Tropical independence
II: The maximal rank conjecture
for quadrics

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Building on our earlier results on tropical independence and shapes of divisors in tropical linear series, we give a tropical proof of the maximal rank conjecture for quadrics. We also prove a tropical analogue of Max Noether's theorem on quadrics containing a canonically embedded curve, and state a combinatorial conjecture about tropical independence on chains of loops that implies the maximal rank conjecture for algebraic curves.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a smooth curve of genus g, and recall that a linear map between finite dimensional vector spaces has *maximal rank* if it is either injective or surjective. The kernel of the restriction map

$$\rho_m : H^0(\mathbb{P}^r, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m)|_X)$$

is the space of homogeneous polynomials of degree m that vanish on X. The conjecture that ρ_m should have maximal rank for sufficiently general embeddings of sufficiently general curves, attributed to Noether in [Arbarello and Ciliberto 1983, p. 4][1], was studied classically by Severi [1915, §10], and popularized by Harris [1982, p. 79].

Maximal rank conjecture. *Suppose X is general and $V \subset \mathcal{L}(D_X)$ is a general linear series of given degree and rank. Then the multiplication maps*

$$\mu_m : \mathrm{Sym}^m V \to \mathcal{L}(m D_X)$$

have maximal rank for all m.

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[1]Noether considered the case of space curves in [Noether 1882, §8]. See also [Castelnuovo et al. 1925, pp. 172–173] for hints toward Noether's understanding of the general problem.

Recall that the general curve of genus g has a linear series of degree d and rank r if and only if the Brill–Noether number $\rho(g, r, d) = g - (r + 1)(g - d + r)$ is nonnegative, and in this case there is an open dense subset of $\mathcal{M}_g$ over which the universal space parametrizing curves with a linear series of degree d and rank r is irreducible. Therefore, it makes sense to talk about a general linear series of degree d and rank r on a general curve of genus g when $\rho(g, r, d)$ is nonnegative.

Our main result gives a combinatorial condition on the skeleton of a curve over a nonarchimedean field to ensure the existence of such a linear series for which μ_2 has maximal rank. Let Γ be a chain of loops connected by bridges with admissible edge lengths, as defined in Section 4. See Figure 1 for a schematic illustration, and note that our conditions on the edge lengths are more restrictive than those in [Cools et al. 2012; Jensen and Payne 2014].

Theorem 1.1. *Let X be a smooth projective curve of genus g over a nonarchimedean field such that the minimal skeleton of the Berkovich analytic space X^{an} is isometric to Γ. Suppose $r \geq 3$, $\rho(g, r, d) \geq 0$, and $d < g + r$. Then there is a very ample complete linear series $\mathcal{L}(D_X)$ of degree d and rank r on X such that the multiplication map $\mu_2 : \mathrm{Sym}^2 \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$ has maximal rank.*

Such curves do exist, over fields of arbitrary characteristic, and the condition that X^{an} has skeleton Γ ensures that X is Brill–Noether–Petri general [Jensen and Payne 2014]. As explained in Section 2, to prove the maximal rank conjecture for fixed g, r, d, and m over an algebraically closed field of given characteristic, it is enough to produce a single linear series $V \subset \mathcal{L}(D_X)$ on a single Brill–Noether–Petri general curve over a field of the same characteristic for which μ_m has maximal rank. In particular, the maximal rank conjecture for $m = 2$, and arbitrary g, r, and d, follows from Theorem 1.1, so we recover the main result of [Ballico 2012a] and extend this from characteristic zero to arbitrary characteristic; whenever the general curve of genus g admits a nondegenerate embedding of degree d in $\mathbb{P}^r$ then the image of a general nondegenerate embedding is contained in the expected number of independent quadrics.

Surjectivity of μ_m for small values of m can often be used, together with uniform position arguments, to prove surjectivity for larger values of m. See, for instance, [Arbarello et al. 1985, pp. 140–141]. When Theorem 1.1 gives surjectivity of μ_2, we apply such uniform position arguments, together with some analysis of a few special cases where uniform position is not known to hold in positive characteristic, to deduce surjectivity of μ_m for all m.

Theorem 1.2. *Let X and D_X be as in Theorem 1.1, and suppose μ_2 is surjective. Then μ_m is surjective for all $m \geq 2$.*

This proves the maximal rank conjecture for all m in the range where μ_2 is surjective, recovering the main result of [Ballico and Fontanari 2010], which

determines when the general embedding of the general curve in characteristic zero is projectively normal, and extending this result to arbitrary characteristic.

Remark 1.3. The tropical methods presented here give a manifestly characteristic free approach to the maximal rank conjecture (see Conjecture 4.6). This is also the first approach to the maximal rank conjecture based on the intrinsic geometry of curves; all prior work depends in one way or another on degenerations of embedded curves in projective space. Most of these papers are written with a characteristic zero hypothesis, which is used, for instance, in uniform position arguments, but in most cases this seems to be a matter of convenience rather than necessity. Our proof of Theorem 1.2 circumvents the cases where uniform position is not known in positive characteristic, and with some care it should be possible to use similar arguments to remove the characteristic zero hypotheses from results such as those in [Ballico and Fontanari 2010; Ballico 2012a] more directly, without this tropical approach.

Remark 1.4. The maximal rank conjecture is known, for all m, when $r = 3$ [Ballico and Ellia 1987a], and in the nonspecial case $d \geq r + g$ [Ballico and Ellia 1987b]. There is a rich history of partial results on the maximal rank conjecture for $m = 2$, including some with significant applications, prior to the work of Ballico and Fontanari mentioned above. Voisin [1992, §4] proved the case of adjoint bundles of gonality pencils and deduced the surjectivity of the Wahl map for generic curves. Teixidor [2003] proved that μ_2 is injective for all linear series on the general curve when $d < g + 2$, over fields of characteristic not equal to two. Farkas proved the case where $\rho(g, r, d)$ is zero and dim $\mathrm{Sym}^2 \mathcal{L}(D_X) = \dim \mathcal{L}(2D_X)$, and used this to deduce an infinite sequence of counterexamples to the slope conjecture [Farkas 2009, Theorem 1.5]. Another special case is Noether's theorem on canonically embedded curves, discussed below. Furthermore, Larson [2012] has proved an analogue of the maximal rank conjecture for hyperplane sections of curves.

This is only a small sampling of prior work related to the maximal rank conjecture. Other notable results include the asymptotic theorem from [Ballico and Ellia 1989]. The difficulties involved in applying the same classical degeneration method to the remaining cases of the conjecture are discussed in [Ballico and Ellia 1989, §11], and the evidence for the conjecture and best known results as of a few years ago are surveyed in [Harris 2009; Ballico 2012b].

Two key tools in the proof of Theorem 1.1 are the lifting theorem from [Cartwright et al. 2015] and the notion of *tropical independence* developed in [Jensen and Payne 2014]. The lifting theorem allows us to realize any divisor D of rank r on Γ as the tropicalization of a divisor D_X of rank r on X. Our understanding of tropical linear series on Γ, together with the nonarchimedean Poincaré–Lelong formula, produces rational functions $\{f_0, \ldots, f_r\}$ in the linear series $\mathcal{L}(D_X)$ whose tropicalizations

$\{\psi_0, \dots, \psi_r\}$ are a specific well-understood collection of piecewise linear functions on Γ. We then show the tropical independence of a large subset of the piecewise linear functions $\{\psi_i + \psi_j\}_{0 \le i \le j \le r}$. Since $\psi_i + \psi_j$ is the tropicalization of $f_i \cdot f_j$, the size of this subset is a lower bound for the rank of μ_2, and this is the bound we use to prove Theorem 1.1.

There is no obvious obstruction to proving the maximal rank conjecture in full generality using this approach, although the combinatorics become more challenging as the parameters increase. We state a precise combinatorial conjecture in Section 4, which, for any given g, r, d, and m, implies the maximal rank conjecture for the same g, r, d, and m. We prove this conjecture not only for $m = 2$, but also for $md < 2g + 4$. (See Theorem 5.3.)

We also present advances in understanding multiplication maps by tropical methods on skeletons other than a chain of loops. Recall that Noether's theorem on canonically embedded curves says that $\mu_2 : \mathrm{Sym}^2 \mathcal{L}(K_X) \to \mathcal{L}(2K_X)$ is surjective whenever X is not hyperelliptic. This may be viewed as a strong form of the maximal rank conjecture for quadrics in the case where $r = g - 1$ and $d = 2g - 2$.

On the purely tropical side, we prove an analogue of Noether's theorem for trivalent, 3-edge-connected graphs.

Theorem 1.5. *Let Γ be a trivalent, 3-edge-connected metric graph. Then there is a tropically independent set of $3g - 3$ functions in $2R(K_\Gamma)$.*

Furthermore, we prove the appropriate lifting statements to leverage this tropical result into a maximal rank statement for canonical embeddings of curves with trivalent and 3-connected skeletons.

Theorem 1.6. *Let X be a smooth projective curve of genus g over a nonarchimedean field such that the minimal skeleton Γ of X^{an} is trivalent and 3-edge-connected with first Betti number g. Then there are $3g - 3$ rational functions in the image of $\mu_2 : \mathrm{Sym}^2 \mathcal{L}(K_X) \to \mathcal{L}(2K_X)$ whose tropicalizations are tropically independent. In particular, μ_2 is surjective.*

The last statement, on surjectivity of μ_2, also follows from Noether's theorem, because trivalent, 3-edge connected graphs are never hyperelliptic [Baker and Norine 2009, Lemma 5.3].

Remark 1.7. The present article is a sequel to [Jensen and Payne 2014], further developing the method of tropical independence. This is just one aspect of the tropical approach to linear series, an array of techniques for handling degenerations of linear series over a one parameter family of curves where the special fiber is not of compact type, combining discrete methods with computations on skeletons of Berkovich analytifications. Seminal works in the development of this theory include [Baker and Norine 2007; Baker 2008; Amini and Baker 2015]. Combined

with techniques from p-adic integration, this method also leads to uniform bounds on rational points for curves of fixed genus with small Mordell–Weil rank [Katz et al. 2015].

This tropical approach is in some ways analogous to the theory of limit linear series, developed by Eisenbud and Harris in the 1980s, which systematically studies the degeneration of linear series to singular curves of compact type [Eisenbud and Harris 1986]. This theory led to simplified proofs of the Brill–Noether and Gieseker–Petri theorems [Eisenbud and Harris 1983], along with many new results about the geometry of curves, linear series, and moduli [Eisenbud and Harris 1987a; 1987b; 1987c; 1989]. Tropical methods have also led to new proofs of the Brill–Noether and Gieseker–Petri theorems [Cools et al. 2012; Jensen and Payne 2014]. Some progress has been made toward building frameworks that include both classical limit linear series and also generalizations of limit linear series for curves not of compact type [Amini and Baker 2015; Osserman 2014; 2016], which are helpful for explaining connections between the tropical and limit linear series proofs of the Brill–Noether theorem. These relations are also addressed in [Jensen and Payne 2014, Remark 1.4] and [Castorena et al. 2014]. The nature of the relations between the tropical approach and more classical approaches for results involving multiplication maps, such as the Gieseker–Petri theorem and other maximal rank results, remain unclear, as do the relations between such basic and essential facts as the Riemann–Roch theorems for algebraic and tropical curves.

Note that several families of curves appearing in proofs of the Brill–Noether and Gieseker–Petri theorems are not contained in the open subset of $\overline{\mathcal{M}}_g$ for which the maximal rank condition holds. For example, the sections of $K3$ surfaces used by Lazarsfeld [1986] in his proof of the Brill–Noether and Gieseker–Petri theorems without degenerations do not satisfy the maximal rank conjecture for $m = 2$ [Voisin 1992, Theorem 0.3 and Proposition 3.2]. Furthermore, the stabilizations of the flag curves used by Eisenbud and Harris are limits of such curves [Farkas and Popa 2005, Proposition 7.2].

2. Preliminaries

Recall that a general curve X of genus g has a linear series of rank r and degree d if and only if the Brill–Noether number

$$\rho(g, r, d) = g - (r+1)(g - d + r)$$

is nonnegative, and the scheme $\mathcal{G}_d^r(X)$ parametrizing its linear series of degree d and rank r is smooth of pure dimension $\rho(g, r, d)$. This scheme is irreducible when $\rho(g, r, d)$ is positive, and monodromy acts transitively when $\rho(g, r, d) = 0$. Therefore, if $U \subset \mathcal{M}_g$ is the dense open set parametrizing such *Brill–Noether–Petri*

general curves, then $\mathcal{G}_d^r(U)$, the universal linear series of rank r and degree d over U, is smooth and irreducible of relative dimension $\rho(g, r, d)$. The *general linear series of degree d and rank r on a general curve of genus g* appearing in the statement of the maximal rank conjecture refers simply to a general point in the irreducible space $\mathcal{G}_d^r(U)$.

When X is Brill–Noether–Petri general and D_X is a basepoint-free divisor of rank at least 1, the basepoint-free pencil trick shows that its multiples mD_X are nonspecial for $m \geq 2$ (see Remark 2.1). Therefore, by standard upper semicontinuity arguments from algebraic geometry and the fact that $\mathcal{G}_d^r$ is defined over $\operatorname{Spec} \mathbb{Z}$, to prove the maximal rank conjecture for fixed g, r, d, and m, over an arbitrary algebraically closed field of given characteristic, it suffices to produce a single Brill–Noether–Petri general curve X of genus g over a field of the same characteristic with a linear series $V \subset \mathcal{L}(D_X)$ of degree d and rank r such that μ_m has maximal rank. As mentioned in the introduction, the maximal rank conjecture is known when the linear series is nonspecial. In the remaining cases, the general linear series is complete, so we can and do assume that $V = \mathcal{L}(D_X)$.

Remark 2.1. Suppose D_X is a basepoint-free special divisor of rank $r \geq 1$ on a Brill–Noether–Petri general curve X. The fact that mD_X is nonspecial for $m \geq 2$ is an application of the basepoint-free pencil trick, as follows. Choose a basepoint-free pencil $V \subset \mathcal{L}(D_X)$. Then the trick identifies $\mathcal{L}(K_X - 2D_X)$ with the kernel of the multiplication map

$$\mu : V \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(K_X).$$

The Petri condition says that this multiplication map is injective, even after replacing V by $\mathcal{L}(D_X)$. Therefore, there are no sections of K_X that vanish on $2D_X$ and hence no sections that vanish on mD_X for $m \geq 2$, which means that mD_X is nonspecial.

Remark 2.2. When $r \geq 3$ and $\rho(g, r, d) \geq 0$, the general linear series of degree d on a general curve of genus g defines an embedding in $\mathbb{P}^r$, and hence the conjecture can be rephrased in terms of a general point of the corresponding component of the appropriate Hilbert scheme. One can also consider analogues of the maximal rank conjecture for curves that are general in a given irreducible component of a given Hilbert scheme, rather than general in moduli. However, the maximal rank condition can fail when the Hilbert scheme in question does not dominate $\mathcal{M}_g$. Suppose, for example, that X is a curve of genus 8 and degree 8 in $\mathbb{P}^3$. Then $h^0(\mathcal{O}_X(2)) = 9$, and hence X is contained in a quadric surface. It follows that the kernel of μ_3 has dimension at least 4, and therefore μ_3 is not surjective. This does not contradict the maximal rank conjecture, since the general curve of genus 8 has no linear series of rank 3 and degree 8.

Proof of Theorem 1.2. We assume μ_2 is surjective. Suppose $r \geq 4$. We begin by showing that μ_3 is surjective. Note the polynomial identity

$$\binom{r+2}{2} - (2d - g + 1) = \binom{d-g}{2} - \binom{g-d+r}{2} - \rho(g, r, d).$$

(This identity reappears as Lemma 8.2, in the special case $\rho(g, r, d) = 0$.) By assumption, the left-hand side is nonnegative, as are $\rho(g, r, d)$ and $g - d + r$. It follows that $d \geq g$. By [Arbarello et al. 1985, Exercise B-6, p. 138][2], it follows that the dimension of the linear series spanned by sums of divisors in $|D_X|$ and $|2D_X|$ is at least

$$\min\{4d - 2g, 3d - g\} = 3d - g.$$

Therefore, if μ_2 is surjective then μ_3 is also surjective.

We now show, by induction on m, that μ_m is surjective for all $m > 3$. Let $V \subset \mathcal{L}(D_X)$ be a basepoint-free pencil. By the basepoint-free pencil trick, we have an exact sequence

$$0 \to \mathcal{L}((m-1)D_X) \to V \otimes \mathcal{L}(mD_X) \to \mathcal{L}((m+1)D_X).$$

Since $(m-1)D_X$ and mD_X are both nonspecial, the image of the right hand map has dimension

$$2(md - g + 1) - ((m-1)d - g + 1) = (m+1)d - g + 1,$$

hence it is surjective.

It remains to consider the cases where $r = 3$. By assumption, the divisor D_X is special, so $d < g + 3$. Furthermore, μ_2 is surjective, so $2d - g + 1 \leq 10$, and $\rho(g, r, d) \geq 0$, so $3g \leq 4d - 12$. This leaves exactly two possibilities for (g, d), namely $(4, 6)$ and $(5, 7)$. In each of these cases, $h^1(\mathcal{O}(D_X)) = 1$ and, since X is Brill–Noether general, $\mathrm{Cliff}(X) = \lfloor (g-1)/2 \rfloor$. Then $d = 2g + 1 - h^1(\mathcal{O}(D_X)) - \mathrm{Cliff}(X)$ and hence $\mathcal{O}(D_X)$ gives a projectively normal embedding, by [Green and Lazarsfeld 1986, Theorem 1]. □

Since we are trying to produce a single sufficiently general curve of each genus over a field of each characteristic, we may, for simplicity, assume that we are working over an algebraically closed field that is spherically complete with respect to a valuation that surjects onto the real numbers. Any metric graph Γ of first Betti number g appears as the skeleton of a smooth projective genus g curve X over such a field (see, for instance, [Abramovich et al. 2015]).

Recall that the skeleton is a subset of the set of valuations on the function field of X, and evaluation of these valuations, also called tropicalization, takes each

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[2]The statement of the exercise is missing a necessary hypothesis, that $\mathcal{D}$ has rank at least 3. The solution following the hint requires the uniform position lemma, which is known for $r \geq 3$ in characteristic zero [Harris 1980] and, over arbitrary fields, when $r \geq 4$ [Rathmann 1987].

rational function f on X to a piecewise linear function with integer slopes on Γ, denoted trop(f).

Our primary tool for using the skeleton of a curve and tropicalizations of rational functions to make statements about ranks of multiplication maps is the notion of tropical independence developed in [Jensen and Payne 2014].

Definition 2.3. A set of piecewise linear functions $\{\psi_0, \ldots, \psi_r\}$ on a metric graph Γ is *tropically dependent* if there are real numbers $b_0, \ldots, b_r$ such that for every point v in Γ the minimum

$$\min\{\psi_0(v) + b_0, \ldots, \psi_r(v) + b_r\}$$

occurs at least twice. If there are no such real numbers then $\{\psi_0, \ldots, \psi_r\}$ is *tropically independent*.

One key basic property of this notion is that if $\{\text{trop}(f_i)\}_i$ is tropically independent on Γ, then the corresponding set of rational functions $\{f_i\}_i$ is linearly independent in the function field of X [Jensen and Payne 2014, §3.1]. Note also that if f and g are rational functions, then $\text{trop}(f \cdot g) = \text{trop}(f) + \text{trop}(g)$.

Remark 2.4. Adding a constant to each piecewise linear function does not affect the tropical independence of a given collection. When $\{\psi_0, \ldots, \psi_r\}$ is tropically dependent, we often replace each ψ_i with $\psi_i + b_i$ and assume that the minimum of the set $\{\psi_0(v), \ldots, \psi_r(v)\}$ occurs at least twice at every point $v \in \Gamma$.

Lemma 2.5. *Let D_X be a divisor on X, and let $\{f_0, \ldots, f_r\}$ be rational functions in $\mathcal{L}(D_X)$. If there exist k multisets $I_1, \ldots, I_k \subset \{0, \ldots, r\}$, each of size m, such that $\left\{\sum_{i \in I_j} \text{trop}(f_i)\right\}_j$ is tropically independent, then the multiplication map*

$$\mu_m : \text{Sym}^m \mathcal{L}(D_X) \to \mathcal{L}(m D_X)$$

has rank at least k.

Proof. The tropicalization of $\prod_{i \in I_j} f_i$ is the corresponding sum $\sum_{i \in I_j} \text{trop}(f_i)$. If these sums $\left\{\sum_{i \in I_j} \text{trop}(f_i)\right\}_j$ are tropically independent then the rational functions $\left\{\prod_{i \in I_j} f_i\right\}_j$ are linearly independent. These k rational functions are in the image of μ_m, and the lemma follows. □

Remark 2.6. If $f_0, \ldots, f_r$ are rational functions in a linear series $\mathcal{L}(D_X)$, and $b_0, \ldots, b_r$ are real numbers, then the pointwise minimum

$$\theta = \min\{\text{trop}(f_0) + b_0, \ldots, \text{trop}(f_r) + b_r\}$$

is the tropicalization of a rational function in $\mathcal{L}(D_X)$. The rational function may be chosen of the form $a_0 f_0 + \cdots + a_r f_r$ where a_i is a sufficiently general element of the ground field such that $\text{val}(a_i) = b_i$.

We will also repeatedly use the following basic fact about the shapes of divisors associated to a pointwise minimum of functions in a tropical linear series.

Shape lemma for minima [Jensen and Payne 2014, Lemma 3.4]. *Let D be a divisor on a metric graph Γ, with $\psi_0, \ldots, \psi_r$ piecewise linear functions in $R(D)$, and let*

$$\theta = \min\{\psi_0, \ldots, \psi_r\}.$$

Let $\Gamma_j \subset \Gamma$ be the closed set where θ is equal to ψ_j. Then $\operatorname{div}(\theta) + D$ contains a point $v \in \Gamma_j$ if and only if v is in either

(1) *the divisor $\operatorname{div}(\psi_j) + D$, or*

(2) *the boundary of Γ_j.*

In [Jensen and Payne 2014], this shape lemma for minima is combined with another lemma about shapes of canonical divisors to reach the contradiction that proves the Gieseker–Petri theorem.

3. Max Noether's theorem

Here we examine functions in the canonical and 2-canonical linear series using trivalent and 3-edge-connected graphs. This section is not logically necessary for the proof of Theorem 1.1, and can be safely skipped by a reader who is interested only in the proof of the maximal rank conjecture for quadrics. Nevertheless, the two are not unrelated and we include this section because, as explained in the introduction, Noether's theorem is a strong form of one case of the maximal rank conjecture for quadrics. Also, the arguments presented here illustrate the potential for applying our methods to the study of linear series and multiplication maps using skeletons other than a chain of loops, which may be important for future work.

Our arguments in this section depend on a careful analysis of the loci where piecewise linear functions attain their minima. Recall that, for a divisor D on a metric graph Γ, the tropical linear series $R(D)$ is the set of piecewise linear functions with integer slope ψ on Γ such that $\operatorname{div}(\psi) + D$ is effective. The tropical linear series $R(D)$ is a tropical module, which means that it is closed under scalar addition and pointwise minimum [Haase et al. 2012, Lemma 4]. For $v \in \Gamma$, we write $\deg_v(D)$ for the coefficient of v in the divisor D, and for a piecewise linear function ψ, we write

$$\Gamma_\psi = \{v \in \Gamma \mid \psi(v) = \min_{w \in \Gamma} \psi(w)\}$$

for the subgraph on which ψ attains its global minimum.

Lemma 3.1. *Let D be a divisor on Γ with $\psi \in R(D)$. Then, for any point $v \in \Gamma_\psi$,*

$$\operatorname{outdeg}_{\Gamma_\psi}(v) \leq \deg_v(D),$$

where $\mathrm{outdeg}_{\Gamma_\psi}(v)$ *denotes the number of tangent vectors based at v that are not contained in Γ_ψ.*

Proof. Since ψ obtains its minimum value at v, all of the outgoing slopes of ψ at v are nonnegative, and those along edges that are not in Γ_ψ are strictly positive. Since all of these slopes are integers and $\mathrm{div}(\psi) + D$ is effective, it follows that $\mathrm{outdeg}_{\Gamma_\psi}(v)$ is at most $\mathrm{deg}_v(D)$. ◻

Recall that the canonical divisor K_Γ is given by

$$\mathrm{deg}_v(K_\Gamma) = \mathrm{val}(v) - 2,$$

where $\mathrm{val}(v)$ is the valence (or number of outgoing edges) of v in Γ. The following lemma restricts the loci on which functions in $R(K_\Gamma)$ attain their minimum.

Lemma 3.2. *Let ψ be a piecewise linear function in $R(K_\Gamma)$. Then the subgraph Γ_ψ on which ψ attains its minimum is a union of edges in Γ and has no leaves.*

Proof. By Lemma 3.1, the outdegree $\mathrm{outdeg}_v(\Gamma_\psi)$ is at most $\mathrm{deg}(v) - 2$ at each point $v \in \Gamma_\psi$. It follows that any edge which contains a point of Γ_ψ in its interior is entirely contained in ψ, and the number of edges in Γ_ψ containing any vertex v is at least two, so Γ_ψ has no leaves. ◻

As a first application, we show that every loop in Γ is the locus where some function in $R(K_\Gamma)$ attains its minimum, and that this function lifts to a canonical section on any totally degenerate curve whose skeleton is Γ. Here, a *loop* is an embedded circle in Γ or, equivalently, a connected subgraph in which every point has valence 2.

Proposition 3.3. *Let Γ be a metric graph and let $\Gamma' \subset \Gamma$ be a loop. Then there is a function $\psi \in R(K_\Gamma)$ such that the subgraph Γ_ψ on which ψ attains its minimum is exactly Γ'.*

Furthermore, if X is a smooth projective curve over a nonarchimedean field such that the minimal skeleton of the Berkovich analytic space X^{an} is isometric to Γ, and K_X is a canonical divisor that tropicalizes to K_Γ, then ψ can be chosen to be $\mathrm{trop}(f)$ for some $f \in \mathcal{L}(K_X)$.

Proof. Let g be the first Betti number of Γ. Choose points $v_1, \ldots, v_{g-1}$ of valence 2 in $\Gamma \smallsetminus \Gamma'$ such that $\Gamma \smallsetminus \{v_1, \ldots, v_{g-1}\}$ is connected. Since K_Γ has rank $g - 1$, there is a divisor $D \sim K_\Gamma$ such that $D - v_1 - \cdots - v_{g-1}$ is effective. Let ψ be a piecewise linear function such that $K_\Gamma + \mathrm{div}(\psi) = D$.

By Lemma 3.1, the subgraph $\Gamma_\psi \subset \Gamma$ where ψ attains its minimum is a union of edges of Γ and has no leaves. Since $\mathrm{ord}_{v_i}(\psi)$ is positive for $1 \le i \le g-1$, it follows that Γ_ψ does not contain any of the points $v_1, \ldots, v_{g-1}$. Being a subgraph of $\Gamma \smallsetminus \{v_1, \ldots, v_{g-1}\}$, the first Betti number of Γ_ψ is at most 1. On the other hand,

every point has valence at least two in Γ_ψ. It follows that Γ_ψ is a loop, and hence must be the unique loop Γ' contained in $\Gamma \smallsetminus \{v_1, \ldots, v_{g-1}\}$.

We now prove the last part of the proposition. Let $p_1, \ldots, p_{g-1}$ be points in X specializing to $v_1, \ldots, v_{g-1}$, respectively. Since K_X has rank $g - 1$, there is a rational function $f \in \mathcal{L}(K_X)$ such that $\mathrm{div}(f) + K_\Gamma - p_1 - \cdots - p_{g-1}$ is effective. From this we see that $\mathrm{div}(\mathrm{trop}(f)) + K_\Gamma - v_1 - \cdots - v_{g-1}$ is effective, and the proposition follows. □

Our next lemma controls the locus where a piecewise linear function in $R(2K_\Gamma)$ attains its minimum, when Γ is trivalent.

Lemma 3.4. *Suppose Γ is trivalent and let ψ be a piecewise linear function in $R(2K_\Gamma)$. Then Γ_ψ is a union of edges in Γ.*

Proof. If $v \in \Gamma_\psi$ lies in the interior of an edge of Γ then, by Lemma 3.1, we have $\mathrm{outdeg}_{\Gamma_\psi}(v) = 0$, so Γ_ψ contains the entire edge. On the other hand, if $v \in \Gamma_\psi$ is a trivalent vertex of Γ then Lemma 3.1 says that $\mathrm{outdeg}_{\Gamma_\psi}(v) \leq 2$. It follows that Γ_ψ contains at least one of the three edges adjacent to v. □

We conclude this section by applying this lemma and the preceding proposition together with Menger's theorem to prove Theorem 1.5, the analogue of Noether's theorem for trivalent 3-connected graphs.

Remark 3.5. A similar application of Menger's theorem is used to prove an analogue of Noether's theorem for graph curves in [Bayer and Eisenbud 1991, §4].

Proof of Theorems 1.5 and 1.6. Assume Γ is trivalent and 3-edge-connected. Let $e \subset \Gamma$ be an edge with endpoints v and w. Since Γ is 3-edge-connected, Menger's theorem says that there are two distinct paths from v to w that do not share an edge and do not pass through e. Equivalently, there are two loops Γ_1 and Γ_2 in Γ such that $\Gamma_1 \cap \Gamma_2 = e$. By Proposition 3.3 there are functions ψ_1 and ψ_2 in $R(K_\Gamma)$ such that $\Gamma_{\psi_i} = \Gamma_i$. We write $\psi_e = \psi_1 + \psi_2$, which is a piecewise linear function in $R(2K_\Gamma)$. Note that $\Gamma_{\psi_e} = e$. Furthermore, again by Proposition 3.3, if X is a curve with skeleton Γ and K_X is a canonical divisor tropicalizing to K_Γ, then we can choose f_1 and f_2 in $\mathcal{L}(K_X)$ such that $\psi_i = \mathrm{trop}(f_i)$, and hence $\psi_e = \mathrm{trop}(f_1 \cdot f_2)$ is the tropicalization of a function in the image of $\mu_2 : \mathrm{Sym}^2(\mathcal{L}(K_X)) \to \mathcal{L}(2K_X)$.

We claim that the set of $3g - 3$ functions $\{\psi_e\}_e$ is tropically independent. Suppose not. Then there are constants b_e such that $\min_e\{\psi_e + b_e\}$ occurs twice at every point of Γ. Let

$$\theta = \min_e\{\psi_e + b_e\},$$

which is a piecewise linear function in $R(2K_\Gamma)$. By Lemma 3.4, the function θ achieves its minimum along an edge, and hence there must be two functions in the set $\{\psi_e + b_e\}_e$ that achieve their minima along this edge. However, by construction, the

functions $\psi_e + b_e$ achieve their minima along distinct edges, which is a contradiction. We conclude that $\{\psi_e\}_e$ is tropically independent, as claimed. □

4. Special divisors on a chain of loops

For the remainder of the paper, we focus our attention on a chain of loops with bridges Γ, as pictured in Figure 1. Here, we briefly recall the classification of special divisors on Γ from [Cools et al. 2012], along with the characterization of vertex avoiding classes and their basic properties.

The graph Γ has $2g + 2$ vertices, one on the left-hand side of each bridge, which we label $w_0, \ldots, w_g$, and one on the right-hand side of each bridge, which we label $v_1, \ldots, v_{g+1}$. There are two edges connecting the vertices v_k and w_k, the top and bottom edges of the k-th loop, whose lengths are denoted ℓ_k and m_k, respectively, as shown schematically in Figure 1. For $1 \le k \le g+1$ there is a bridge connecting w_k and v_{k+1}, which we refer to as the k-th bridge β_k, of length n_k. Throughout, we assume that Γ has admissible edge lengths in the following sense, which is stronger than the genericity conditions in [Cools et al. 2012; Jensen and Payne 2014].

Definition 4.1. The graph Γ has *admissible edge lengths* if

$$4gm_k < \ell_k \ll \min\{n_{k-1}, n_k\} \text{ for all } k,$$

and there are no nontrivial linear relations $c_1m_1 + \cdots + c_gm_g = 0$ with integer coefficients of absolute value at most $g + 1$.

Remark 4.2. The inequality $4gm_k < \ell_k$ is required to ensure that the shapes of the functions ψ_i and ψ_{ij} are as described in Sections 6 and 7. Both inequalities are used in the proof of Lemma 6.2, and the required upper bound on ℓ_k depends on the size of the multisets. For multisets of size m, we assume $2m\ell_k < \min\{n_{k-1}, n_k\}$. In particular, for Theorem 1.1, the inequality $4\ell_k < \min\{n_{k-1}, n_k\}$ would suffice. The condition on integer linear relations is used in the proof of Proposition 7.6.

The special divisor classes on a chain of loops, i.e., the classes of effective divisors D such that $r(D) > \deg(D) - g$, are explicitly classified in [Cools et al. 2012]. Every effective divisor on Γ is equivalent to an effective w_0-reduced divisor, which has d_0 chips at the vertex w_0, together with at most one chip on every loop.
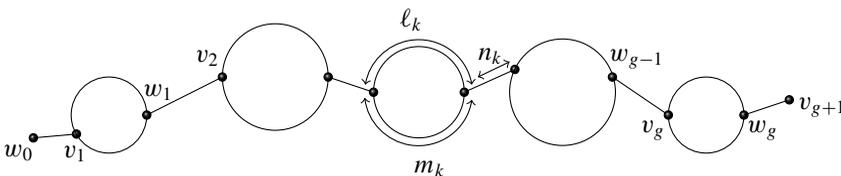

Figure 1. The graph Γ.

We may therefore associate to each equivalence class the data $(d_0, x_1, x_2, \ldots x_g)$, where $x_i \in \mathbb{R}/(\ell_i + m_i)\mathbb{Z}$ is the distance from v_i to the chip on the i-th loop in the counterclockwise direction, if such a chip exists, and $x_i = 0$ otherwise. The associated lingering lattice path in $\mathbb{Z}^r$, whose coordinates we number from 0 to $r - 1$, is a sequence of points $p_0, \ldots, p_g$ starting at

$$p_0 = (d_0, d_0 - 1, \ldots, d_0 - r + 1).$$

We write $p_i(j)$ for the j-th coordinate of p_i. With this notation, the i-th step in the lingering lattice path is given by

$$p_i - p_{i-1} = \begin{cases} (-1, -1, \ldots, -1) & \text{if } x_i = 0, \\ e_j & \text{if } x_i = (p_{i-1}(j) + 1)m_i \mod \ell_i + m_i \\ & \text{and both } p_{i-1} \text{ and } p_{i-1} + e_j \text{ are in } \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases}$$

where $e_0, \ldots e_{r-1}$ are the basis vectors in $\mathbb{Z}^r$ and $\mathcal{C}$ is the set of lattice points in the open Weyl chamber

$$\mathcal{C} = \{y \in \mathbb{Z}^r \mid y_0 > \cdots > y_{r-1} > 0\}.$$

By [Cools et al. 2012, Theorem 4.6], a divisor D on Γ has rank at least r if and only if the associated lingering lattice path lies entirely in the open Weyl chamber $\mathcal{C}$.

Remark 4.3. Although the lingering lattice path associated to D, as defined above and in [Cools et al. 2012], is a sequence of points in $\mathbb{Z}^r$ with coordinates labeled from 0 to $r - 1$, we find it convenient to consider this $\mathbb{Z}^r$ as being embedded in $\mathbb{Z}^{r+1}$, with coordinates labeled from 0 to r, as the sublattice in which the last coordinate is zero. In other words, we set $p_j(r) = 0$ for all j.

The steps in the direction 0 are referred to as *lingering steps*, and the number of lingering steps cannot exceed the Brill–Noether number $\rho(g, r, d)$. In the case where $\rho(g, r, d) = 0$, such lattice paths are in bijection with rectangular tableaux of size $(r + 1) \times (g - d + r)$. This bijection is given as follows. We label the columns of the tableau from 0 to r and place i in the j-th column when the i-th step is in the direction e_j, and we place i in the last column when the i-th step is in the direction $(-1, \ldots, -1)$.

An open dense subset of the special divisor classes of degree d and rank r on Γ are *vertex avoiding*, in the sense of [Cartwright et al. 2015, Definition 2.3], which means that

- the associated lingering lattice path has exactly $\rho(g, r, d)$ lingering steps,

- for any i, $x_i \neq m_i \mod (\ell_i + m_i)$, and

- for any i and j, $x_i \neq (p_{i-1}(j))m_i \mod (\ell_i + m_i)$.

Vertex avoiding classes come with a useful collection of canonical representatives. If D is a divisor of rank r on Γ whose class is vertex avoiding, then there is a unique effective divisor $D_i \sim D$ such that $\deg_{w_0}(D_i) = i$ and $\deg_{v_{g+1}}(D_i) = r - i$. Equivalently, D_i is the unique divisor equivalent to D such that $D_i - i w_0 - (r-i) v_{g+1}$ is effective. Furthermore,

- the divisor D_i has no points on any of the bridges,

- for $i < r$, the divisor D_i fails to have a point on the j-th loop if and only if the j-th step of the associated lingering lattice path is in the direction e_i,

- the divisor D_r fails to have a point on the j-th loop if and only if the j-th step of the associated lingering lattice path is in the direction $(-1, \ldots, -1)$.

Notation 4.4. Throughout, we let X be a smooth projective curve of genus g whose analytification has skeleton Γ. For the remainder of the paper, we let D be a w_0-reduced divisor on Γ of degree d and rank r whose class is vertex avoiding, D_X a lift of D to X, and ψ_i a piecewise linear function on Γ such that $D + \operatorname{div}(\psi_i) = D_i$. By a *lift* of D to X, we mean that D_X is a divisor of degree d and rank r on X whose tropicalization is D.

Note that ψ_i is uniquely determined up to an additive constant, and for $i < r$ the slope of ψ_i along the bridge β_j is $p_j(i)$. In this context, being w_0-reduced means that $D = D_r$, so the function ψ_r is constant. In particular, the functions $\psi_0, \ldots, \psi_r$ have distinct slopes along bridges, so $\{\psi_0, \ldots, \psi_r\}$ is tropically independent. Recall that, for convenience, we set $p_j(r) = 0$ for all j.

Proposition 4.5. *There is a rational function $f_i \in \mathcal{L}(D_X)$ such that $\operatorname{trop}(f_i) = \psi_i$.*

Proof. The proof is identical to the proof of [Jensen and Payne 2014, Proposition 6.5], which is the special case where $\rho(g, r, d) = 0$. □

When $\rho(g, r, d) = 0$, all divisor classes of degree d and rank r are vertex avoiding. Note that, since $\{\psi_0, \ldots, \psi_r\}$ is tropically independent of size $r + 1$, the set of rational functions $\{f_0, \ldots, f_r\}$ is a basis for $\mathcal{L}(D_X)$.

For a multiset $I \subset \{0, \ldots, r\}$ of size m, let $D_I = \sum_{i \in I} D_i$ and let ψ_I be a piecewise linear function such that $mD + \operatorname{div} \psi_I = D_I$. By construction, the function ψ_I is in $R(mD)$ and agrees with $\sum_{i \in I} \psi_i$ up to an additive constant.

Conjecture 4.6. *Suppose $r \geq 3$, $\rho(g, r, d) \geq 0$, and $d < g + r$. Then there is a divisor D of rank r and degree d whose class is vertex avoiding on a chain of loops Γ with generic edge lengths, and a tropically independent subset $\mathcal{A} \subset \{\psi_I \mid \#I = m\}$ of size*

$$\#\mathcal{A} = \min\left\{\binom{r+m}{m}, \, md - g + 1\right\}.$$

The conjecture is trivial for $r = 0$ and easy for $r = 1$, since the functions $k\psi_0$ have distinct nonzero slopes on every bridge and hence $\{0, \psi_0, \ldots, m\psi_0\}$ is tropically independent. Yet another easy case is $m = 1$, since $\{\psi_0, \ldots \psi_r\}$ is tropically independent. In the remainder of the paper we prove the conjecture for $m = 2$ and for $md < 2g + 4$.

Proposition 4.7. *For any fixed g, r, d, and m, the maximal rank conjecture follows from Conjecture 4.6.*

Proof. Choose a smooth projective curve X over a nonarchimedean field whose skeleton is Γ. Then X is Brill–Noether–Petri general [Jensen and Payne 2014] and D lifts to a divisor D_X of degree d and rank r on X [Cartwright et al. 2015]. We may assume $r \geq 1$, and it follows that mD_X is nonspecial for $m \geq 2$ by Remark 2.1. By Lemma 2.5, the rank of μ_m is at least as large as any set $\mathcal{A}$ such that $\{\psi_I \mid I \in \mathcal{A}\}$ is tropically independent. Therefore, Conjecture 4.6 implies that $\mu_m : \operatorname{Sym}^m \mathcal{L}(D_X) \to \mathcal{L}(mD_X)$ has maximal rank and, as discussed in Section 2, the maximal rank conjecture for g, r, d, and m follows. $\square$

5. Two points on each loop

Let D be a w_0-reduced vertex avoiding divisor on Γ. We continue to use the notation established in the previous section and recall, in particular, that since D is vertex avoiding there are piecewise linear functions ψ_i, unique up to an additive constant, such that $D + \operatorname{div} \psi_i$ is the unique effective divisor equivalent to D such that $\deg_{w_0}(D_i) = i$ and $\deg_{v_{g+1}}(D_i) = r - i$. Furthermore, since D is w_0-reduced, $D = D_r$ and ψ_r is a constant function.

We now show that any nontrivial tropical dependence among the piecewise linear functions $\psi_I = \sum_{i \in I} \psi_i$, for multisets I of size m, gives rise to a divisor equivalent to mD with degree at least 2 at w_0, degree at least 2 at v_{g+1}, and degree at least 2 on each loop. As a consequence, we deduce Theorem 5.3, which confirms Conjecture 4.6 and the maximal rank conjecture for $md < 2g + 4$.

Lemma 5.1. *Let I and J be distinct multisets of size m. Then, for each loop γ° in Γ, the restrictions $D_I|_{\gamma^\circ}$ and $D_J|_{\gamma^\circ}$ are distinct.*

Proof. Suppose γ° is the j-th loop. Let q_i be the point on γ° whose distance from v_j in the counterclockwise direction is $x_j - p_{j-1}(i)m_j$. Then the degree of q_i in D_I is equal to the multiplicity of i in the multiset I, unless the j-th step of the lingering lattice path is in the direction e_i, in which case the degree of q_i in D_I is zero. It follows that the multiset I can be recovered from the restriction $D_I|_{\gamma^\circ}$. $\square$

Let θ be the piecewise linear function

$$\theta = \min_I\{\psi_I\},$$

$$\gamma_0 \qquad \gamma_1 \qquad\qquad \gamma_2 \qquad\qquad\qquad \gamma_g \qquad\qquad \gamma_{g+1}$$
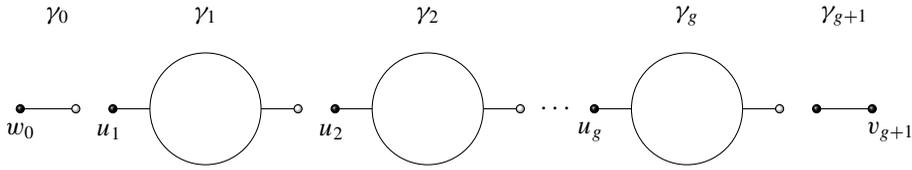
Figure 2. Decomposition of the graph Γ into locally closed pieces $\{\gamma_k\}$.

which is in $R(mD)$, and let Δ be the corresponding effective divisor

$$\Delta = mD + \operatorname{div}\theta.$$

By Lemma 5.1, no two functions ψ_I can agree on an entire loop, so if the minimum occurs everywhere at least twice on a loop, then there are at least three functions ψ_I that achieve the minimum at some point of the loop. We will study θ and Δ by systematically using observations like this one, examining behavior on each piece of Γ and controlling which functions ψ_I can achieve the minimum at some point in each loop.

Recall that, for $0 \leq k \leq g$, the k-th bridge β_k connects w_k to v_{k+1}. Let u_k be the midpoint of β_{k-1}. We then decompose Γ into $g + 2$ locally closed subgraphs $\gamma_0, \ldots, \gamma_{g+1}$, as follows. The subgraph γ_0 is the half-open interval $[w_0, u_1)$. For $1 \leq i \leq g$, the subgraph γ_i, which includes the i-th loop of Γ, is the union of the two half-open intervals $[u_i, u_{i+1})$, which contain the top and bottom edges of the i-th loop, respectively. Finally, the subgraph γ_{g+1} is the closed interval $[u_{g+1}, v_{g+1}]$. We further write γ_i° for the i-th embedded loop in Γ, which is a closed subset of γ_i, for $1 \leq i \leq g$. The decomposition

$$\Gamma = \gamma_0 \sqcup \cdots \sqcup \gamma_{g+1}$$

is illustrated by Figure 2.

Proposition 5.2. *Suppose the minimum of $\{\psi_I(v)\}_I$ occurs at least twice at every point v in Γ. Then $\deg_{w_0}(\Delta)$, $\deg_{v_{g+1}}(\Delta)$, and $\deg(\Delta|_{\gamma_i^\circ})$ are all at least 2.*

Proof. Note that exactly one function ψ_I has slope mr on the first bridge; this is the function corresponding to the multiset $I = \{0, \ldots, 0\}$. Similarly, the only multiset that gives slope $mr - 1$ is $\{1, 0, \ldots, 0\}$. Therefore, if the minimum occurs twice along the first bridge, then the outgoing slope of θ at w_0 is at most $mr - 2$, and hence $\deg_{w_0}(\Delta) \geq 2$, as required. Similarly, we have $\deg_{v_{g+1}}(\Delta) \geq 2$.

It remains to show that $\deg(\Delta|_{\gamma_i^\circ}) \geq 2$ for $1 \leq i \leq g$. Choose a point $v \in \gamma_i^\circ$. By assumption, there are at least two distinct multisets I and I' such that both ψ_I and $\psi_{I'}$ obtain the minimum on some closed interval containing v. By Lemma 5.1, the functions ψ_I and $\psi_{I'}$ do not agree on all of γ_i°, so there is another point $v' \in \gamma_i^\circ$ where at least one of these two functions does not obtain the minimum. Without

loss of generality, assume that ψ_I does not obtain the minimum at v'. Then ψ_I obtains the minimum on a proper closed subset of γ_i°, and since γ_i° is a loop, this set has outdegree at least two. By the shape lemma for minima (see Section 2), it follows that $(\mathrm{div}(\theta) + mD)|_{\gamma_i^\circ}$ has degree at least two.   □

As an immediate application of this proposition, we prove Conjecture 4.6 for $md < 2g + 4$.

Theorem 5.3. *If $md < 2g + 4$ then $\{\psi_I \mid \#I = m\}$ is tropically independent.*

Proof. Suppose that $\{\psi_I\}_I$ is tropically dependent. After adding a constant to each ψ_I, we may assume that the minimum $\theta(v) = \min_I \psi_I(v)$ occurs at least twice at every point v in Γ. By Proposition 5.2, the restriction of $\Delta = mD + \mathrm{div}(\theta)$ to each of the $g + 2$ locally closed subgraphs $\gamma_k \subset \Gamma$ has degree at least two. Therefore the degree of Δ is at least $2g + 4$, and the theorem follows.   □

In particular, the maximal rank conjecture holds for $md < 2g + 4$. This partially generalizes the case where $m = 2$ and $d < g + 2$, proved by Teixidor i Bigas [2003]. Note, however, that [Teixidor 2003] proves that the maximal rank condition holds for *all* divisors of degree less than $g + 2$, whereas Theorem 5.3 implies this statement only for a general divisor.

6. Permissible functions

In the preceding section, we introduced a decomposition of Γ as the disjoint union of locally closed subgraphs $\gamma_0, \ldots, \gamma_{g+1}$ and proved that if $\theta(v) = \min_I \psi_I(v)$ occurs at least twice at every point v in γ_i then the degree of $\Delta = mD + \mathrm{div}(\theta)$ restricted to γ_i is at least 2. These degrees of restrictions $\Delta|_{\gamma_i}$ appear repeatedly throughout the rest of the paper, so we fix

$$\delta_i = \deg(\Delta|_{\gamma_i}).$$

By Proposition 5.2, we have $\delta_i \geq 2$ for all i.

We now discuss how the nonnegative integer vector $\delta = (\delta_0, \ldots, \delta_{g+1})$ restricts the multisets I such that ψ_I can achieve the minimum on the k-th loop of Γ.

For $a \leq b$, let $\Gamma_{[a,b]}$ be the locally closed, connected subgraph

$$\Gamma_{[a,b]} = \gamma_a \sqcup \cdots \sqcup \gamma_b.$$

Note that the degrees of divisors in a tropical linear series restricted to such subgraphs are governed by the slopes of the associated piecewise linear functions, as follows.

Suppose $\Gamma' \subset \Gamma$ is a closed connected subgraph and ψ is a piecewise linear function with integer slopes on Γ. Then $\mathrm{div}(\psi|_{\Gamma'})$ has degree zero and the multiplicity of each boundary point $v \in \partial\Gamma'$ is the sum of the incoming slopes at v, along the edges in Γ'. Now $\mathrm{div}(\psi)|_{\Gamma'}$ agrees with $\mathrm{div}(\psi|_{\Gamma'})$ except at the boundary points

and a simple computation at the boundary points of the locally closed subgraph γ_k, for $1 \leq k \leq g$ shows that

$$\deg(\operatorname{div}(\psi)|_{\gamma_k}) = \sigma_k(\psi) - \sigma_{k+1}(\psi),$$

where $\sigma_k(\psi)$ is the incoming slope of ψ from the left at u_k. Similarly,

$$\deg(\operatorname{div}(\psi)|_{\Gamma_{[0,k]}}) = -\sigma_{k+1}(\psi).$$

Our indexing conventions for lingering lattice paths are chosen for consistency with [Cools et al. 2012], and with this notation we have

$$\sigma_k(\psi_i) = p_{k-1}(i).$$

These slopes, and the conditions on the edge lengths on Γ, lead to restrictions on the multisets I such that ψ_I achieves the minimum at some point in the k-th loop γ_k°.

Definition 6.1. Let $I \subset \{0, \ldots, r\}$ be a multiset of size m. We say that ψ_I is δ-*permissible* on γ_k° if

$$\deg(D_I|_{\Gamma_{\leq k-1}}) \geq \delta_0 + \cdots + \delta_{k-1}$$

and

$$\deg(D_I|_{\Gamma_{\leq k}}) \leq \delta_0 + \cdots + \delta_k.$$

We say that ψ_I is δ-*permissible* on $\Gamma_{[a,b]}$ if there is some $k \in [a, b]$ such that ψ_I is δ-permissible on γ_k°.

Lemma 6.2. *If $\psi_I(v) = \theta(v)$ for some $v \in \gamma_k^\circ$ then ψ_I is δ-permissible on γ_k°.*

Proof. Recall that the edge lengths of Γ are assumed to be admissible, in the sense of Definition 4.1.

Suppose $\psi_I(v) = \theta(v)$ for some point v in γ_k°. We claim that the slope of ψ_I along the bridge β_{k-1} to the left of the loop is at most the incoming slope of θ from the left at u_{k-1}. Indeed, if the slope of ψ_I is strictly greater than that of θ then, since $\psi_I(u_{k-1}) \geq \theta(u_{k-1})$ and the slope of θ can only decrease when going from u_{k-1} to v_k, the difference $\psi_I(v_k) - \theta(v_k)$ will be at least the distance from u_{k-1} to v_k, which is $n_{k-1}/2$.

The slopes of ψ_I and θ along the bottom edge are between 0 and mg, and the slopes along the top edge are between 0 and m. Since $\ell_k > 4gm_k$ by assumption, it follows that $|\psi_I - \theta|$ changes by at most $m\ell_k$ between v_k and any other point in γ_k°. Assuming $2m\ell_k < n_{k-1}$, this proves the claim.

Note that the incoming slopes of ψ_I and θ from the left at u_k are

$$\deg(mD|_{\Gamma_{[0,k-1]}}) - \deg(D_I|_{\Gamma_{[0,k-1]}}), \quad \text{and} \quad \deg(mD|_{\Gamma_{[0,k-1]}}) - \delta_0 - \cdots - \delta_{k-1},$$

respectively. Therefore, the claim implies that $\deg(D_I|_{\Gamma_{[0,k-1]}}) \geq \delta_0 + \cdots + \delta_{k-1}$.

A similar argument using slopes along the bridge β_k to the right of γ_k° and assuming $2m\ell_k < n_k$ shows that $\deg(D_I|_{\Gamma_{\leq k}}) \leq \delta_0 + \cdots + \delta_k$, and the lemma follows. $\square$

Our general strategy for proving Conjecture 4.6 in the case $m = 2$ is to choose the set $\mathcal{A}$ carefully, assume that the minimum occurs everywhere at least twice, and then bound $\delta_0 + \ldots + \delta_i$ inductively, moving from left to right across the graph. By induction, we assume a lower bound on $\delta_0 + \cdots + \delta_i$. Then, for a carefully chosen $j > i$, we consider $\delta_0 + \cdots + \delta_j$. If this is too small, then Lemma 6.2 severely restricts which functions ψ_I can achieve the minimum on loops in $\Gamma_{[i+1,j]}$, making it impossible for the minimum to occur everywhere at least twice unless the bottom edge lengths $m_{i+1}, \ldots, m_j$ satisfy a nontrivial linear relation with small integer coefficients. We deduce a lower bound on $\delta_0 + \cdots + \delta_j$ and continue until we can show $\delta_0 + \cdots + \delta_{g+1} > 2d$, a contradiction. We give a first taste of this type of argument in Lemma 6.4 and Example 6.6. Example 6.7 illustrates how similar techniques may be applied to understand the kernel of μ_m when it is not injective. A more general (and more technical) version of the key step in this argument, using the assumption that a small number of functions ψ_I achieve the minimum everywhere at least twice on $\Gamma_{[i+1,j]}$ to produce a nontrivial linear relation with small integer coefficients, appears in the proof of Proposition 7.6.

Notation 6.3. For the remainder, we fix $m = 2$, and I and I_j will always denote multisets of size 2 in $\{0, \ldots, r\}$, which we identify with pairs (i, j) with $0 \leq i \leq j \leq r$. We write ψ_{ij} for the piecewise linear function $\psi_i + \psi_j$ corresponding to the multiset $I = \{i, j\}$.

Lemma 6.4. *Suppose that $\delta_k = 2$ and $\theta(v) = \min\{\psi_{I_1}(v), \psi_{I_2}(v), \psi_{I_3}(v)\}$ occurs at least twice at every point in γ_k°. Then, $\theta|_{\gamma_k^\circ} = \psi_{I_j}|_{\gamma_k^\circ}$, for some $1 \leq j \leq 3$.*

Proof. By Lemma 5.1, no two of the functions may obtain the minimum on all of γ_k°. After renumbering, we may assume that ψ_{I_3} obtains the minimum on some but not all of the loop. Let v be a boundary point of the locus where ψ_{I_3} obtains the minimum. Since there are only three functions that obtain the minimum, one must obtain the minimum in a neighborhood of v. After renumbering we may assume that this is ψ_{I_1}. We claim that θ is equal to ψ_{I_1} on the whole loop. If not, then by the shape lemma for minima, $D + \operatorname{div} \theta$ would contain the two points in the boundary of the locus where ψ_{I_1} obtains the minimum, in addition to v, contradicting the assumption that δ_k, the degree of $D + \operatorname{div} \theta$ on γ_k, is 2. $\square$

Remark 6.5. It follows from Lemma 6.4 that, under the given hypotheses, the tropical dependence on the k-th loop is essentially unique, in the sense that if b_1, b_2, and b_3 are real numbers such that

$$\theta(v) = \min\{\psi_{I_1}(v) + b_1, \psi_{I_2}(v) + b_2, \psi_{I_3}(v) + b_3\}$$
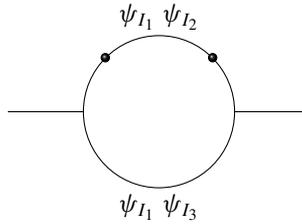
Figure 3. An illustration of the regions where different functions
obtain the minimum in the situation of Lemma 6.4.

occurs at least twice at every point on the k-th loop, then $b_1 = b_2 = b_3$. Furthermore,
since each ψ_{I_j} has constant slope along the bottom edge of γ_k and no two agree on
the entire top edge, there must be one pair that agrees on the full bottom edge and
part of the top edge and another pair that agrees on part of the top edge, as shown
in Figure 3. Note that the divisor $D + \mathrm{div}\,\theta$ consists of two points on the top edge
and one (but not both) of these points may lie at one of the end points, v_k or w_k.

Before we turn to the proof of the main theorem, we illustrate the techniques
involved with a pair of examples.

Example 6.6. Suppose $g = 10$, and let D be the divisor of rank 4 and degree 12
corresponding to the tableau pictured in Figure 4. We note that this special case of
the maximal rank conjecture for $m = 2$ is used to produce a counterexample to the
slope conjecture in [Farkas and Popa 2005].

Assume that the minimum $\theta = \min\{\psi_I\}$ occurs at least twice at every point of Γ.
By Proposition 5.2, the divisor $\Delta = \mathrm{div}(\theta) + 2D$ has degree at least two on each of
the 12 locally closed subgraphs γ_k. Since $\deg(2D) = 24$, the degree of Δ on each
of these subgraphs must be exactly 2. In other words, $\delta = (2, \ldots, 2)$.

In the lingering lattice path for D, we have

$$p_4 = (6, 5, 2, 1, 0), \quad p_5 = (6, 5, 3, 1, 0), \quad p_6 = (6, 5, 4, 1, 0).$$

The δ-permissible functions ψ_{ij} on $\Gamma_{[5,6]}$ are those such that either

$$p_4(i) + p_4(j) \le 6 \quad \text{and} \quad p_5(i) + p_5(j) \ge 6, \quad \text{or}$$
$$p_5(i) + p_5(j) \le 6 \quad \text{and} \quad p_6(i) + p_6(j) \ge 6.$$

1	3	5	7	9
2	4	6	8	10

Figure 4. The tableau corresponding to the divisor D in Example 6.6.
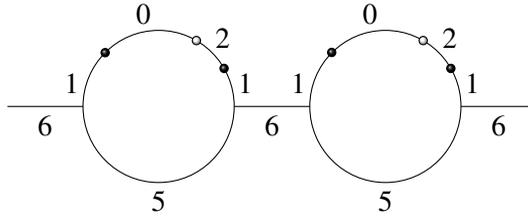
Figure 5. The shape of ψ_{04} and ψ_{13} on $\Gamma_{[5,6]}$.
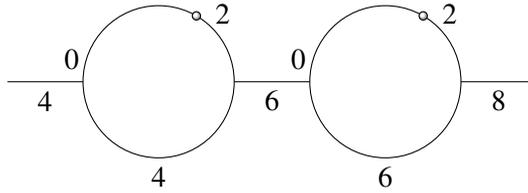
Figure 6. The shape of ψ_{22} on $\Gamma_{[5,6]}$.

There are only 3 such pairs: $(0, 4)$, $(1, 3)$, and $(2, 2)$. These functions are illustrated in Figures 5 and 6. Each domain of linearity is labeled with the slope of ψ_{ij} from left to right. The point of D is marked with a white circle. There are no points on $\Gamma_{[5,6]}$ where the function ψ_{22} has positive order of vanishing. The points where ψ_{04} has positive order of vanishing lie on either side of the white circle, and similarly for ψ_{13}. These points are marked with black circles. For the function ψ_{04}, these black circles occur closer to the white circle than they do for ψ_{13}. The region on which these two functions disagree on γ_5 is the disjoint union of two line segments, each of length m_5. Similarly, the region on which these two functions disagree on γ_6 is again the disjoint union of two line segments, each of length m_6.

By Lemma 6.4, in order for the minimum to occur at least twice at every point of $\Gamma_{[5,6]}$, on each of the two loops there must be a single function ψ_{ij} that obtains the minimum at every point. Because the slope of ψ_{22} along the bottom edge differs from that of ψ_{13} and ψ_{04}, we see that, on either loop, the function that obtains the minimum at every point cannot be ψ_{22}. Similarly, because on each loop the function ψ_{04} has slope 1 on a region where both of the other functions have slope 2, we see that the function that obtains the minimum at every point cannot be ψ_{04}. We therefore see that ψ_{13} obtains the minimum at every point of $\Gamma_{[5,6]}$, and ψ_{04} must achieve the minimum on both bottom edges. Let q_5 and q_6 be the points of D on γ_5 and γ_6, respectively, as shown in Figure 7.

The regions of the graph are labeled by the pairs of functions ψ_{ij}, $\psi_{i'j'}$ that obtain the minimum on that region. For each i, the change in value $\psi_i(q_6) - \psi_i(q_5)$ may be expressed as a function of the entries in the lattice path and the lengths of the edges in Γ. Specifically, as we travel from q_5 to q_6, the slopes of ψ_{22} and ψ_{13}
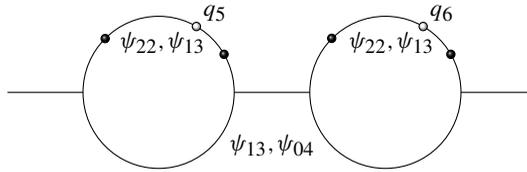
Figure 7. Regions of $\Gamma_{[5,6]}$ on which the functions obtain the minimum.

differ by 1 on an interval of length m_5 along the top edge of γ_5, and again on an interval of length m_6 along the top edge of γ_6. This computation shows that

$$(\psi_{22}(q_5) - \psi_{13}(q_5)) - (\psi_{22}(q_6) - \psi_{13}(q_6)) = m_5 - m_6.$$

Since ψ_{13} and ψ_{22} agree at q_5 and q_6, it follows that m_5 must equal m_6, contradicting the hypothesis that Γ has admissible edge lengths in the sense of Definition 4.1.

We conclude that the minimum cannot occur everywhere at least twice, so $\{\psi_I\}_I$ is tropically independent. Therefore, for any curve X with skeleton Γ and any lift of D to a divisor D_X of rank 4, the map

$$\mu_2 : \mathrm{Sym}^2 \, \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$$

is injective.

We now consider an example illustrating our approach via tropical independence when μ_2 is not injective. Recall that the canonical divisor on a nonhyperelliptic curve of genus 4 gives an embedding in $\mathbb{P}^3$ whose image is contained in a unique quadric. This is the special case of the maximal rank conjecture where g, r, d, and m are 4, 3, 6, and 2, respectively.

Example 6.7. Suppose $g = 4$ and $m = 2$. Note that the class of the canonical divisor $D = K_\Gamma$ is vertex avoiding of rank 3. Since Γ is the skeleton of a curve whose canonical embedding lies on a quadric, the functions ψ_I are tropically dependent, and we may assume $\min_I \psi_I(v)$ occurs at least twice at every point $v \in \Gamma$.

Let $\theta(v) = \min_I \psi_I(v)$, and let $\Delta = 2K_\Gamma + \mathrm{div}\,\theta$. By Proposition 5.2, the degree δ_k of Δ on γ_k is at least 2 for $k = 0, \ldots, 5$. Since $\deg(\Delta) = 12$, it follows that $\delta = (2, \ldots, 2)$.

The lingering lattice path associated to K_Γ is given by

$$p_0 = (3, 2, 1, 0), \quad p_1 = (4, 2, 1, 0), \quad p_2 = (4, 3, 1, 0),$$
$$p_3 = (4, 3, 2, 0), \quad p_4 = (3, 2, 1, 0).$$

Since $\delta_0 = \delta_1 = 2$, the δ-permissible functions ψ_{ij} on γ_1 are those such that

$$p_0(i) + p_0(j) \leq 4 \quad \text{and} \quad p_1(i) + p_1(j) \geq 4.$$
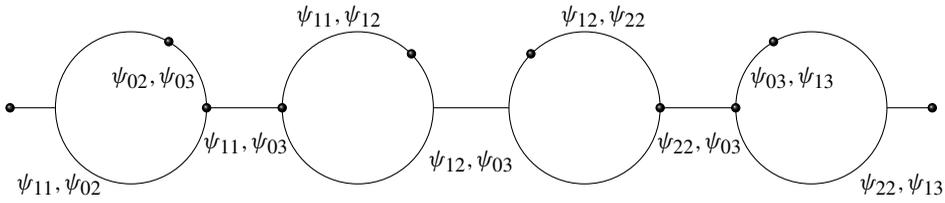
Figure 8. The unique tropical dependence for the canonical linear system when $g = 4$ and $m = 2$.

There are only three such pairs: $(0, 2)$, $(1, 1)$, and $(0, 3)$. In a similar way, we see that there are precisely three δ-permissible functions on each loop γ_k. By Lemma 6.4 and Remark 6.5, the tropical dependence among the three functions that achieve the minimum on each loop is essentially unique. Figure 8 illustrates the combinatorial structure of this dependence.

Since this dependence among the functions that realize the minimum at some point in Γ is essentially unique, omitting any one of the six functions that appear leaves a tropically independent set of size 9. Therefore, the map

$$\mu_2 : \mathrm{Sym}^2 \, \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$$

has rank at least 9. Since $\mathcal{L}(2D_X)$ has dimension 9, it follows that μ_2 is surjective.

7. Shapes of functions, excess degree, and linear relations among edge lengths

In this section, and in Section 8, below, we assume that $\rho(g, r, d) = 0$. All of the essential difficulties appear already in this special case. The case $\rho(g, r, d) > 0$ is treated in Section 9 through a minor variation on these arguments.

We now proceed with the more delicate and precise combinatorial arguments required to prove Theorem 1.1. With g, r, and d fixed, and assuming $d - g \leq r$, we must produce a divisor D of degree d and rank r on Γ, together with a set

$$\mathcal{A} \subset \{(i, j) \mid 0 \leq i \leq j \leq r\}$$

of size

$$\#\mathcal{A} = \min\left\{ \binom{r+2}{2}, 2d - g + 1 \right\},$$

such that the corresponding collection of rational functions

$$\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$$

is tropically independent.

Notation 7.1. The quantity $g - d + r$ appears repeatedly throughout, so we set

$$s = g - d + r,$$

which simplifies various formulas. The condition that $\rho(g, r, d) = 0$ means that $g = (r+1)s$.

We now specify the divisor D that we will use to prove Conjecture 4.6 for $m = 2$ when $\rho(g, r, d) = 0$. The set $\mathcal{A}$ is described in Section 8.

Notation 7.2. For the remainder of this section and Section 8, let D be the divisor of degree d and rank r on Γ corresponding to the standard tableau with $r + 1$ columns and s rows in which the numbers $1, \ldots, s$ appear in the leftmost column; $s + 1, \ldots, 2s$ appear in the next column, and so on. We number the columns from zero to r, so the ℓ-th column contains the numbers $\ell s + 1, \ldots, (\ell + 1)s$. The specific case $g = 10$, $r = 4$, $d = 12$ is illustrated in Figure 4 from Example 6.6.

Remark 7.3. Our choice of divisor is particularly convenient for the inductive step in the proof of Theorem 1.1, in which we divide the graph Γ into the $r + 1$ regions $\Gamma_{[\ell s+1, (\ell+1)s]}$, for $0 \leq \ell \leq r$, and move from left to right across the graph, one region at a time, studying the consequences of the existence of a tropical dependence. Since the numbers $\ell s + 1, \ldots, (\ell + 1)s$ all appear in the ℓ-th column, the slopes of the functions ψ_i, for $i \neq \ell$, are the same along all bridges and bottom edges, respectively, in the subgraph $\Gamma_{[\ell s+1, (\ell+1)s]}$. Only the slopes of ψ_ℓ are changing in this region.

Here we describe the *shape* of the function ψ_i, by which we mean the combinatorial configuration of regions on the loops and bridges on which ψ_i has constant slope, as well as the slopes from left to right on each region. These data determine (and are determined by) the combinatorial configurations of the points in $D_i = D + \mathrm{div}(\psi_i)$.

Fix $0 \leq \ell \leq r$. Suppose $\ell s + 1 \leq k \leq (\ell + 1)s$, so γ_k° is a loop in the subgraph $\Gamma_{[\ell s+1, (\ell+1)s]}$. Recall from Section 4 and Section 6 that if $\ell \neq r$ then D contains one point on the top edge of γ_k°, at distance

$$p_{k-1}(\ell) = \sigma_k(\ell)$$

in the counterclockwise direction from w_k, where $\sigma_k(\ell)$ is the slope of ψ_ℓ along the bridge β_k.

Case 1: The shape of ψ_i, for $i < \ell$. If $i < \ell$ then $D_i = D + \mathrm{div}\, \psi_i$ contains one point on the top edge of γ_k°, at distance $(r + s - i - 1 - \sigma_k(\ell)) \cdot m_k$ from v_k, the left endpoint of γ_k°. This is illustrated schematically in Figure 9. The point of D_i on the top edge of γ_k° is marked with a black circle, and the point of D is marked with a white circle. (In the case where $\ell = r$, the white circle is located at the right-hand vertex w_k.) Each region of constant slope is labeled with the slope of ψ_i from left to right. The slope of ψ_i from left to right along each bridge adjacent to γ_k° is $r + s - i$, and the slope along the bottom edge is $r + s - i - 1$.
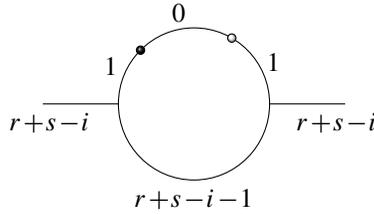
Figure 9. The shape of ψ_i on γ_k, for $i < \ell$.
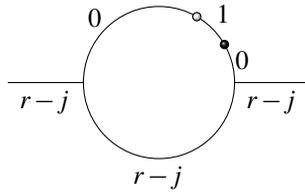

Figure 10. The shape of ψ_j on γ_k, for $j > \ell$.

Case 2: The shape of ψ_j, for $j > \ell$. If $j > \ell$ then $D_j = D + \operatorname{div} \psi_j$ contains one point on the top edge of γ_k°, at distance $(\sigma_k(\ell) - r + j)$ from w_k, as shown in Figure 10. The slope of ψ_j along the bottom edge and both adjacent bridges is $r - j$.

Case 3: The shape of ψ_ℓ. The divisor D_ℓ has no points on γ_k°, as shown in Figure 11. Note that this is the only case in which the slope is not the same along the two bridges adjacent to γ_k°.

We use the shapes of the functions ψ_i to control the set of pairs (i, j) such that ψ_{ij} is δ-permissible on certain loops, as follows. Suppose $\{\psi_{ij}\}$ is tropically dependent, so there are constants b_{ij} such that $\min\{\psi_{ij}(v) + b_{ij}\}$ occurs at least twice at every point $v \in \Gamma$. Replacing ψ_{ij} with $\psi_{ij} + b_{ij}$, we may assume $\min\{\psi_{ij}(v)\}$ occurs at least twice at every point. Let

$$\theta = \min_{ij}\{\psi_{ij}\}, \quad \Delta = 2D + \operatorname{div}(\theta), \quad \text{and} \quad \delta_i = \deg(\Delta|_{\gamma_i}).$$
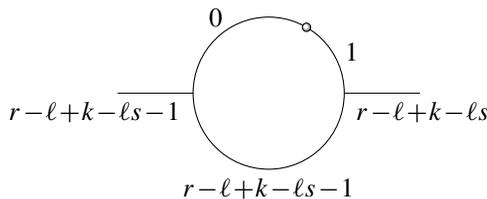
Figure 11. The shape of ψ_ℓ on γ_k.

By Proposition 5.2, each δ_i is at least 2, and some may be strictly greater. We keep track of the *excess degree function*

$$e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

It contains exactly the same information as δ, but in a form that is somewhat more convenient for our inductive arguments in Section 8. Note that $e(k)$ is positive and nondecreasing as a function of k.

In the induction step, we study the δ-permissible functions ψ_{ij} on subgraphs

$$\Gamma_{[a(\ell),b(\ell)]} \subseteq \Gamma_{[\ell s+1,(\ell+1)s]},$$

where $a(\ell)$ and $b(\ell)$ are given by

$$a(\ell) = \begin{cases} \ell s + 1 & \text{for } \ell \leq \lceil r/2 \rceil, \\ \ell(s+1) - \lceil r/2 \rceil + 1 & \text{for } \ell > \lceil r/2 \rceil, \end{cases}$$

and

$$b(\ell) = \begin{cases} \ell(s+1) - \lceil r/2 \rceil + s & \text{for } \ell \leq \lceil r/2 \rceil, \\ (\ell+1)s & \text{for } \ell > \lceil r/2 \rceil. \end{cases}$$

Note that the subgraph $\Gamma_{[a(\ell),b(\ell)]}$ is only well-defined if $a(\ell) \leq b(\ell)$. This is the case when ℓ is in the range

$$\max\left\{0, \left\lceil \frac{r}{2} \right\rceil - s\right\} \leq \ell < \min\left\{r, \left\lceil \frac{r}{2} \right\rceil + s\right\}.$$

We focus on the situation where $e(\ell s)$ and $e((\ell+1)s)$ are both equal to $\ell - s + \lceil r/2 \rceil$, which is the critical case for our argument.

Lemma 7.4. *Suppose*

$$e(\ell s) = e((\ell+1)s) = \ell - s + \left\lceil \frac{r}{2} \right\rceil,$$

for some $0 \leq \ell \leq r$. If ψ_{ij} is δ-permissible on $\Gamma_{[a(\ell),b(\ell)]}$, then either

(1) $i < \ell < j$, *and* $i + j = \ell + \lceil r/2 \rceil$, *or*

(2) $i = j = \ell$.

Proof. Note that, by our choice of D,

$$\deg(D_i|_{\Gamma_{[0,k]}}) = \begin{cases} i+k & \text{for } i > \ell, \\ i + \ell s & \text{for } i = \ell, \\ i+k-s & \text{for } i < \ell. \end{cases}$$

Also, since $e(k)$ is nondecreasing,

$$e(k) = \ell - s + \left\lceil \frac{r}{2} \right\rceil$$

for all k in $[\ell s, (\ell+1)s]$, and in particular for k in $[a(\ell), b(\ell)]$.

We now prove the lemma in the case where $\ell \leq \lceil r/2 \rceil$. The proof in the case where $\ell > \lceil r/2 \rceil$ is similar. Suppose $i \geq \ell$, $j > \ell$, and $k \in [a(\ell), b(\ell)]$. Then

$$\deg(D_{ij}|_{\Gamma_{[0,k]}}) \geq i + j + k + \ell s$$
$$> 2\ell + k + \ell s.$$

On the other hand, we have

$$\deg(\Delta|_{\Gamma_{[0,k]}}) = 2k + \ell - s + \left\lceil \frac{r}{2} \right\rceil$$
$$\leq 2\ell + k + \ell s,$$

where the inequality is given by using $k \leq b(\ell)$ and $b(\ell) = \ell(s+1) - \lceil r/2 \rceil + s$. Combining the two displayed inequalities shows that $\deg(D_{ij}|_{\Gamma_{[0,k]}})$ is greater than $\deg(\Delta|_{\Gamma_{[0,k]}})$, and hence ψ_{ij} is not δ-permissible on γ_k°.

A similar argument shows that, if $i < \ell$, $j \leq \ell$, and $k \in [a(\ell), b(\ell)]$, then

$$\deg(D_{ij}|_{\Gamma_{[0,k-1]}}) \leq i + j + \ell s + k - 1 - s$$
$$< 2\ell + \ell s + k - 1 - s.$$

On the other hand, since $\ell \leq \lceil r/2 \rceil$ by hypothesis, and $k \geq \ell s + 1$, we have

$$\deg(\Delta|_{\Gamma_{[0,k-1]}}) = 2k - 2 + \ell - s + \left\lceil \frac{r}{2} \right\rceil$$
$$\geq 2k - 2 + 2\ell - s$$
$$\geq 2\ell + \ell s + k - 1 - s.$$

In this case, we conclude that $\deg(D_{ij}|_{\Gamma_{[0,k-1]}})$ is less than $\deg(\Delta|_{\Gamma_{[0,k-1]}})$, and hence ψ_{ij} is not δ-permissible on γ_k°.

We have shown that, if ψ_{ij} is δ-permissible on $\Gamma_{[a(\ell),b(\ell)]}$, then either $i = j = \ell$ or $i < \ell < j$. It remains to show that if $i < \ell < j$ then $i + j = \ell + \lceil r/2 \rceil$. Suppose $i < \ell < j$. Then

$$\deg(D_{ij}|_{\Gamma_{[0,k]}}) = i + j + 2k - s.$$

If ψ_{ij} is δ-permissible on γ_k°, then this is less than or equal to $\deg(\Delta|_{\Gamma_{[0,k]}})$, which is $2k + \ell - s + \lceil r/2 \rceil$. It follows that $i + j \leq \ell + \lceil r/2 \rceil$. Similarly, if ψ_{ij} is δ-permissible on γ_k° then $\deg(D_{ij}|_{\Gamma_{[0,k-1]}}) \geq \deg(\Delta|_{\Gamma_{[0,k-1]}})$, and it follows that $i + j \geq \ell + \lceil r/2 \rceil$. Therefore, $i + j = \ell + \lceil r/2 \rceil$, as required. $\square$

We continue with the notation from Lemma 7.4, with ℓ a fixed integer between 0 and r, and $[a(\ell), b(\ell)]$ the corresponding subinterval of $[\ell s + 1, (\ell+1)s]$, when this is nonempty. We also fix a subset $\mathcal{A} \subset \{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i \leq j \leq r\}$ and suppose that $\theta(v) = \min\{\psi_{ij}(v) \mid (i, j) \in \mathcal{A}\}$ occurs at least twice at every point. Equivalently, in the set up of Lemma 7.4, we assume that $b_{ij} \gg 0$ for (i, j) not in $\mathcal{A}$.

Remark 7.5. The following proposition is the key technical step in our inductive argument, and may be seen as a generalization of the following two simple facts. In order for the minimum to be achieved everywhere at least twice, on a chain of zero loops (i.e., a single edge), at least two functions are required, and on a chain of one loop, at least three functions are required (Lemma 5.1).

Proposition 7.6. *Suppose* $e(a(\ell)) = e(b(\ell)) = \ell - s + \lceil r/2 \rceil$. *Then there are at least* $b(\ell) - a(\ell) + 3$ *functions* ψ_{ij}, *with* $(i, j) \in \mathcal{A}$, *that are* δ-*permissible on* $\Gamma_{[a(\ell),b(\ell)]}$.

Proof. Let $a = a(\ell)$ and $b = b(\ell)$. Assume that there are at most $b - a + 2$ functions that are δ-permissible on $\Gamma_{[a,b]}$. We will show that the bottom edge lengths m_k for $k \in [a, b]$ satisfy a linear relation with small integer coefficients, contradicting the admissibility of the edge lengths of Γ (Definition 4.1).

Since $e(k)$ is nondecreasing, the assumption that $e(a) = e(b)$ implies that Δ contains exactly two points on each loop in $\Gamma_{[a,b]}$, and no points in the interiors of the bridges. It follows that θ has constant slope on each of these bridges. As discussed in Section 6, the slope at the midpoint of β_k is determined by the degree of $\operatorname{div} \theta$ on $\Gamma_{[0,k]}$, and one computes that this slope is $2r - e(k)$. Therefore, the slope of θ is constant on every bridge in $\Gamma_{[a,b]}$, and equal to

$$\sigma := 2r - \ell + s - \left\lceil \frac{r}{2} \right\rceil.$$

We begin by describing the shapes of the δ-permissible functions ψ_{ij} on $\Gamma_{[a,b]}$. By Lemma 7.4, the δ-permissible functions ψ_{ij} satisfy either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. Suppose $i < \ell < j$. In this case, the shape of ψ_{ij} on the subgraph γ_k is as pictured in Figure 12.

Note that the shape of ψ_{ij} is determined by the shapes of ψ_i and ψ_j, as shown in Figures 9 and 10, respectively. The point q_k of D on γ_k° is marked with a white circle. The fact that the slopes of ψ_{ij} along the bridges are equal to σ is due to the condition $i + j = \ell + \lceil r/2 \rceil$.

We now describe the shape of the function $\psi_{\ell\ell}$. Note that the slope of $\psi_{\ell\ell}$ along the bridge $\beta_{\ell s}$ is $2r - 2\ell$, and the slope increases by two along each successive bridge
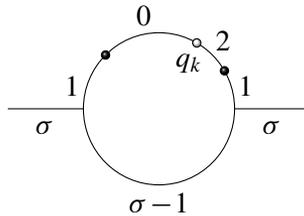

Figure 12. The shape of ψ_{ij} on γ_k, for $i < \ell < j$.
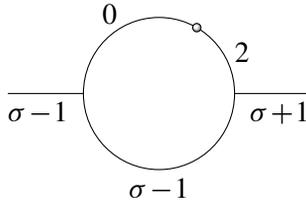
**Figure 13.** The shape of $\psi_{\ell\ell}$ on γ_h, when σ is odd.
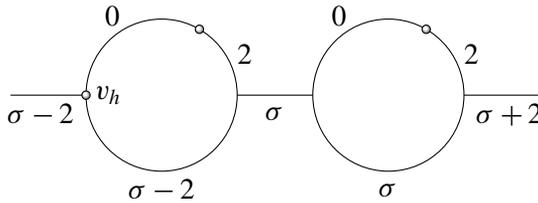
Figure 14. The shape of $\psi_{\ell\ell}$ on $\Gamma_{[h,h+1]}$, when σ is even.

β_k, for $k \in [\ell s + 1, (\ell + 1)s]$. It follows that if σ is odd then $\psi_{\ell\ell}$ is δ-permissible on only one loop γ_h°, as shown in Figure 13.

If σ is even, then $\psi_{\ell\ell}$ is δ-permissible on two consecutive loops γ_h and γ_{h+1}, as shown in Figure 14. We choose h so that γ_h° is the leftmost loop on which $\psi_{\ell\ell}$ is δ-permissible. We will use v_h as a point of reference for the remaining calculations in the proof of the proposition. (The values of ψ_{ij} and $\psi_{\ell\ell}$ at every point in $\Gamma_{[a,b]}$ are determined by the shape computations above and the values at v_h.)

For the permissible functions ψ_{ij} with $i < \ell < j$, the slopes along the bridges and bottom edges are independent of (i, j). One then computes directly that

$$\psi_{ij}(q_k) - \psi_{i'j'}(q_k) = \psi_{ij}(v_h) - \psi_{i'j'}(v_h) + (i' - i)m_k. \tag{1}$$

Similarly, one computes

$$\psi_{ij}(q_h) - \psi_{\ell\ell}(q_h) = \psi_{ij}(v_h) - \psi_{\ell\ell}(v_h) + (r + s - i - 1 - \sigma_h(\ell))m_h, \tag{2}$$

and, when σ is even,

$$\psi_{ij}(q_{h+1}) - \psi_{\ell\ell}(q_{h+1}) = \psi_{ij}(v_h) - \psi_{\ell\ell}(v_h) + (r + s - i - 1 - \sigma_{h+1}(\ell))m_{h+1} + m_h. \tag{3}$$

We use these expressions, together with the tropical dependence hypothesis (our standing assumption that $\min\{\psi_{ij}(v) \mid (i, j) \in \mathcal{A}\}$ occurs at least twice at every point) to produce a linear relation with small integer coefficients among the bottom edge lengths $m_a, \ldots, m_b$, as follows.

Let $\mathcal{A}' \subset \mathcal{A}$ be the set of pairs (i, j) such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We now build a graph whose vertices are the pairs $(i, j) \in \mathcal{A}'$, and whose edges record the pairs that achieve the minimum at one of the points q_k or at the point v_h. Say

$\psi_{i_0 j_0}$ and $\psi_{i_0' j_0'}$ achieve the minimum at v_h. Then we add an edge e_0 from (i_0, j_0) to (i_0', j_0') in the graph. Associated to this edge, we have the equation

$$\psi_{i_0 j_0}(v_h) - \psi_{i_0' j_0'}(v_h) = 0. \qquad (E_0)$$

Next, for $a \le k \le b$, say $\psi_{i_k j_k}$ and $\psi_{i_k' j_k'}$ achieve the minimum at q_k. Then we add an edge e_k from (i_k, j_k) to (i_k', j_k') and, associated to this edge, we have the equation

$$\psi_{i_k j_k}(v_h) - \psi_{i_k' j_k'}(v_h) = \alpha_k m_k + \lambda_k m_{k-1}, \qquad (E_k)$$

where α_k and λ_k are small positive integers determined by formula (1), (2), or (3), according to whether one of the pairs is equal to (ℓ, ℓ) and, if so, whether k is equal to h or $h + 1$. Note that, in every case, α_k is nonzero.

The graph now has $b - a + 2$ edges and, by hypothesis, it has at most $b - a + 2$ vertices. Therefore, it must contain a loop. If the edges $e_{k_1}, \ldots e_{k_t}$ form a loop then we can take a linear combination of the equations $E_{k_1}, \ldots, E_{k_t}$, each with coefficient ± 1, so that the left-hand sides add up to zero. This gives a linear relation among the bottom edge lengths $m_{k_1}, \ldots, m_{k_t}$, with small integer coefficients. Furthermore, if $k_t > k_j$ for all $j \ne t$, then m_{k_t} appears with nonzero coefficient in E_{k_t}, and does not appear in E_{k_j} for $j < t$, so this linear relation is nontrivial. Finally, note that $|\alpha_k| \le r + s \le g$ for all k, and λ_k is either 0 or 1, so the coefficient of each edge length m_k is an integer of absolute value less than or equal to $g + 1$. This contradicts the hypothesis that Γ has admissible edge lengths, and proves the proposition. $\qquad \square$

8. Proof of Theorem 1.1 for $\rho(g, r, d) = 0$

In this section, we continue with the assumption from Section 7 that $\rho(g, r, d) = 0$ and prove Conjecture 4.6 for $m = 2$, applying an inductive argument that relies on Lemma 7.4 and Proposition 7.6 in the inductive step. The case $\rho(g, r, d) > 0$ is handled by a minor variation on these arguments in Section 9.

Remark 8.1. Wang [2015] has recently shown that the maximal rank conjecture for $m = 2$ follows from the special case where $\rho(g, r, d) = 0$. Our proof of Theorem 1.1 does not rely on this reduction. We prove Conjecture 4.6 for $m = 2$ and arbitrary $\rho(g, r, d)$.

We separate the argument into two cases, according to whether or not μ_2 is injective. The following identity is used to characterize the range of cases in which μ_2 is injective and to count the set $\mathcal{A}$ that we define in the remaining cases.

Lemma 8.2. *Suppose $s \le r$. Then*

$$\binom{r+2}{2} - \binom{r-s}{2} + \binom{s}{2} = 2d - g + 1.$$

Proof. The lemma follows from a series of algebraic manipulations. Expand the left-hand side as a polynomial in r and s, collect terms, and apply the identities $s = g - d + r$ and $g = (r+1)s$.   □

It follows immediately from Lemma 8.2 that

$$\binom{r+2}{2} \leq 2d - g + 1 \quad \text{if and only if} \quad r - s \leq s.$$

In particular, the maximal rank conjecture predicts that μ_2 is injective for a general linear series on a general curve exactly when $r \leq 2s$. We now proceed with the proof that $\{\psi_{ij} \mid 0 \leq i \leq j \leq r\}$ is tropically independent in the injective case.

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r \leq 2s$. We must show that the set of functions $\{\psi_{ij} \mid 0 \leq i \leq j \leq r\}$ is tropically independent. Suppose not. Then there are constants b_{ij} such that the minimum

$$\theta(v) = \min_{ij}\{\psi_{ij}(v) + b_{ij}\}$$

occurs at least twice at every point $v \in \Gamma$. We continue with the notation from Section 7, setting

$$\Delta = 2D + \operatorname{div}\theta, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

As described above, our strategy is to bound the excess degree function $e(k) = \delta_0 + \cdots + \delta_k - 2k$ inductively, moving from left to right across the graph.

More precisely, we claim that

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil \quad \text{for } \ell \leq r. \tag{4}$$

We prove this claim by induction on ℓ, using Lemma 7.4 and Proposition 7.6. To see that the theorem follows from the claim, note that the claim implies that

$$\deg(\Delta) \geq 2g + r - s + \left\lceil \frac{r}{2} \right\rceil + 2.$$

Since $d = g + r - s$, this gives $\deg(\Delta) \geq 2d + s - \lfloor r/2 \rfloor + 2$, a contradiction, since $r \leq 2s$. It remains to prove the claim (4).

The claim is clear for $\ell = 1$, since $e(k) \geq \delta_0 \geq 2$ for all k and $\lceil r/2 \rceil \leq s$, by assumption. We proceed by induction on ℓ. Assume that $\ell < 2r - s - 1 - \lceil r/2 \rceil$ and

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil.$$

We must show that $e((\ell + 1)s) \geq \ell - s + \lceil r/2 \rceil + 1$. If $e(\ell s) > \ell - s + \lceil r/2 \rceil$ then there is nothing to prove, since e is nondecreasing. It remains to rule out the possibility that $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$.

Suppose that $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$. Fix $a = a(\ell)$ and $b = b(\ell)$ as in Section 7. By Lemma 7.4, if ψ_{ij} is δ-admissible on $\Gamma_{[a,b]}$ then either $i = j = \ell$

or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We consider two cases and use Proposition 7.6 to reach a contradiction in each case.

Case 1: If $1 \le \ell \le \lceil r/2 \rceil$ then there are exactly $\ell + 1$ possibilities for i, and j is uniquely determined by i. In this case $b - a = \ell + s - \lceil r/2 \rceil - 1$. Since $r \le 2s$, this implies that the number of δ-permissible functions is at most $b - a + 2$, which contradicts Proposition 7.6, and the claim follows.

Case 2: If $\lceil r/2 \rceil < \ell < r$ then there are exactly $r - \ell + 1$ possibilities for j, and i is uniquely determined by j. In this case, $b - a = s - \ell + \lceil r/2 \rceil + 1$, which is at least $r - \ell - 1$, since $r \le 2s$. Therefore, the number of δ-permissible functions on $\Gamma_{[a,b]}$ is at most $b - a + 2$, which contradicts Proposition 7.6, and the claim follows.

This completes the proof of Conjecture 4.6 (and hence Theorem 1.1) in the case where $m = 2$, $\rho(g, r, d) = 0$, and $r \le 2s$. $\qquad\qquad\qquad\qquad\qquad\square$

Our proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r > 2s$ is similar to the argument above, bounding the excess degree function $e(\ell s)$ by induction on ℓ, with Lemma 7.4 and Proposition 7.6 playing a key role in the inductive step. The one essential new feature is that we must specify the subset $\mathcal{A}$. The description of this set, and the argument that follows, depend in a minor way on the parity of r, so we fix

$$\epsilon(r) = \begin{cases} 0 & \text{if } r \text{ is even,} \\ 1 & \text{if } r \text{ is odd.} \end{cases}$$

Let $\mathcal{A}$ be the subset of the integer points in the triangle $0 \le i \le j \le r$ that are not in any of the following three regions:

(1) the half-open triangle where $j \ge i + 2$ and $i + j < r - 2s + \epsilon(r)$,

(2) the half-open triangle where $j \ge i + 2$ and $i + j > r + 2s$,

(3) the closed chevron where $r - s + \epsilon(r) \le i + j \le r + s$, and either

$$i \le \tfrac{1}{2}(r - 2s - 2 + \epsilon(r)) \quad \text{or} \quad j \ge \tfrac{1}{2}(r + 2s + 2).$$

Figure 15 illustrates the case $g = 36$, $r = 11$, $d = 44$, and $s = 3$. The points of $\mathcal{A}$ are marked with black dots, the three regions are shaded gray, and the omitted integer points are marked with white circles.

Remark 8.3. There are many possible choices for $\mathcal{A}$, as one can see even in relatively simple cases, such as Example 6.7. We present one particular choice that works uniformly for all g, r, and d. (In the situation of Example 6.7, the two half-open triangles are empty, and the closed chevron contains a single integer point, namely $(0, 3)$.) The essential property for the purposes of our inductive argument is the number of points (i, j) in $\mathcal{A}$, with $i \ne j$, on each diagonal line $i + j = k$, for $0 \le k \le 2r$. The argument presented here works essentially verbatim for any other subset of the integer points in the triangle with this property, and can be adapted

to work somewhat more generally. We have made no effort to characterize those subsets that are tropically independent, since producing a single such subset is sufficient for the proof of Theorem 1.1.

Remark 8.4. Our choice of $\mathcal{A}$, suitably interpreted, works even in the injective case. When $r - s \leq s$, the shaded regions are empty, since the half space $i \leq \frac{1}{2}(r - 2s - 2 + \epsilon(r))$ lies entirely to the left of the triangle $0 \leq i \leq j \leq r$, and the half space $j \geq \frac{1}{2}(r + 2s + 2)$ lies above it.

We now verify that the set $\mathcal{A}$ described above has the correct size.

Lemma 8.5. *The size of $\mathcal{A}$ is* $\quad \#\mathcal{A} = 2d - g + 1$.

Proof. As shown in Figure 15, moving the lower left triangle vertically and the upper right triangle horizontally by integer translations, we can assemble the shaded regions to form a closed triangle minus a half-open triangle. These translations show that the two half-open triangles plus the convex hull of the chevron shape are scissors congruent to a triangle of side length $r - s - 2$ that contains $\binom{r-s}{2}$ integer points. The difference between the chevron shape and its convex hull is
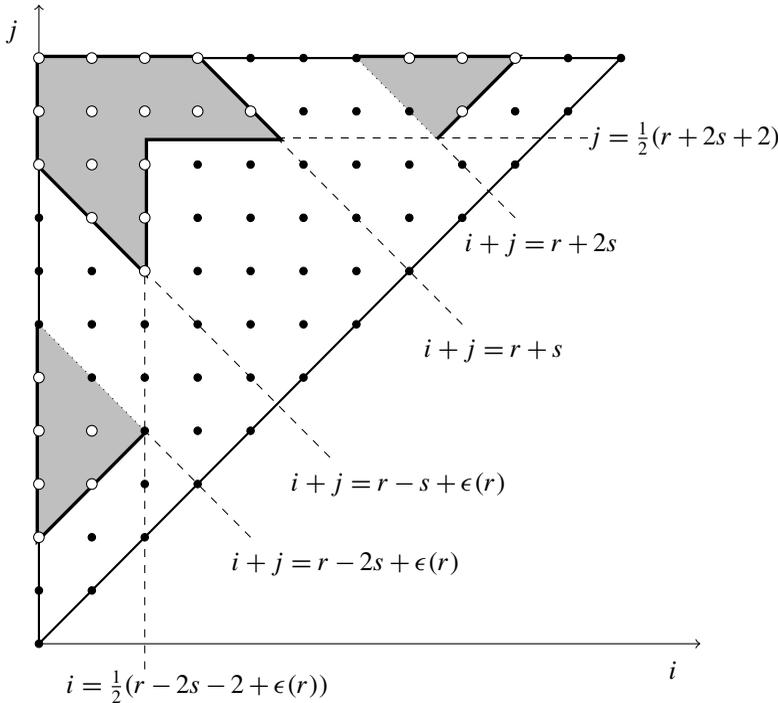
Figure 15. Points in the set $\mathcal{A}$ are marked by black dots. The integer points in the triangle $0 \leq i \leq j \leq r$ that are omitted from $\mathcal{A}$ are marked with white circles.

a half-open triangle that contains $\binom{s}{2}$ integer points. Therefore, the shaded region contains exactly $\binom{r-s}{2} - \binom{s}{2}$ lattice points, and the proposition then follows from the identity in Lemma 8.2. $\qquad\square$

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r > 2s$. We will show that

$$\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$$

is tropically independent. Suppose not. Then there are constants b_{ij} such that $\theta(v) = \min_{(i,j) \in \mathcal{A}}\{\psi_{ij}(v) + b_{ij}\}$ occurs at least twice at every point v in Γ. Let

$$\Delta = \operatorname{div}(\theta) + 2D, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

Note that $\deg_{w_0}(\Delta)$ is $2r - \sigma_0(\theta)$, where $\sigma_0(\theta)$ is the outgoing slope of θ at w_0. Since the minimum is achieved twice at every point, this slope must agree with the slope $\sigma_0(\psi_{ij}) = 2r - i - j$ for at least two pairs $(i, j) \in \mathcal{A}$. The points in the half-open triangle where $j \geq i + 2$ and $i + j > r + 2s$ are omitted from $\mathcal{A}$, so there is only one pair $(i, j) \in \mathcal{A}$ such that $i + j = k$, for $k < r - 2s + \epsilon(r)$. It follows that $\deg_{w_0}(\Delta) \geq r - 2s + \epsilon(r)$. Similarly, $\deg_{v_{g+1}}(\Delta) \geq r - 2s$.

We claim that

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil \quad \text{for} \quad \ell \leq \left\lfloor \frac{r}{2} \right\rfloor + s + 1. \tag{5}$$

Note that the assumption $r > 2s$ implies that $\lfloor r/2 \rfloor + s + 1 \leq r$. Since e is a nondecreasing function of k, and $\deg_{v_{g+1}}(\Delta) \geq r - 2s$, the claim implies that

$$\deg(\Delta) \geq 2g + \left(\left\lfloor \frac{r}{2} \right\rfloor + s + 1 - s + \left\lceil \frac{r}{2} \right\rceil \right) + r - 2s.$$

Collecting terms gives $\deg(\Delta) \geq 2g + 2r - 2s + 1 = 2d + 1$, a contradiction.

It remains to prove claim (5). Since $\delta_0 \geq r - 2s + \epsilon(r)$, the claim holds for $\ell \leq \lceil r/2 \rceil - s$. We proceed by induction on ℓ. Assume that $e(\ell s) \geq \ell - s + \lceil r/2 \rceil$ and $\ell \leq \lfloor r/2 \rfloor + s$. We must show that $e((\ell + 1)s) \geq \ell - s + \lceil r/2 \rceil + 1$. If $e(\ell s) > \ell - s + \lceil r/2 \rceil$ then there is nothing to prove, since e is nondecreasing. It remains to rule out the possibility that $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$.

Suppose $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$. Fix $a = a(\ell)$ and $b = b(\ell)$ as in Section 7. By Lemma 7.4, if ψ_{ij} is δ-admissible on $\Gamma_{[a,b]}$ then either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We consider three cases.

Case 1: If $\lceil r/2 \rceil - s \leq \ell \leq \lfloor r/2 \rfloor$ then there are $\lceil r/2 \rceil - s$ pairs (i, j) with $i \neq j$ and $i + j = \ell + \lceil r/2 \rceil$ that are contained in the closed chevron and hence omitted from $\mathcal{A}$. This leaves

$$\ell + 1 - \left\lceil \frac{r}{2} \right\rceil + s = b - a + 2$$

pairs $(i, j) \in \mathcal{A}$ such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We can then apply Proposition 7.6, and the claim follows.

Case 2: If $\lfloor r/2 \rfloor < \ell < r/2 + s$ then there are $\lfloor r/2 \rfloor - s$ pairs (i, j) with $i + j = \ell + \lceil r/2 \rceil$ that are in the closed chevron and hence omitted from $\mathcal{A}$. This leaves

$$r - \ell + 1 - \left\lfloor \frac{r}{2} \right\rfloor + s = b - a + 2$$

pairs $(i, j) \in \mathcal{A}$ such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We can then apply Proposition 7.6, and the claim follows.

Case 3: If $\ell = r/2 + s$, then there are $r/2 - s$ pairs (i, j) with $i + j = r + s$ that are contained in the closed chevron and hence omitted from $\mathcal{A}$. This leaves one pair $(i, j) \in \mathcal{A}$ such that ψ_{ij} has slope r on the bridge $\beta_{(r/2+s)s}$. It follows that θ cannot have slope r at any point of this bridge. If $e((\ell + 1)s) \leq r$, however, then the inductive hypothesis implies that $e(\ell s) = e(\ell s + 1) = r$, hence θ has constant slope r on this bridge, a contradiction, and the claim follows. $\qquad\square$

Remark 8.6. In Case 3 of the above argument, the formulas for $a(\ell)$ and $b(\ell)$ would give $a(\ell) = b(\ell) + 1$, so the subgraph $\Gamma_{[a(\ell),b(\ell)]}$ might be thought of as a chain of $b(\ell) - a(\ell) + 1 = 0$ loops. The inductive step in this case is then an application of a degenerate version of Proposition 7.6 for a chain of zero loops, i.e., for a single edge. See also Remark 7.5.

9. Proof of Theorem 1.1 for $\rho(g, r, d) > 0$

Fix $\rho = \rho(g, r, d)$, $g' = g - \rho$, and $d' = d - \rho$. Let Γ' be a chain of g' loops with admissible edge lengths. Note that $\rho(g', r, d') = 0$. Therefore, the constructions in Sections 7 and 8 produce a divisor D' on Γ' of rank r and degree d' whose class is vertex avoiding, together with a set $\mathcal{A}'$ of integer points (i, j) with $0 \leq i \leq j \leq r$ of size

$$\# \mathcal{A}' = \min \left\{ \binom{r+2}{2}, 2d' - g' + 1 \right\} = \min \left\{ \binom{r+2}{2}, 2d - g + 1 - \rho \right\},$$

such that the collection of piecewise linear functions $\{\psi'_{ij} \in R(D') \mid (i, j) \in \mathcal{A}'\}$ is tropically independent.

We use Γ', D', and $\mathcal{A}'$ as starting points to construct a chain of g loops with admissible edge lengths Γ, a divisor D of degree d and rank r whose class is vertex avoiding, and a set $\mathcal{A}$ with size $\# \mathcal{A} = \min \left\{ \binom{r+2}{2}, 2d - g + 1 \right\}$ such that $\{\psi_{ij} \in R(D) \mid (i, j) \in \mathcal{A}\}$ is tropically independent. Note that

$$g = g' + \rho, \quad d = d' + \rho, \quad \text{and} \quad \# \mathcal{A} - \# \mathcal{A}' = \min \left\{ \rho, \binom{r+2}{2} - \# \mathcal{A}' \right\}.$$

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) > 0$, and $\binom{r+2}{2} \geq 2d - g + 1$. We construct Γ, D and $\mathcal{A}$ by adding ρ new loops to Γ', ρ new points to D', and ρ new points to $\mathcal{A}'$. Any collection of ρ points in the complement of $\mathcal{A}'$ will work, but the location of the new loops added to Γ' depends on the set $\mathcal{A} \setminus \mathcal{A}'$.

Recall that the complement of $\mathcal{A}'$ consists of the integer points in the closed chevron, the lower left half-open triangle, and the upper right half-open triangle, as shown in Figure 15. Suppose $\mathcal{A} \smallsetminus \mathcal{A}'$ consists of ν new points in the chevron, ν_1 new points in the lower left half-open triangle, and ν_2 new points in the upper right half-open triangle. Then construct Γ from Γ' by adding ν_1 new loops to the left end of Γ', ν_2 new loops to the right end of Γ', and ν new loops in the middle of the chain, at locations that are specified as follows.

For $\lceil r/2 \rceil - s \leq \ell < \lceil r/2 \rceil + s$, let $a(\ell)$ and $b(\ell)$ be as defined in Section 7. For each new element (i, j) from the chevron, we add a corresponding loop to the end of the subgraph $\Gamma'_{[a(\ell),b(\ell)]}$, where ℓ is the unique integer such that $i + j = \ell + \lceil r/2 \rceil$. In other words, if there are t points (i, j) in $\mathcal{A} \smallsetminus \mathcal{A}'$ such that $i + j = \ell + \lceil r/2 \rceil$, we add t new loops immediately to the right of the $b(\ell)$-th loop in Γ'.

Let $\alpha(k)$ denote the number of new points $(i, j) \in \mathcal{A} \smallsetminus \mathcal{A}'$ such that $i + j \leq k$. We construct our divisor D' so that it has one chip on each of the new loops. The new loops correspond to lingering steps in the associated lattice path, and the location of the points on the new loops are chosen in specific regions on the top edges, as described below, and sufficiently general so that the class of D' is vertex avoiding.

Just as in Sections 7 and 8, we suppose that $\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$ is tropically dependent, choose constants b_{ij} such that the minimum

$$\theta(v) = \min_{ij}\{\psi_{ij}(v) + b_{ij})\}$$

occurs at least twice at every point v in Γ, and fix

$$\Delta = \mathrm{div}(\theta) + 2D, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

We again fix $s = g - d + r$, which is the same as $g' - d' + r$. We claim that

(1) $\delta_0 + \cdots + \delta_{\nu_1} \geq r - 2s + \epsilon(r)$,

(2) $e(\ell s + \alpha(\ell + \lceil r/2 \rceil)) \geq \ell - s + \lceil r/2 \rceil$ for $\lfloor r/2 \rfloor - s + \epsilon(r) \leq \ell \leq \lfloor r/2 \rfloor + s + 1$,

(3) $\delta_{g - \nu_2 + 1} + \cdots + \delta_{g+1} \geq r - 2s$.

Just as in the proof for $\rho = 0$ and $r > 2s$, the claim implies that $\deg(\Delta) \geq 2d + 1$, which is a contradiction. It remains to prove the claim, which we do inductively, moving from left to right across the graph.

To prove (1), we show that $e(\alpha(k)) \geq k$ for $0 \leq k \leq r - 2s + \epsilon(r)$. For $k = 0$, there is nothing to prove, and we proceed by induction on k. Let $a = \alpha(k) + 1$ and $b = \alpha(k + 1)$. As in the $\rho = 0$ case, we must rule out the possibility that $e(a) = e(b) = k$. Just as in Lemma 7.4, if $e(a) = e(b) = k$ then the δ-permissible functions on $\Gamma_{[a,b]}$ are exactly those ψ_{ij} such that $i + j = k$. We choose the location of the new points on the loops in $\Gamma_{[a,b]}$ so that the functions ψ_i for $i \leq k/2$ have the combinatorial shape shown in Figure 9, on each loop in $\Gamma_{[a,b]}$, and those for

$i > k/2$ have the combinatorial shape shown in Figure 10. It follows that each δ-permissible ψ_{ij} has the combinatorial shape shown in Figure 12. By construction, there are exactly $b - a + 2$ pairs $(i, j) \in \mathcal{A}$ such that $i + j = k$. Then, just as in Proposition 7.6, we conclude that $e(b) \geq k + 1$, which proves (1). (The argument in this case is somewhat simpler than in Proposition 7.6, since the combinatorial shapes appearing in Figures 13 and 14 do not occur.) The proof of (3) is similar.

It remains to prove (2). Note that (2) follows from (1) for $\ell = \lfloor r/2 \rfloor - s + \epsilon(r)$. We proceed by induction on ℓ. Let $a = a(\ell) + \alpha(\ell + \lceil r/2 \rceil - 1)$ and let $b = b(\ell) + \alpha(\ell + \lceil r/2 \rceil)$. As in the $\rho = 0$ case, it suffices to rule out the possibility that $e(a) = e(b) = \ell - s + \lceil r/2 \rceil$.

Suppose $e(a) = e(b) = \ell - s + \lceil r/2 \rceil$. Then, just as in Lemma 7.4, if ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$, then either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We choose the location of the points on the new loops in $\Gamma_{[a,b]}$ so that ψ_{ij} has the shape shown in Figure 12 for $i < \ell < j$. Then, just as in Proposition 7.6, it follows that there must be at least $b - a + 3$ functions that are δ-permissible on $\Gamma_{[a,b]}$. However, by construction, there are only $b - a + 2$ functions that are δ-permissible on $\Gamma_{[a,b]}$, a contradiction. We conclude that $e(b) > \ell - s + \lceil r/2 \rceil$, as required. This completes the proof of the claim, and the theorem follows. $\qquad\square$

Remark 9.1. The analogue of (1) in the case $\rho(g, r, d) = 0$ and $r > 2s$ is the lower bound $\delta_0 \geq r - 2s + \epsilon(r)$ which comes from having only one pair $(i, j) \in \mathcal{A}$ such that ψ_{ij} has a given slope σ at w_0, for $0 \leq \sigma < r - 2s + \epsilon(r)$. This bound may be seen as coming from $r - 2s + \epsilon(r)$ applications of the degenerate version of Proposition 7.6 for a chain of zero loops, i.e., a single edge. As we add points to $\mathcal{A}$ and add loops to the left of w_0, these chains of zero loops become actual chains of loops, and we then use the usual version of Proposition 7.6. A similar remark applies to (3).

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) > 0$, and $\binom{r+2}{2} \leq 2d - g + 1$. Again, it suffices to construct a divisor D on Γ of rank r and degree d whose class is vertex avoiding such that all of the functions ψ_{ij} are tropically independent. Let

$$\eta = \min\left\{\rho, 2d - g + 1 - \binom{r+2}{2}\right\}.$$

By the arguments in the preceding case, on the chain of $g - \eta$ loops with bridges, there exists a vertex avoiding divisor D' of rank r and degree $d - \eta$ such that the functions ψ_{ij} are tropically independent. We construct a divisor D on Γ of rank r and degree d by specifying that $D|_{\Gamma_{[0,g-\eta]}} = D'$, and the remaining η steps of the corresponding lattice path are all lingering, with the points on the last η loops chosen sufficiently general so that the class of D is vertex avoiding. Then the restrictions of the functions ψ_{ij} to $\Gamma_{[0,g-\eta]}$ are tropically independent, so the functions themselves are tropically independent as well. $\qquad\square$

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dave.h.jensen@gmail.com   Department of Mathematics, 719 Patterson Office Tower, Lexington, KY 40506-0027, United States

sam.payne@yale.edu   Mathematics Department, Yale University, 10 Hillhouse Ave, New Haven, CT 06511, United States

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