOSITION 5.2. *We have the following equalities between conformal block divisors*:

$$\mathbb{D}(\mathfrak{sl}_2,\ell,\ell\omega_1^n) = \ell\mathbb{D}(\mathfrak{sl}_2,1,\omega_1^n) = \frac{\ell}{k}\mathbb{D}(\mathfrak{sl}_{2k},1,\omega_k^n).$$

Proof. The second assertion is a direct application of [GiG, Prop. 5.1], which states that

$$\mathbb{D}(\mathfrak{sl}_r,1,(\omega_{z_1},\dots,\omega_{z_n})) = \frac{1}{k}\mathbb{D}(\mathfrak{sl}_{rk},1,(\omega_{kz_1},\dots,\omega_{kz_n})).$$

For the first assertion, let $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_2,\ell,\ell\omega_1^n)$. It suffices to consider intersection numbers of $\mathbb{D}$ with F-curves of the form $F_i = F_{n-i-2,i,1,1}$. Then

$$\mathbb{D}\cdot F_{i_1,i_2,i_3,i_4} = \sum_{\vec{u}\in P_\ell^4} \deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(u_1\omega_1,u_2\omega_1,u_3\omega_1,u_4\omega_1))\big)\prod_{k=1}^4 r_\ell(\ell^{i_k},t),$$

where $P_\ell = \{0,1,\dots,\ell\}$. When $i_3=i_4=1$, we may use the two-point fusion rule for $\mathfrak{sl}_2$ to obtain

$$\mathbb{D}\cdot F_i = \sum_{0\le u_1,u_2\le\ell} \deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(u_1\omega_1,u_2\omega_1,\ell\omega_1,\ell\omega_1))\big) r_\ell(\ell^{n-i-2},u_1)r_\ell(\ell^i,u_2).$$

By the case $I=\{n\}$ if n is even and $I=\emptyset$ if n is odd in Lemma 5.1, we see that $r_\ell(\ell^j,t)=0$ if $0<t<\ell$. Hence

$$\mathbb{D}\cdot F_i = \sum_{u_1,u_2=0,\ell} \deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(u_1\omega_1,u_2\omega_1,\ell\omega_1,\ell\omega_1))\big) r_\ell(\ell^{n-i-2},u_1)r_\ell(\ell^i,u_2).$$

But by [Fa, Prop. 4.2],

$$\deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(0,0,\ell\omega_1,\ell\omega_1))\big) = \deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(0,\ell\omega_1,\ell\omega_1,\ell\omega_1))\big) = 0$$

and $\deg\big(\mathbb{V}(\mathfrak{sl}_2,\ell,(\ell\omega_1,\ell\omega_1,\ell\omega_1,\ell\omega_1))\big) = \ell$. Thus

$$\mathbb{D}\cdot F_i = \ell r_\ell(\ell^{n-i-2},\ell)r_\ell(\ell^i,\ell).$$

It therefore suffices to show that $r_\ell(\ell^t, \ell) = r_\ell(1^t, 1)$, but this follows by induction via the factorization rules and the propagation of vacua. □

For the majority of values of k such that $1 < k < \ell$, the divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ appears to give a map to a Hassett space. To establish our evidence for this, we begin with the following lemma.

LEMMA 5.3. *Suppose that* $1 < k < \ell$. *Then* $r_\ell(k^i, t) = 0$ *if and only if either* $ki + t$ *is odd or one of the following holds*:

(i) $2k \le \ell$ *and* $i < \max\left\{\frac{t}{k}, 2 - \frac{t}{k}\right\}$;
(ii) $2k > \ell$, i *is even, and* $i < \max\left\{\frac{t}{\ell-k}, 2 - \frac{t}{\ell-k}\right\}$;
(iii) $2k > \ell$, i *is odd, and* $i < \max\left\{\frac{\ell-t}{\ell-k}, 2 - \frac{\ell-t}{\ell-k}\right\}$.

Proof. Each of (i)–(iii) follows from the case-by-case analysis of Lemma 5.1 and the remark that follows it. □

We now consider which S_n-invariant F-curves have trivial intersection with the divisors in question.

PROPOSITION 5.4. *Suppose that* $1 < k < \frac{3}{4}\ell$, *and let* $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$. *Assume that* n *is even and that* $\ell \le \frac{kn}{2} - 1$. (*Recall that, by Remark 4.7, this assumption is necessary for the nontriviality of* $\mathbb{D}$.) *If* $a \le b \le c \le d$, *then* $\mathbb{D} \cdot F_{a,b,c,,d} = 0$ *if and only if* $a + b + c \le \frac{\ell+1}{k}$.

Proof. By [Fa, Prop. 4.7], the map associated to $\mathbb{D}$ factors through the map $\overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. Therefore, if $a + b + c \le \frac{\ell+1}{k}$ then $\mathbb{D} \cdot F_{a,b,c,d} = 0$. Hence it suffices to show the converse. We assume throughout that $a + b + c > \frac{\ell+1}{k}$.

By [Fa, Prop. 2.7], we have

$$\mathbb{D} \cdot F_{a,b,c,d} = \sum_{\vec{u} \in P_\ell^4} \deg\bigl(\mathbb{V}(\mathfrak{sl}_2, \ell, (u_1\omega_1, u_2\omega_1, u_3\omega_1, u_4\omega_1))\bigr) \times r_\ell(k^a, u_1) r_\ell(k^b, u_2) r_\ell(k^c, u_3) r_\ell(k^d, u_4).$$

Since each term in this sum is nonnegative, it is enough to show that a single term is nonzero.

We first consider the case that $2k \le \ell$. Set

$$w_a = \begin{cases} \min\{ka, \ell\} & \text{if } ka \equiv \ell \pmod 2, \\ \min\{ka, \ell - 1\} & \text{if } ka \not\equiv \ell \pmod 2. \end{cases}$$

Note that, by assumption, both $k(a+b+c+d) = kn$ and $k(a+b+c)+\ell$ are strictly greater than $2\ell + 1$. So it is straightforward to check that $w_a + w_b + w_c + w_d > 2\ell$ and $\ell + 1 > w_d$. Note further that $2\ell + 2 + 2w_a > 2w_a + w_c + w_d$ and $2w_a \ge 4$. Hence there is an integer w_b' such that $w_b' \equiv w_b \pmod 2$ and $2\ell < w_a + w_b' + w_c + w_d < 2\ell + 2 + 2w_a$. Then $w_a + w_b' + w_c + w_d \equiv w_a + w_b + w_c + w_d \equiv k(a+b+c+d) \equiv 0 \pmod 2$. Thus $\deg\bigl(\mathbb{V}(\mathfrak{sl}_2, \ell, (w_a\omega_1, w_b'\omega_1, w_c\omega_1, w_d\omega_1))\bigr) \ne 0$ by [Fa, Prop. 4.2]. It therefore suffices to show that $r_\ell(k^a, w_a) \ne 0$. In this case, however, Lemma 5.3 tells us that $r_\ell(k^a, w_a) = 0$ only if $a < \max\{w_a/k, 2 - w_a/k\} \le \max\{a, 2\}$, which is possible only if $a = 1$. But then $w_a = k$, so $r_\ell(k^a, w_a) = r_\ell(k, k) = 1 \neq 0$ by the two-point fusion rule. Therefore, $r_\ell(k^a, w_a)$ is always nonzero.

We next consider the case $2k > \ell$. Since here $\frac{\ell+1}{k} < 3$, we must show that no F-curves are contracted. Set

$$w_a = \begin{cases} k & \text{if } a \text{ is odd,} \\ 2(\ell - k) & \text{if } a \text{ is even.} \end{cases}$$

Again we have that $\ell + 1 > \max\{w_a, w_b, w_c, w_d\}$, $2\ell < w_a + w_b + w_c + w_d < 2\ell + 2 + 2\min\{w_a, w_b, w_c, w_d\}$, and $w_a + w_b + w_c + w_d \equiv 0 \pmod 2$. Thus $\deg\big(\mathbb{V}(\mathfrak{sl}_2, \ell, (w_a\omega_1, w_b\omega_1, w_c\omega_1, w_d\omega_1))\big) \neq 0$ by [Fa, Prop. 4.2]. If a is odd, we see that $r_\ell(k^a, w_a) = 0$ if and only if $a < 1$. If a is even, we see that $r_\ell(k^a, w_a) = 0$ if and only if $a < 2$. It follows that no F-curves are contracted. □

By Proposition 5.4, the F-curves that have trivial intersection with $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ are precisely those that are contracted by the morphism $\rho_{(\frac{k}{\ell+1})^n} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. However, this is not enough to conclude that $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ is actually the pullback of an ample divisor from this Hassett space, although this would follow from a well-known conjecture (see [KeMc, Ques. 1.1]).

Theorem 5.5. *Assume that n is even, $1 < k < \frac{3}{4}\ell$, and $\ell \le \frac{kn}{2} - 1$. If the F-Conjecture holds (see [KeMc, Ques. 1.1]), then the divisor $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ is the pullback of an ample class via the morphism $\rho_{(\frac{k}{\ell+1})^n} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,(\frac{k}{\ell+1})^n}$. In particular, if $\frac{\ell+1}{3} < k < \frac{3}{4}\ell$ then $\mathbb{D}$ is ample.*

We note further that Proposition 5.4 does not cover all of the possible cases of symmetric-weight $\mathfrak{sl}_2$ conformal block divisors. In particular, if $k \ge \frac{3}{4}\ell$ then $\mathbb{D}(\mathfrak{sl}_2, \ell, k\omega_1^n)$ may have trivial intersection with an F-curve when all of the legs contain an even number of marked points. Such is the case, for example, with the divisor $\mathbb{D}(\mathfrak{sl}_2, 4, 3\omega_1^8)$, which has zero intersection with the F-curve $F(2, 2, 2, 2)$ and positive intersection with every other F-curve. It is not difficult to see that the associated birational model is the Kontsevich–Boggi compactification of $\mathrm{M}_{0,8}$ (see [GiJM, Sec. 7.2] for details on this moduli space).

5.2. *Birational Properties of $\mathfrak{sl}_r$ Conformal Blocks*

In every known case, the birational model associated to conformal block divisors is in fact a compactification of $\mathrm{M}_{0,n}$. That is, the associated morphism restricts to an isomorphism on the interior. We pose this as a conjecture.

Conjecture 5.6. *If $\mathbb{D}$ is a nontrivial conformal block divisor of $\mathfrak{sl}_r$ with strictly positive weights, then $\mathbb{D}$ separates all points on $\mathrm{M}_{0,n}$. More precisely: for any two distinct points $x_1, x_2 \in \mathrm{M}_{0,n}$, the morphism*

$$H^0(\overline{\mathrm{M}}_{0,n}, \mathbb{D}) \to \mathbb{D}|_{x_1} \oplus \mathbb{D}|_{x_2}$$

is surjective.

If true, this conjecture would have several interesting consequences. Among them is the following simple description of the maps associated to conformal block divisors. Let $\mathbb{D}$ be a conformal block divisor of $\mathfrak{sl}_r$ and let $\rho_{\mathbb{D}}\colon \overline{\mathrm{M}}_{0,n} \to X$ be the associated morphism. Consider a boundary stratum

$$\prod_{i=1}^{m} \mathrm{M}_{0,k_i} \hookrightarrow \overline{\mathrm{M}}_{0,n}.$$

By factorization (see [Fa, Prop. 2.4]), the pullback of an $\mathfrak{sl}_r$ conformal block divisor to $\overline{\mathrm{M}}_{0,k_m}$ and its interior M_{0,k_m} is an effective sum of $\mathfrak{sl}_r$ conformal block divisors. If all of the divisors in this sum are trivial, then the restriction of $\rho_{\mathbb{D}}$ to this boundary stratum forgets a component of the curve:

$$\begin{tikzcd} \prod_{i=1}^{m} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \prod_{i=1}^{m-1} \mathrm{M}_{0,k_i} \arrow[r] & X. \end{tikzcd}$$

If the only nontrivial divisors in this sum have weight 0 on some subset of the attaching points, then these divisors are pullbacks of nontrivial conformal block divisors via the map that forgets these points. Therefore, the restriction of $\rho_{\mathbb{D}}$ to this boundary stratum forgets these attaching points:

$$\begin{tikzcd} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \mathrm{M}_{0,k_i-j} \arrow[r] & X. \end{tikzcd}$$

Finally, if any of the nontrivial conformal block divisors in this sum has strictly positive weights, then (by Conjecture 5.6) the restriction of $\rho_{\mathbb{D}}$ to the interior of this stratum is an isomorphism:

$$\begin{tikzcd} \mathrm{M}_{0,k_i} \arrow[r] \arrow[d, "\cong"] & \overline{\mathrm{M}}_{0,n} \arrow[d, "\rho_{\mathbb{D}}"] \\ \mathrm{M}_{0,k_i} \arrow[r] & X. \end{tikzcd}$$

In summary, Conjecture 5.6 implies that the image of a boundary stratum $\prod_{i=1}^{m} \mathrm{M}_{0,k_i}$ in X is isomorphic to

$$\prod_{i=1}^{a} \mathrm{M}_{0,k_i} \times \prod_{i=a+1}^{b} \mathrm{M}_{0,k_i-j_i}$$

for some $1 \le a \le b \le n$ and $1 \le j_i \le k_i - 3$.

In this way, the morphisms associated to $\mathfrak{sl}_r$ conformal blocks are reminiscent of Smyth's modular compactifications [Sm]. Each of those compactifications can be described by assigning, to each boundary stratum, a collection of "forgotten" components. In a similar way, the morphism $\rho_{\mathbb{D}}$ appears to assign to each boundary stratum a collection of forgotten components and forgotten points of attachment. It follows that, if Conjecture 5.6 holds, one can understand the morphism $\rho_{\mathbb{D}}$ completely from such combinatorial data.

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