Gonality of expander graphs

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ABSTRACT

We provide lower bounds on the gonality of a graph in terms of its spectral and edge expansion. As a consequence, we see that the gonality of a random 3-regular graph is asymptotically almost surely greater than one seventh its genus.

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1. Introduction

In this paper, we study the relationship between the gonality of a graph G and its expansion properties. In particular, we provide lower bounds on the gonality $gon(G)$ in terms of the Cheeger constant $h(G)$ and a graph invariant $B_u(G)$ that we define here. We refer the reader to Section 2 for definitions of the Cheeger constant and the terms appearing in Definition 1.1.

Definition 1.1. For any $u \in (0, \frac{1}{2}]$, we define $B_u(G)$ be the smallest degree of an effective divisor D such that every connected component of $V(G) \smallsetminus \text{supp}(D)$ has size at most $u|V(G)|$.

Our main result is the following.

Theorem 1.2. *Let G be a graph. For any $u \in (0, \frac{1}{2}]$, we have*

$$gon(G) \geq \min \{B_u(G), h(G)u|V(G)|\} .$$

The function $B_u(G)$ is decreasing in u, while $h(G)u|V(G)|$ is increasing in u. As a consequence of Theorem 1.2, we obtain the following lower bound on the gonality of a random 3-regular graph.

Theorem 1.3. *Let G be a random 3-regular graph on n vertices. Then*

$$\text{gon}(G) \geq 0.072n$$

asymptotically almost surely.

Note that the genus of a 3-regular graph on n vertices is $\frac{1}{2}n + 1$. Theorem 1.3 therefore implies that the expected gonality of a random 3-regular graph is at least one seventh its genus. This result is obtained by bounding the invariants $h(G)$ and $B_u(G)$ that appear in Theorem 1.2 for random 3-regular graphs. This yields a lower bound on the gonality of such graphs that depends on u, which is optimized for some u between 0.35 and 0.40.

Our primary motivation for studying the gonality of graphs comes from the theory of algebraic curves. In [3], Baker develops a theory of specialization of divisors from curves to graphs. This theory allows one to establish results in algebraic geometry using combinatorial techniques, and conversely, to discover new combinatorics using algebraic geometry. Because the top-dimensional strata of the moduli space of tropical curves correspond to 3-regular graphs, such graphs are considered

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to be the closest analogues of algebraic curves. It is therefore natural to compare invariants of random 3-regular graphs to the analogous invariants of general curves. (For more on the moduli space of tropical curves, we refer the reader to [7].)

By [17,18], the gonality of any curve of genus g is at most $\lfloor\frac{g+3}{2}\rfloor$. Moreover, the Brill–Noether theorem implies that equality holds for the general curve [15]. In other words, the set of genus g curves of gonality exactly $\lfloor\frac{g+3}{2}\rfloor$ is a dense open subset of the moduli of space of curves M_g. One could hope to address the analogous questions for 3-regular graphs. First, what is the maximum gonality of a 3-regular graph? Second, what is the expected gonality of a random 3-regular graph? Theorem 1.3 represents progress toward the second of these questions.

Theorem 1.3 is an improvement on several earlier results. In [25], van Dobben de Bruyn and Gijswijt show that the gonality of a graph is bounded below by its treewidth. In [19], the authors show that a random 3-regular graph on n vertices almost surely has bisection width at least $\frac{10}{99}n$. As noted in [23], this implies that such a graph almost surely has treewidth at least $\frac{5}{99}n$. Amini and Kool use the relation between gonality and treewidth [1, Corollary 5.1] to produce a lower bound on graph gonality in terms of the algebraic connectivity. They then use this to show that the gonality of a random k-regular graph is bounded above and below by linear functions in n. While [1] do not attempt to bound the implied constants, their approach cannot yield a bound higher than $\frac{5}{99}n$, due to their reliance on the treewidth bound. In a related direction, [1] shows that the expected gonality of an Erdős–Rényi random graph on n vertices is also bounded above and below by functions that are linear in n. An improvement of this appears in [11], where it is shown that the expected gonality of such graphs is in fact asymptotic to n.

The argument of the present paper borrows heavily from the proof in [25] that gonality is bounded below by treewidth. The treewidth of a graph can be defined in terms of certain collections of connected subsets of the vertices, known as brambles. One example of a bramble is the set of all connected subsets of $V(G)$ of size at least $\frac{1}{2}|V(G)|$, which implies that the treewidth of a graph G, and hence its gonality, is at least $B_{\frac{1}{2}}(G)$. To improve on this, we note that [25] in fact uses the treewidth in two separate ways. First, the treewidth is a lower bound on the degree of an effective divisor that intersects every element of a bramble. Second, it provides a lower bound on the size of a cut that separates two elements of this bramble. If we consider instead the set of all connected subsets of size at least $u|V(G)|$ for some $u < \frac{1}{2}$ (which is not a bramble), then the minimum size of a cut that separates these subsets decreases, while the minimum size of an effective divisor that intersects every subset increases. By varying u, it is possible to obtain a higher bound.

The invariant $B_u(G)$ is somewhat mysterious, and much of the paper is devoted to finding bounds for it in terms of more familiar graph invariants, including the u-Cheeger constant $h_u(G)$ and the algebraic connectivity λ_2. We again refer the reader to Section 2 for definitions of these graph invariants. Specifically, we establish the following results.

Theorem 1.4. *Let G be a k-regular graph. Then, for any $u \in (0, \frac{1}{2}]$, we have*

$$\text{gon}(G) \geq \min\left\{\frac{h_u(G)}{k+h_u(G)}, h(G)u\right\}|V(G)|.$$

Theorem 1.5. *Let G be a graph, and let d be the maximum valence of a vertex in G. Then*

$$\text{gon}(G) \geq \min\left\{\frac{1}{4}, \frac{1}{2\lambda_2}\left[-(7\lambda_2+9d)+3\sqrt{9\lambda_2^2+14d\lambda_2+9d^2}\right]\right\}|V(G)|.$$

In the literature there are at least two notions of the gonality of a graph. Our interest here is in the *divisorial gonality*, which is the smallest degree of a divisor on the graph with positive rank. A related invariant is the *tree gonality*, which is the smallest degree of a harmonic morphism from the graph to a tree. The divisorial gonality of a graph is always less than or equal to the tree gonality, since the pullback of a vertex via a harmonic morphism has positive rank. Thus, any lower bound on the divisorial gonality, such as those provided in Theorems 1.2, 1.4 and 1.5, is also a lower bound on the tree gonality.

In [9, Theorem C], the authors provide a lower bound on the stable tree gonality of a graph G, which is the minimal tree gonality of any refinement of G, in terms of the algebraic connectivity λ_2. Theorem 1.5 can be seen as a version of this result for the gonality of the graph itself, without passing to any refinement. In the case of primary interest to us, namely that of random 3-regular graphs, the bound provided by Theorem 1.4 is stronger than that of Theorem 1.5. Nevertheless, Theorem 1.5 has at least two advantages. First, while Theorem 1.4 applies only to regular graphs, Theorem 1.5 applies to arbitrary graphs. Second, while computation of the u-Cheeger constant is NP-hard, the algebraic connectivity can be computed, to any degree of accuracy, in polynomial time. For this reason we expect that Theorem 1.5 may be more useful for applications, as it is more efficient.

While this paper is primarily concerned with lower bounds on graph gonality, we make a few brief remarks on upper bounds. It follows from Baker's specialization lemma that, if G is a graph of genus g, then there exists a positive integer e such that the refinement obtained by subdividing each edge of G into e edges has gonality at most $\lfloor\frac{g+3}{2}\rfloor$. We note that this argument uses the Kleiman–Laksov result on algebraic curves and to date there is no known purely combinatorial proof. The question of whether this statement holds without refinement – that is, whether the integer e can always be taken to be 1 – remains open, though it has been proven in small genus cases [2]. Cools and Draisma have shown that, for every 3-regular graph G, there exists an open cone of edge lengths in $\mathbb{R}_{>0}^{E(G)}$ whose integer points correspond to discrete graphs with gonality at most $\lfloor\frac{g+3}{2}\rfloor$ [8]. Without passing to refinements, much less is known. As far as we are aware, the best known upper bound for the gonality of a graph is its genus. Specifically, if G is a graph of genus g and E is any effective divisor of degree $g - 2$,

then by Riemann–Roch the divisor $K_G - E$ has rank at least 1. It follows that $\text{gon}(G) \leq g$. To obtain another upper bound, note that if G is a simple graph, the complement of an independent set has rank at least 1, and thus $\text{gon}(G) \leq |V(G)| - \alpha(G)$, where $\alpha(G)$ is the size of the largest independent set in G. For a 3-regular graph G, however, this bound is smaller than g if and only if G is bipartite. In this case $|V(G)| - \alpha(G) = g - 1$, so for bipartite 3-regular graphs this upper bound differs by only 1 from the bound obtained by Riemann–Roch.

2. Preliminaries

2.1. Divisor theory of graphs

In this section, we outline the basic definitions and properties of divisors on graphs. Throughout, our graphs will always be assumed to be connected with no loops, though possibly with multiple edges. For more details, we refer the reader to [3,4].

Definition 2.1. A *divisor D* on a graph G is a formal $\mathbb{Z}$-linear combination of vertices of G.

Divisors are alternatively known as "chip-configurations", because we may view a divisor as consisting of stacks of chips on the vertices of G. We write $D(v)$ for the coefficient of the vertex v in the divisor D, or alternatively the number of chips of D at v. We define an equivalence relation on the set of divisors on G using chip-firing moves, which we define here.

Definition 2.2. Let D be a divisor on a graph G and fix $v \in V(G)$. The divisor D' obtained from D by *firing* the vertex v is defined by

$$D'(w) = \begin{cases} D(v) - \text{val}(v) & \text{if } w = v \\ D(w) + \varepsilon(w, v) & \text{if } w \neq v \end{cases}$$

where $\varepsilon(w, v)$ is the number of edges between w and v and $\text{val}(v)$ is the valence of v.

If the vertices of a graph G are labeled $v_1, \ldots, v_n$, we define the *Laplacian* of G, denoted $L(G)$, to be the $n \times n$ matrix with entries

$$L(G)_{ij} = \begin{cases} \text{val}(v_i) & \text{if } i = j \\ -\varepsilon(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

Two divisors D and D' are said to be *linearly equivalent* if D' can be obtained from D by a sequence of chip-firing moves. Equivalently, D and D' are equivalent if their difference is contained in the image of $\mathbb{Z}^{V(G)}$ under the graph Laplacian $L(G)$. This defines an equivalence relation on the set of divisors on G.

We now recall some definitions related to positivity of divisors on graphs.

Definition 2.3. A divisor D is called *effective* if $D(v) \geq 0$ for all $v \in V(G)$.

Definition 2.4. The *complete linear series* of a divisor D is

$$|D| = \{D' \sim D | D' \text{ is effective}\}.$$

The *support* of an effective divisor D is

$$\text{supp}(D) = \{v \in V(G) | D(v) > 0\}.$$

Perhaps the most important invariant of a divisor on a graph is its Baker–Norine rank.

Definition 2.5. A divisor D is said to have *rank at least r* if, for every effective divisor E of degree r, $D - E$ is equivalent to an effective divisor. The *rank* of D is the largest integer r such that D has rank at least r.

Our main interest in this paper is a graph invariant known as the gonality.

Definition 2.6. The *gonality* of a graph G is the smallest degree of a divisor on G with positive rank.

While it is NP-hard to compute the gonality of a graph [14], the complexity arises from the sheer number of divisors that one would be required to check. If we are given a divisor, there is a relatively simple algorithm that determines whether it has positive rank. To follow the algorithm, one first needs the notion of a reduced divisor.

Definition 2.7. Let $v \in V(G)$. A divisor D is said to be *v-reduced* if:

(1) the divisor D is "effective away from v" – that is, for all $w \in V(G) \setminus \{v\}$, $D(w) \geq 0$, and
(2) for any nonempty $U \subset V(G) \setminus \{v\}$, the divisor obtained by firing all vertices in U is not effective away from v.

Each divisor has a unique v-reduced representative. Moreover, if a divisor E is equivalent to an effective divisor, then its v-reduced representative is effective. With these facts in hand, it becomes relatively easy to check that an effective divisor E has positive rank. The divisor E has positive rank if and only if the v-reduced divisor equivalent to $E - v$ is effective for all vertices v. Of course, it is crucial in this process that we be able to compute the v-reduced divisor equivalent to $E - v$. To find v-reduced divisors quickly, we have the following algorithm, known as *Dhar's Burning Algorithm*. (For a detailed account of this algorithm, we refer the reader to [12] or the more recent [5].)

(1) Given a divisor D that is effective away from v (for example, $E - v$ where E is effective), start a fire at the vertex v, placing it in the set of burnt vertices.
(2) For each burnt vertex, place each edge adjacent to it in the set of burnt edges.
(3) If the number of burnt edges adjacent to a vertex w exceeds the number of chips on w, place w in the set of burnt vertices. Then repeat step 2. Otherwise, if no such vertex exists, proceed to step 4.
(4) If the entire set of vertices burn, then D is v-reduced. Otherwise, the divisor obtained by firing all unburnt vertices is effective away from v. Replace D with this equivalent divisor and repeat the process.

2.2. Expander graphs

In this paper we relate the gonality to the expansion properties of a graph. One of the most well-known measures of graph expansion is known variously as the Cheeger constant, isoperimetric number, or edge expansion constant. Throughout, if G is a graph and $U \subseteq V(G)$, we define ∂U to be the set of edges with exactly one endpoint in U.

Definition 2.8 ([21])**.** The *Cheeger constant* of a graph G is defined to be

$$h(G) := \min_{0 < |U| \le \frac{1}{2}|V(G)|} \frac{|\partial U|}{|U|}.$$

More generally, for any $u \in (0, \frac{1}{2}]$, the *u-Cheeger constant* of G is defined to be

$$h_u(G) := \min_{0 < |U| \le u|V(G)|} \frac{|\partial U|}{|U|}.$$

For values of u greater than $\frac{1}{2}$, we vacuously define $h_u(G)$ to be infinity.

A second measure of graph expansion is the algebraic connectivity, which we define here.

Definition 2.9. The *algebraic connectivity* λ_2 of a graph G is defined to be the second smallest[1] eigenvalue of the graph Laplacian $L(G)$.

Note that the smallest eigenvalue of the graph Laplacian is zero, and by the Matrix Tree Theorem, this eigenvalue has multiplicity one if and only if the graph is connected. This suggests that λ_2 is a measure of a graph's connectivity. For k-regular graphs, the algebraic connectivity is related to the Cheeger constant by the well-known Cheeger inequalities:

$$\frac{1}{2}\lambda_2 \le h(G) \le \sqrt{2k\lambda_2}.$$

One advantage of the algebraic connectivity is that the Cheeger constant is NP-hard to compute (see [21, Theorem 3.1]), whereas the algebraic connectivity of a graph with n vertices can be computed to any degree of accuracy in $O(n^3)$ time. (In particular, finding the eigenvalues of a matrix has the same computational complexity as matrix multiplication. See [10] for further discussion.)

The algebraic connectivity can be used to bound the size of vertex separators, which we define below.

Definition 2.10 ([22])**.** Let G be a graph. A set $C \subset V(G)$ is said to *separate* vertex sets $A, B \subset V(G)$ if

(1) A, B and C partition $V(G)$ and
(2) no vertex of A is adjacent to a vertex of B.

Lemma 2.11 ([22, Theorem 2.8])**.** *Let G be a graph and denote by d the maximal valence of a vertex of G. If C separates vertex sets A and B, then*

$$|C| \ge \frac{4\lambda_2|A||B|}{d|V(G)| - \lambda_2|A \cup B|}.$$

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[1] We note with mild exasperation that there is no apparent consistency in the combinatorics literature concerning the indices of the eigenvalues of the graph Laplacian. They are equally likely to be ordered smallest to largest as largest to smallest, or indexed starting with zero rather than one. Such issues nevertheless pale in comparison to the wealth of literature that refers simply to the "second eigenvalue" without indicating whether this means second smallest or second largest.

An important class of expander graphs is that of random regular graphs. A model for such graphs can be found in [6]. We say that a property of a random k-regular graph holds *asymptotically almost surely* if the probability that the property holds approaches one as n approaches infinity.

3. Gonality and edge expansion

Our main observation is the following. If, for every representative of a divisor class D on a graph G, there exists a "large" connected component of $V(G) \smallsetminus \text{supp}(D)$, then D cannot have positive rank. This is made precise by Theorem 1.2.

Proof of Theorem 1.2. Let D be a divisor of positive rank and assume that $\deg(D) < B_u(G)$. For any representative $D' \in |D|$, we let $U_{D'}$ denote the union of all connected components of $V(G) \smallsetminus \text{supp}(D')$ of size greater than $u|V(G)|$. Note that, by the definition of $B_u(G)$, for any representative D', the set $U_{D'}$ is nonempty.

Choose $D_0 \in |D|$ such that U_{D_0} is minimal. That is, for any $D' \in |D|$, $U_{D'}$ is not strictly contained in U_{D_0}. Now, pick a vertex $v \in U_{D_0}$. By [25, Lemma 1.3], there exist firing sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq V(G) \smallsetminus \{v\}$ such that the divisor obtained from D_{i-1} by firing A_i is an effective divisor D_i, and D_n is v-reduced. Since by assumption D has positive rank, $D_n(v) > 0$.

Since firing these sets A_i eventually puts at least one chip in U_{D_0}, there must exist an index i such that $U_{D_i} \neq U_{D_0}$. Let i be the smallest such index. Since U_{D_0} was chosen to be minimal, $U_{D_i} \not\subset U_{D_0} = U_{D_{i-1}}$. It follows that there exists some connected component $U \subset U_{D_i}$ that intersects the support of D_{i-1}. Since U does not intersect the support of D_i, we conclude from [25, Lemma 2.2] that $U \subset A_i$. Because of this containment, we have $|U| \leq |A_i|$. At the same time, by the definition of U_{D_i}, every connected component of U_{D_i} has size strictly greater than $u|V(G)|$, hence $|A_i| \geq |U| > u|V(G)|$.

Let $U' \subset U_{D_0}$ be the connected component that contains v, and note that by assumption we have $U' \subset V(G) \smallsetminus A_i$. Since $|A_i| \leq |V(G) \smallsetminus U'|$, we have $|V(G) \smallsetminus A_i| \geq |U'| > u|V(G)|$. Since $\partial A_i = \partial(V(G) \smallsetminus A_i)$ and both A_i and $V(G) \smallsetminus A_i$ have size greater than $u|V(G)|$, we see that $|\partial A_i| > h(G)u|V(G)|$. Finally, since D_i is effective, we must have $D_{i-1}(w) \geq \text{outdeg}_w A_i$ for every vertex $w \in A_i$. Thus, $\deg(D) > |\partial A_i|$, and so $\deg(D) > h(G)u|V(G)|$. $\square$

We note the following, which follows directly from the proof of Theorem 1.2. This is slightly stronger than the statement that the gonality is bounded below by $B_{\frac{1}{2}}(G)$, because it determines a property of a divisor that realizes the gonality.

Corollary 3.1. *Let D be a divisor class of positive rank on a graph G. Then there exists a representative for D such that every connected component of $V(G) \smallsetminus \text{supp}(D)$ has size at most $\frac{1}{2}|V(G)|$.*

Proof. If not, then in the proof of Theorem 1.2, we see that both A_i and $V(G) \smallsetminus A_i$ have size greater than $\frac{1}{2}|V(G)|$, which is impossible. $\square$

The following lemma provides a bound, for regular graphs, on the invariant $B_u(G)$ introduced in Definition 1.1.

Lemma 3.2. *Let G be a k-regular graph, and let D be an effective divisor such that*

$$\deg(D) < \frac{h_u(G)}{k + h_u(G)}|V(G)|.$$

Then there exists a connected component of $V(G) \smallsetminus \text{supp}(D)$ with size greater than $u|V(G)|$.

Proof. Assume that every connected component of $V(G) \smallsetminus \text{supp}(D)$ has size less than or equal to $u|V(G)|$. Let U be a connected component of $V(G) \smallsetminus \text{supp}(D)$. Note that the number of edges with at least one end in U, here denoted $e(U)$, is

$$e(U) = \frac{1}{2}(k|U| + |\partial U|) \geq \frac{k + h_u(G)}{2}|U|.$$

To see this, note that the number of half-edges with one end in U is exactly $k|U|$. Since the internal edges of U contribute twice to this sum and the edges that leave U contribute only once, we arrive at the equality above. The inequality is given by the definition of the u-Cheeger constant.

Note that $e(V(G) \smallsetminus \text{supp}(D))$ is the number of edges with at most one endpoint in the support of D. Thus

$$e(V(G) \smallsetminus \text{supp}(D)) = \sum_U e(U),$$

where the sum is over connected components of $V(G) \smallsetminus \text{supp}(D)$. By the above, therefore, we have

$$\begin{aligned} e(V(G) \smallsetminus \text{supp}(D)) &\geq \sum_U \frac{k + h_u(G)}{2}|U| \\ &= \frac{k + h_u(G)}{2}|V(G) \smallsetminus \text{supp}(D)| \\ &> \frac{k + h_u(G)}{2}\left(1 - \frac{h_u(G)}{k + h_u(G)}\right)|V(G)| = \frac{k}{2}|V(G)|. \end{aligned}$$

Since the total number of edges in G is $\frac{k}{2}|V(G)|$, this is impossible. Therefore, there exists a connected component of the complement of supp(D) of size greater than $u|V(G)|$. □

Together, Theorem 1.2 and Lemma 3.2 yield a lower bound on the gonality of regular graphs in terms of the Cheeger constant.

Proof of Theorem 1.4. By Lemma 3.2, for any effective divisor D with deg(D) $< \frac{h_u(G)}{k+h_u(G)}|V(G)|$, there exists a connected component of $V(G) \smallsetminus$ supp(D) of size greater than $u|V(G)|$. Since all effective divisors of this degree have this property, we see that

$$\frac{h_u(G)}{k+h_u(G)}|V(G)| \leq B_u(G).$$

The result then follows from Theorem 1.2. □

Proof of Theorem 1.3. By [19], the Cheeger constant of a random 3-regular graph is asymptotically almost surely bounded below by $\frac{1}{4.95}$. Similarly, [20, Theorem 3] provides lower bounds above which the u-Cheeger constant of a random regular graph is bounded asymptotically almost surely. For 3-regular graphs, this bound is optimized for $u \approx 0.36$, at which point it is 0.24. By Theorem 1.4, therefore, the gonality of a random 3-regular graph is asymptotically almost surely greater than or equal to

$$\min \left\{ \frac{0.24}{3.24}, \frac{0.36}{4.95} \right\} |V(G)| = 0.072|V(G)|. \square$$

While our main interest is in 3-regular graphs, we note the following amusing fact.

Proposition 3.3. *Let* gon(k) *be the supremum over all ℓ such that the gonality of a random k-regular graph on n vertices is asymptotically almost surely at least ℓn. Then*

$$\lim_{k \to \infty} \text{gon}(k) \geq \frac{1}{3}.$$

Proof. Let $i(k)$ be the supremum over all ℓ such that the Cheeger constant of a random k-regular graph is asymptotically almost surely at least ℓk. By [6], $\lim_{k \to \infty} i(k) \geq \frac{1}{2}$. It follows that

$$\lim_{k \to \infty} \text{gon}(k) \geq \lim_{k \to \infty} \frac{i(k)k}{k+i(k)k} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}. \square$$

4. Gonality and spectral expansion

Determining the Cheeger constant of a graph is an NP-complete problem [16]. Using the algebraic connectivity, however, we can provide a lower bound on graph gonality that can be computed in polynomial time. This bound also does not require the graph to be regular, and can therefore be applied in a more general setting than Theorem 1.4. To obtain this bound, we first prove a proposition about divisors that act as separators.

Proposition 4.1. *Let D be a divisor on a graph G such that all connected components of $V(G) \smallsetminus$ supp(D) have size less than $\frac{1}{2}|V(G)|$. If deg(D) $\leq \frac{1}{4}|V(G)|$, then there exist sets $A, B \subset V(G)$ such that* supp(D) *separates A and B and*

$$\frac{1}{3}|V(G) \smallsetminus \text{supp}(D)| \leq |A| \leq \frac{2}{3}|V(G) \smallsetminus \text{supp}(D)|.$$

Proof. Let A be the maximal union of connected components of $V(G) \smallsetminus$ supp(D) such that $|A| \leq \frac{2}{3}|V(G) \smallsetminus$ supp(D)$|$, and let B be the union of the remaining connected components. If $\frac{1}{3}|V(G) \smallsetminus$ supp(D)$| \leq |A|$, then we are done. We may therefore suppose that $|A| < \frac{1}{3}|V(G) \smallsetminus$ supp(D)$|$. In this case, we move a component U of B into A.

Note that, by our assumption that A is maximal, we must have $|A \cup U| > \frac{2}{3}|V(G) \smallsetminus$ supp(D)$|$. Then, since each connected component of $V(G) \smallsetminus$ supp(D) has size less than half the vertices of G, we see that $\frac{1}{3}|V(G) \smallsetminus$ supp(D)$| \leq |U| \leq \frac{1}{2}|V(G)|$. By assumption, we have

$$\frac{1}{2}|V(G)| \leq \frac{2}{3} \left[|V(G)| - \text{deg}(D) \right] \leq \frac{2}{3}|V(G) \smallsetminus \text{supp}(D)|.$$

Thus, we may let $A = U$, and let B be the union of the remaining connected components. □

We now use Proposition 4.1 in conjunction with Lemma 2.11 to find a lower bound for the gonality entirely in terms of the algebraic connectivity, the maximum valence of a vertex in G, and $|V(G)|$.

Proof of Theorem 1.5. Let D be a divisor class on G of positive rank. If $\deg(D) \geq \frac{1}{4}|V(G)|$, there is nothing to prove. Otherwise, by Corollary 3.1, there exists a representative for D such that every connected component of $V(G) \smallsetminus \text{supp}(D)$ has size at most $\frac{1}{2}|V(G)|$. Without loss of generality, we will assume that D is this representative. By Proposition 4.1 we have that $\text{supp}(D)$ separates vertex sets A and B such that

$$\frac{1}{3}|V(G) \smallsetminus \text{supp}(D)| \leq |A| \leq \frac{2}{3}|V(G) \smallsetminus \text{supp}(D)|.$$

Let $x = \frac{|A|}{|V(G) \smallsetminus \text{supp}(D)|}$. By assumption, $\frac{1}{3} \leq x \leq \frac{2}{3}$, and

$$|A||B| = x(1-x)|V(G) \smallsetminus \text{supp}(D)|^2.$$

On the interval $[\frac{1}{3}, \frac{2}{3}]$, the function $x(1-x)$ has a single critical point at $x = \frac{1}{2}$, but since this is a downward opening parabola we know this critical point is a maximum. Therefore, $x(1-x)$ attains its minimum value at the endpoints $x = \frac{1}{3}, \frac{2}{3}$. Thus we have that

$$x(1-x)|V(G) \smallsetminus \text{supp}(D)|^2 \geq \frac{2}{9}(|V(G)| - |\text{supp}(D)|)^2.$$

Employing Lemma 2.11 we have

$$|\text{supp}(D)| \geq \frac{4\lambda_2|A||B|}{d|V(G)| - \lambda_2|A \cup B|} \geq \frac{\frac{8\lambda_2}{9}(|V(G)| - |\text{supp}(D)|)^2}{d|V(G)| - \lambda_2(|V(G)| - |\text{supp}(D)|)}.$$

Solving for $|\text{supp}(D)|$, we obtain:

$$\lambda_2|\text{supp}(D)|^2 + (7\lambda_2 + 9d)|V(G)||\text{supp}(D)| - 8\lambda_2|V(G)|^2 \geq 0.$$

We may now apply the quadratic formula.

$$|\text{supp}(D)| \geq \frac{1}{2\lambda_2}\left[-(7\lambda_2 + 9d)|V(G)| + \sqrt{(7\lambda_2 + 9d)^2|V(G)|^2 + 32\lambda_2^2|V(G)|^2}\right]$$

$$= \frac{|V(G)|}{2\lambda_2}\left[-(7\lambda_2 + 9d) + 3\sqrt{9\lambda_2^2 + 14d\lambda_2 + 9d^2}\right]. \quad \square$$

Note that the Laplacian of a graph can be constructed easily given a set of vertices and edge relations, and using the QR algorithm, eigenvalues of the Laplacian can be approximated to any degree of accuracy in $O(n^3)$ time. This means that given an arbitrary graph, we can use Theorem 1.5 to obtain a lower bound on the gonality in a relatively small amount of time.

Although it is not as good as our previous bound, we record here what Theorem 1.5 can tell us about random 3-regular graphs.

Corollary 4.2. *Let G be a random 3-regular graph. Then, for any $\epsilon > 0$, we have*

$$\text{gon}(G) \geq (0.0486 - \epsilon)|V(G)|$$

asymptotically almost surely.

Proof. By [13], for any $\epsilon > 0$, the algebraic connectivity of a random k-regular graph is asymptotically almost surely bounded below by $k - 2\sqrt{k-1} - \epsilon$. Taking $k = 3$, we evaluate the right hand side of the inequality in Theorem 1.5 to obtain:

$$\text{gon}(G) \geq \frac{|V(G)|}{2(3 - 2\sqrt{2})}\left[-(7(3 - 2\sqrt{2}) + 27) + 3\sqrt{9(3 - 2\sqrt{2})^2 + 42(3 - 2\sqrt{2}) + 81} - \epsilon\right]$$

$$= (0.0486 - \epsilon)|V(G)|. \quad \square$$

5. An example

As an example, we compute these bounds for the Pappus graph pictured in Fig. 1. The Pappus graph is notable for having the largest Cheeger constant of any 3-regular graph with at most 20 vertices [24]. As such, it is a good candidate for testing our results.

As a first estimate of the gonality of the Pappus graph, we apply Theorem 1.4. The u-Cheeger constants for the Pappus graph are provided in Table 1. Evaluating the expression

$$\min\left\{\frac{h_u(G)}{3 + h_u(G)}|V(G)|, h(G)u|V(G)|\right\}$$

for all u in Table 1, we find that the bound given by Theorem 1.4 is optimized for $u = \frac{1}{3}$, giving a bound of $\frac{1}{4}|V(G)|$. Thus $\text{gon}(G) \geq 4.5$.

Table 1
u-Cheeger constants of the Pappus graph.

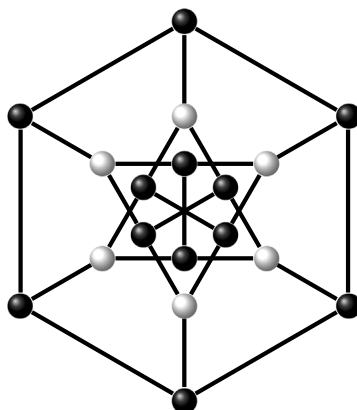| u	$h_u(G)$
1/18	3
2/18	2
3/18	5/3
4/18	3/2
5/18	7/5
6/18	1
7/18	1
8/18	1
9/18	7/9

Fig. 1. The Pappus graph, together with a divisor of degree 6 and rank 1, highlighted in white.

Applying Theorem 1.5, we obtain the same estimate. The second smallest eigenvalue of the Pappus graph is $\lambda_2 = 3 - \sqrt{3}$. We may now evaluate the bound given by Theorem 1.5.

$$\text{gon}(G) \geq \min\left\{ \frac{18}{4}, \frac{18}{2(3-\sqrt{3})}\left[-7(3-\sqrt{3}+27) + 3\sqrt{9(3-\sqrt{3})^2 + 42(3-\sqrt{3}) + 81} \right] \right\}$$

$$= \min\{4.5, 5.04\} = 4.5.$$

By refining our approach, we obtain a better estimate. In the proof of Proposition 4.1, it is necessary to assume that $\deg(D) \leq \frac{1}{4}|V(G)|$ because otherwise the connected component U of $V(G) \smallsetminus \text{supp}(D)$, which has size at most $\frac{1}{2}|V(G)|$, may not necessarily have size at most $\frac{2}{3}|V(G) \smallsetminus \text{supp}(D)|$. In the case of the Pappus graph, however, if $\deg(D) = 5$ then no connected component of $V(G) \smallsetminus \text{supp}(D)$ can have size exactly 9. Thus, in this case we may replace the assumption that $\deg(D) \leq 4.5$ by the assumption that $\deg(D) < 6$. Theorem 1.5 then shows that $\text{gon}(G) \geq \min\{6, 5.04\} > 5$.

Since the gonality is an integer, we know that the gonality of the Pappus graph is at least 6. As discussed in the introduction, it is expected that this is the highest possible gonality of a graph of genus 10. Specifically, it follows from Brill–Noether theory and Baker's specialization lemma that, for any graph G of genus 10, there exists a refinement of G with gonality at most $\lfloor \frac{10+3}{2} \rfloor = 6$. In general, it is unknown whether the bound of $\lfloor \frac{g+3}{2} \rfloor$ holds for the original graph, without refinement. We now show that the gonality of the Pappus graph is exactly 6.

Proposition 5.1. Let G be the Pappus graph. Then $\text{gon}(G) = 6$.

Proof. We have already shown that $\text{gon}(G) \geq 6$, so it suffices to exhibit a divisor of degree 6 and rank at least 1 on G.

Let D be the sum of the white vertices on the Pappus graph illustrated in Fig. 1. Note that if we fire every vertex in the complement of the outer ring once, these chips will migrate to the outer ring. Therefore it suffices to show that we can get at least one chip to any vertex in the innermost ring. By symmetry it suffices to check that we can get a chip to a single vertex in the innermost ring. To do this, we will employ Dhar's burning algorithm on D, with v being a vertex in the innermost ring.

Now, the fire burns the edge connecting v to an adjacent vertex in the inner ring. Each of these vertices is connected to two distinct vertices in the middle ring by one edge each. Since the support of our divisor is the middle ring, the fire must stop having only burned v and the adjacent vertex, so we fire all other vertices once. This chip firing move puts a chip on v, which shows that D has rank at least 1. □

Acknowledgments

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