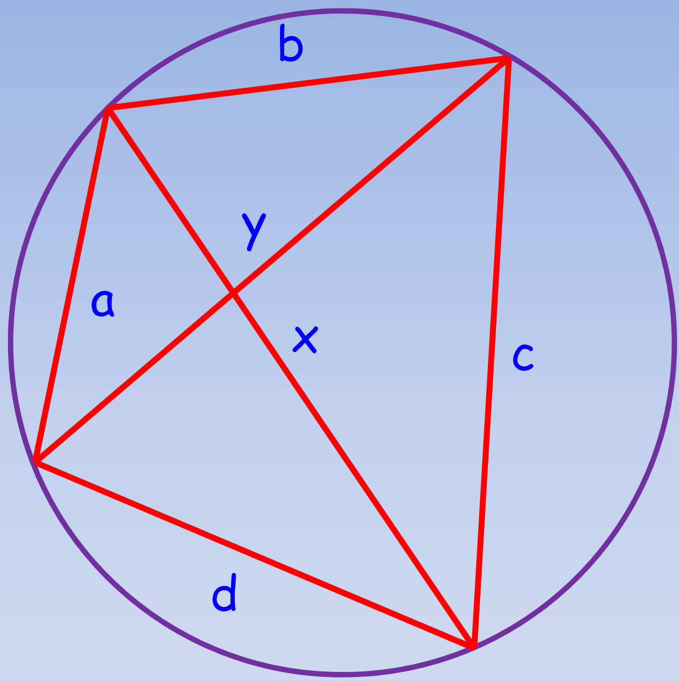
Quadrilaterals

MA 341 – Topics in Geometry
Lecture 21

Ptolemy's Theorem

Let a, b, c, and d be the lengths of consecutive sides of a cyclic quadrilateral and let x and y be the lengths of the diagonals. Then

$$ac + bd = xy.$$

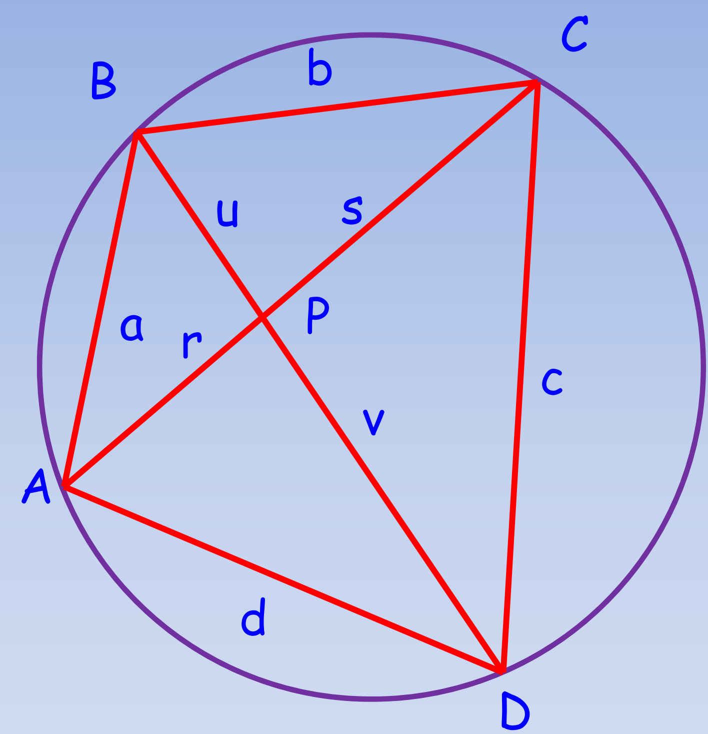


Ptolemy's Theorem

We have: $\Delta ABP \sim \Delta CDP$ & $\Delta BCP \sim \Delta DAP$.

So

$$\frac{a}{c} = \frac{u}{s} = \frac{r}{v} \quad \text{and} \quad \frac{b}{d} = \frac{u}{r} = \frac{s}{v}$$



$as = uc$, $br = ud$, and $uv = rs$.

$sa^2 + rb^2 = uac+ubd = u(ac+bd)$

$xu^2+xrs = xu^2+xuv=xu(u+v)=uxy$

By Stewart's Theorem

$$u(ac + bd) = sa^2 + rb^2 = xu^2 + xrs = uxy$$

$$ac + bd = xy$$

The Converse of Ptolemy's Theorem

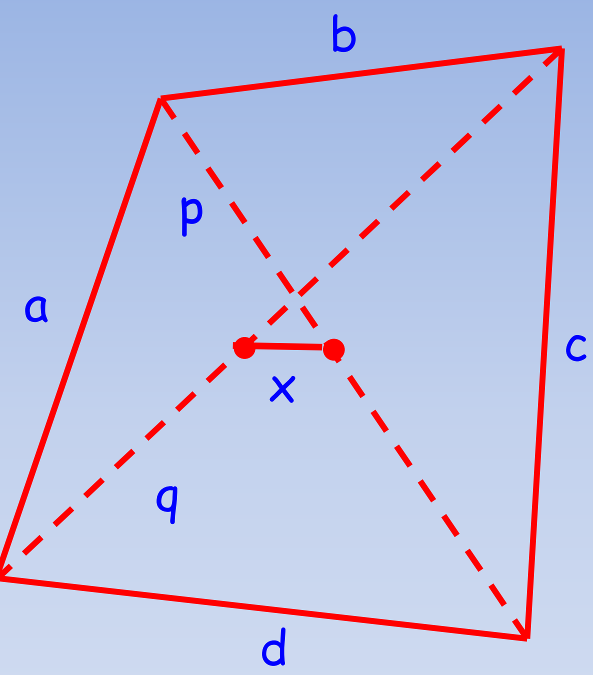
Let a, b, c, and d be the lengths of consecutive sides of a quadrilateral and let x and y be the lengths of the diagonals. If

$$ac + bd = xy,$$

then the quadrilateral is a cyclic quadrilateral.

Euler's Theorem

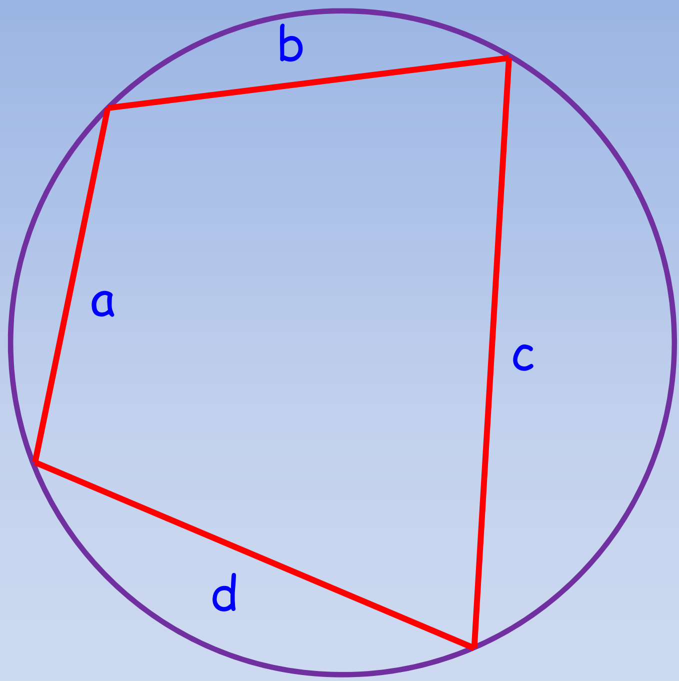
Let a, b, c, and d be the lengths of consecutive sides of a quadrilateral, m and n lengths of diagonals, and x the distance between midpoints of diagonals. Then



$$a^2 + b^2 + c^2 + d^2 = m^2 + n^2 + 4x^2$$

Brahmagupta's Theorem

There is an analog of Heron's Formula for special quadrilaterals.

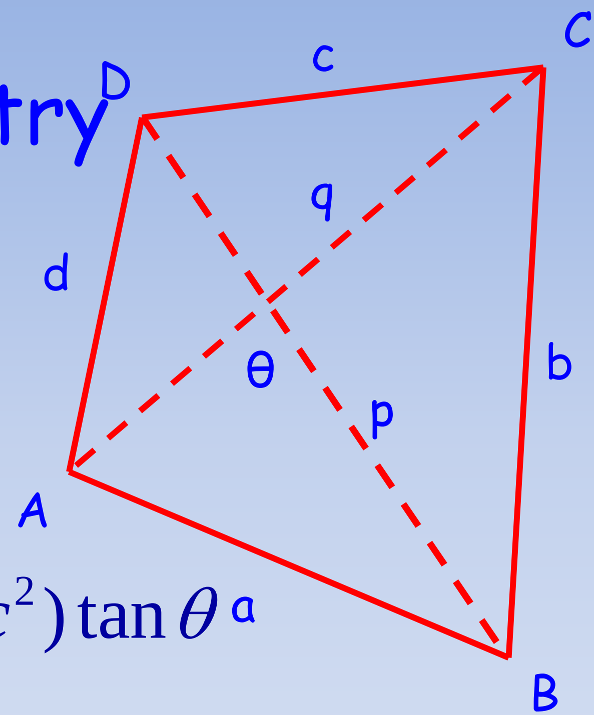


Let a, b, c, and d be lengths of consecutive sides of cyclic quadrilateral, then

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Area of a Quadrilateral

Using triangle trigonometry you can show that



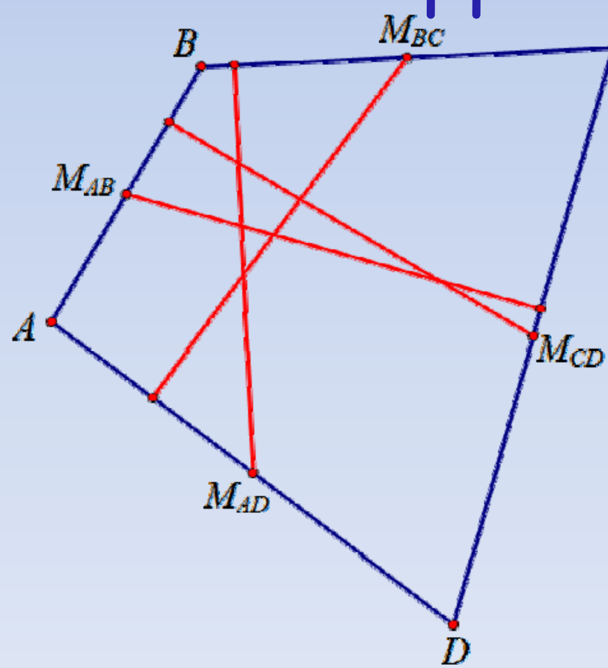
$$A = \frac{1}{2} pq \sin\theta = \frac{1}{4}(b^2 + d^2 - a^2 - c^2)\tan\theta$$

$$= \frac{1}{4}\sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos\tfrac{1}{2}(A+C)}$$

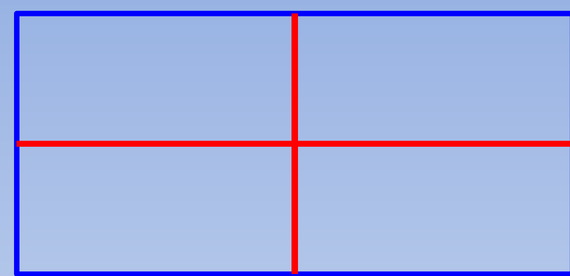
Maltitudes

For a quadrilateral the maltitude (midpoint altitude) is a perpendicular through the midpoint of one side perpendicular to the opposite side.



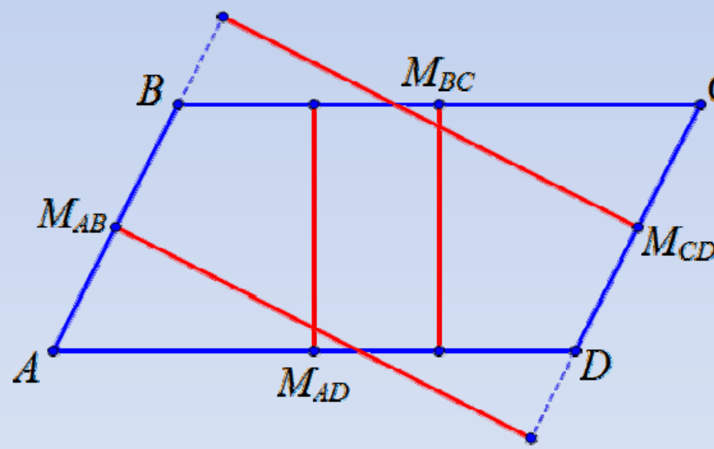
Maltitudes

Rectangle



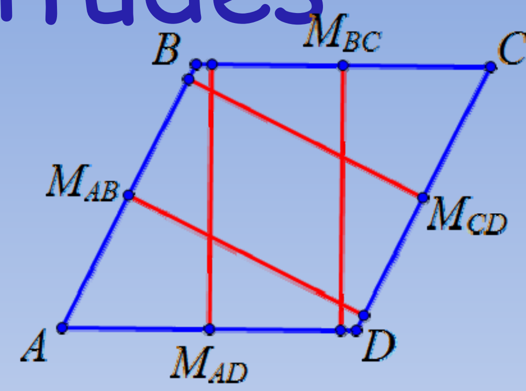
Square – same

Parallelogram

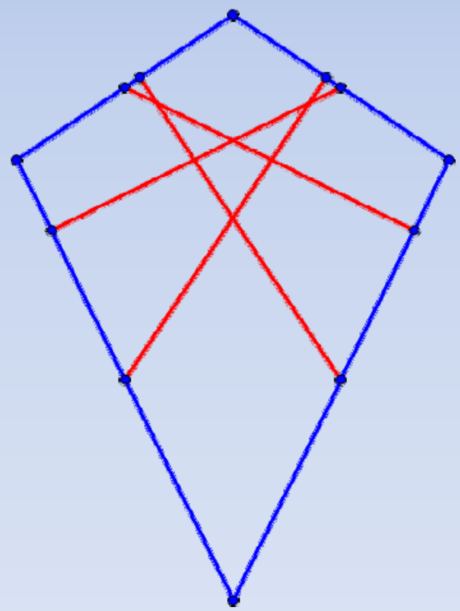


Maltitudes

Rhombus

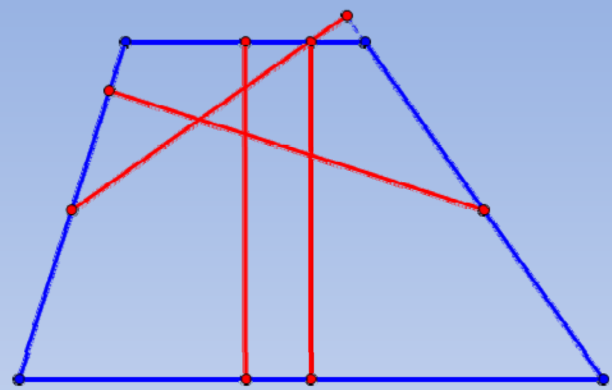


Kite

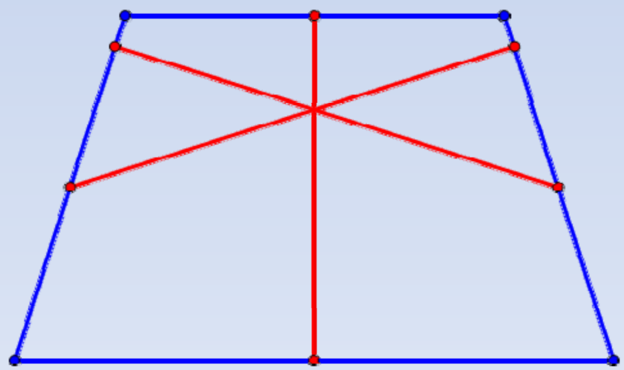


Maltitudes

Trapezoid

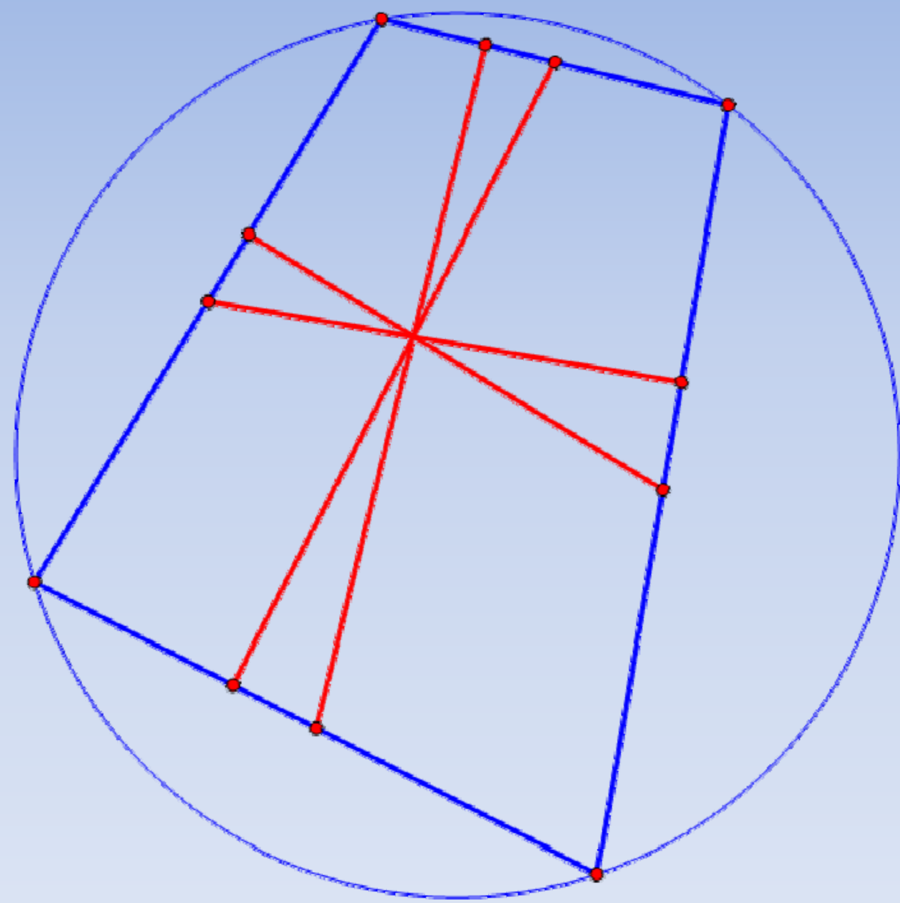


Isosceles trapezoid



Maltitudes

Cyclic quadrilaterals



Quadrilaterals and Circles

- For a cyclic quadrilateral, area is easy and there are nice relationships
- Maybe maltitudes of cyclic quadrilateral are concurrent
- Can we tell when a quadrilateral is cyclic?
- Can we tell when a quadrilateral has an inscribed circle?

Theorem

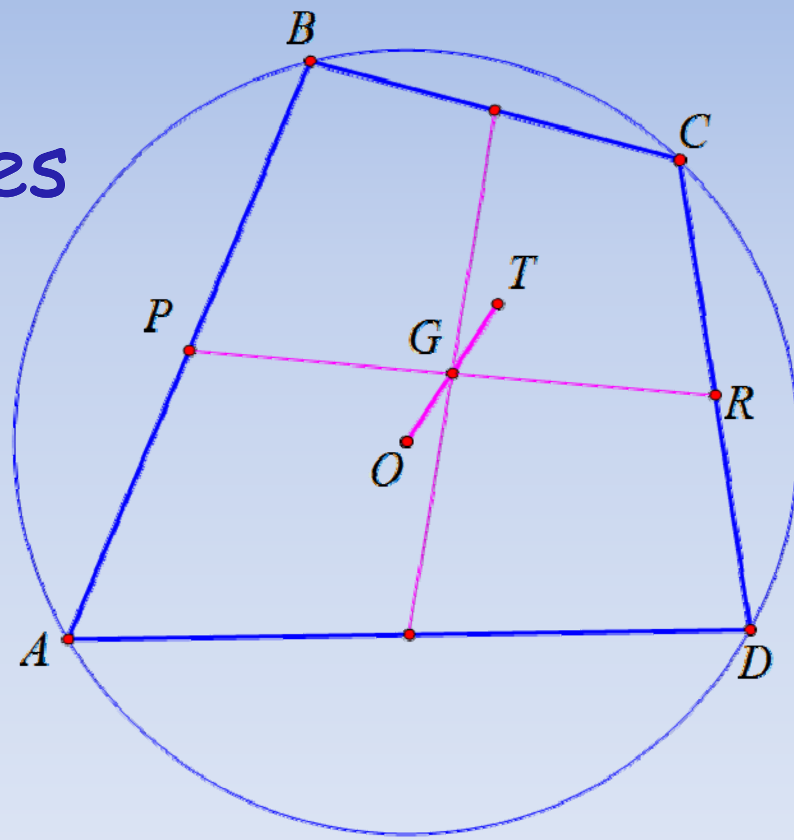
For a cyclic quadrilateral the maltitudes intersect in a single point, called the anti-center.

Proof

Let O = center of circle

G = centroid,

intersection of midlines



Let T = point on ray OG so that OG = GT

Proof

PG = RG

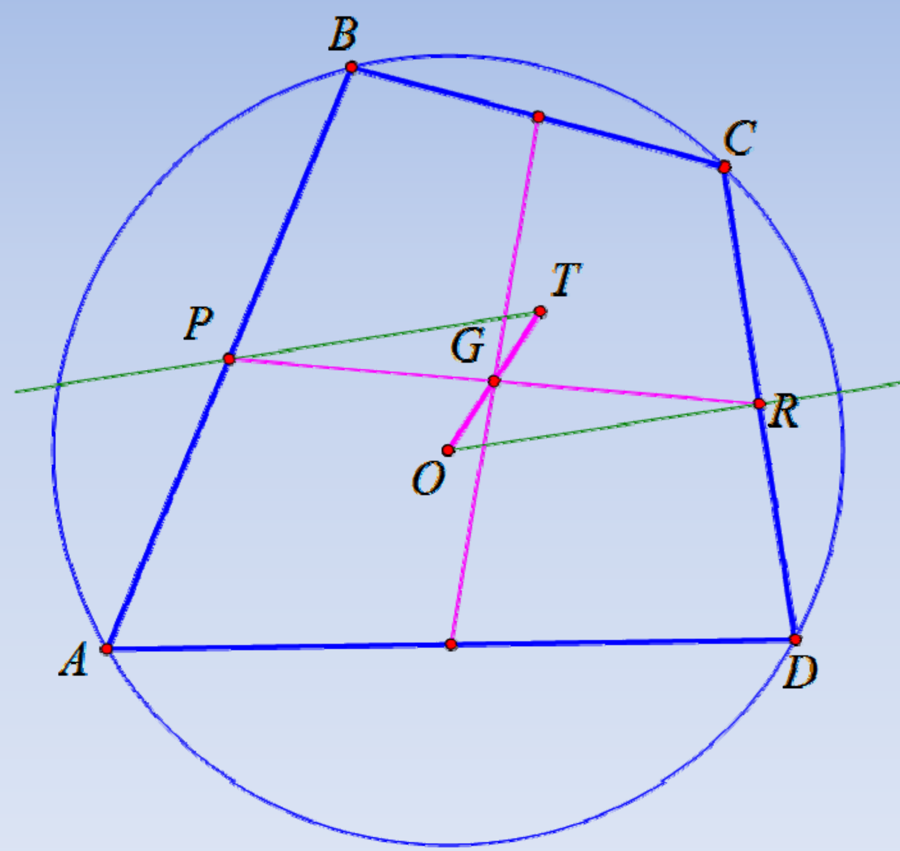
OG = GT

$\angle PGT = \angle RGO$

$\Delta PGT = \Delta RGO$

$\angle PTG = \angle ROG$

PT || OR



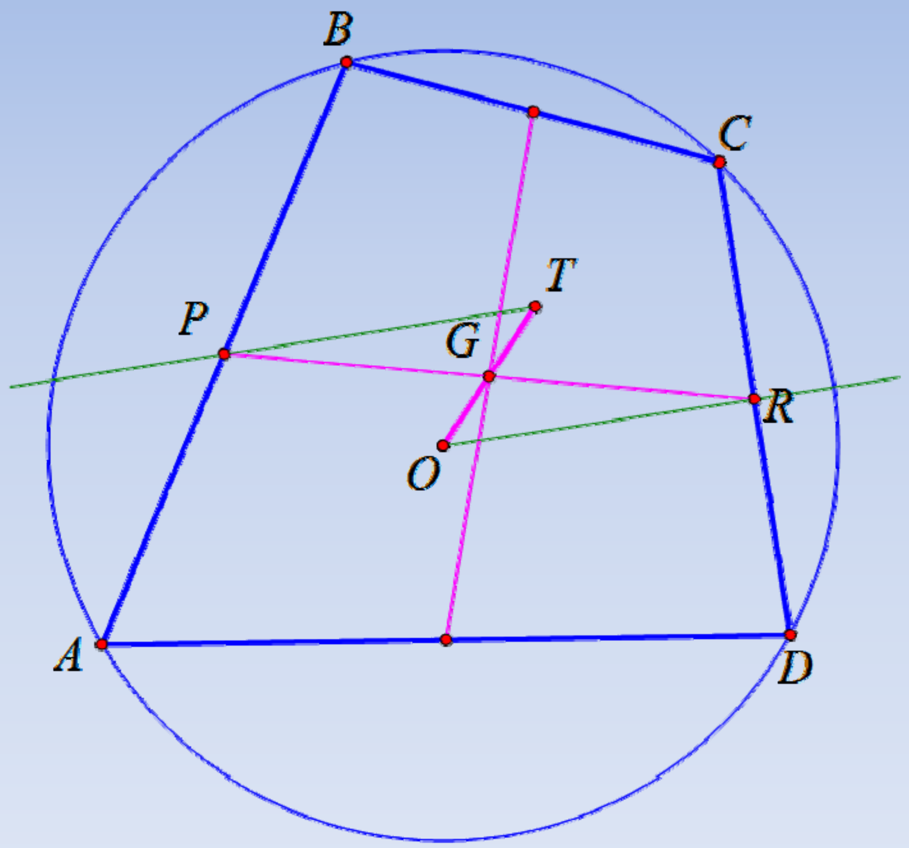
Proof

OR ⊥ CD

Thus, PT ⊥ CD

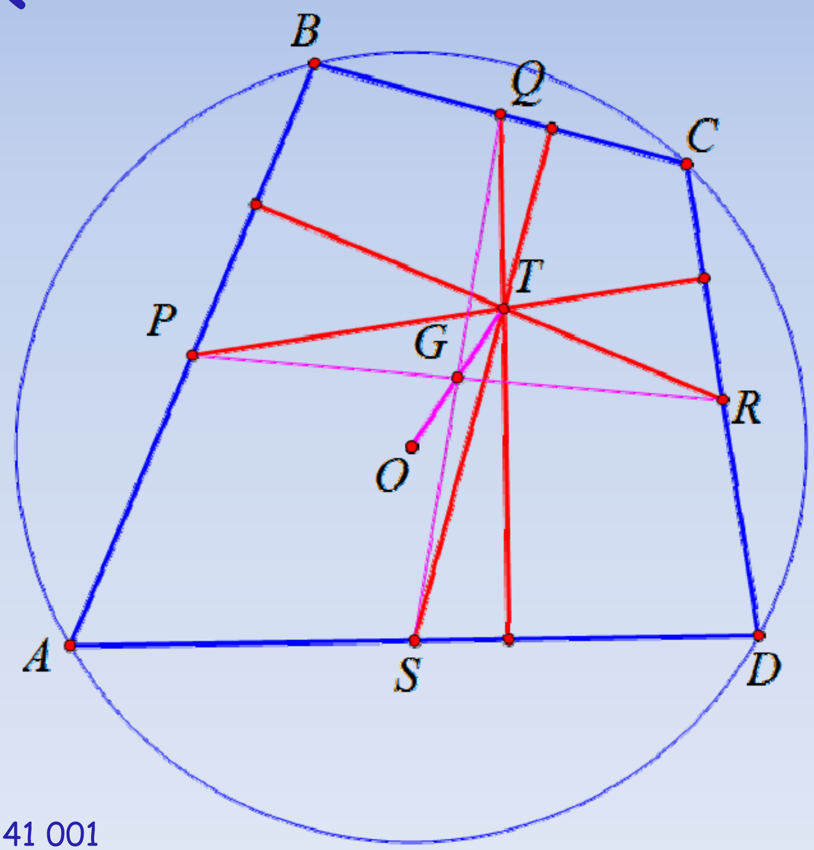
T lies on maltitude through P

Use ΔRGT = ΔPGO to show T lies on maltitude through R



Proof

Using other midline we show T lies on maltitudes through Q and S.



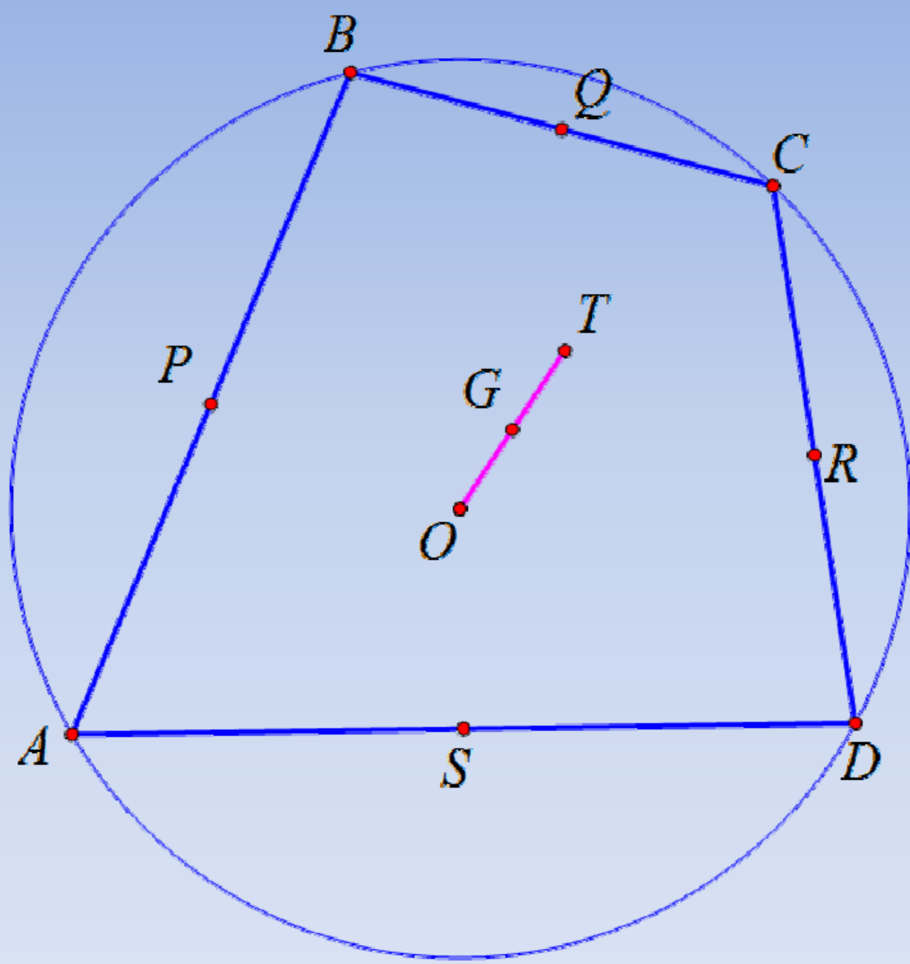
Anticenter

T = anticenter = intersection of maltitudes

G = midpoint

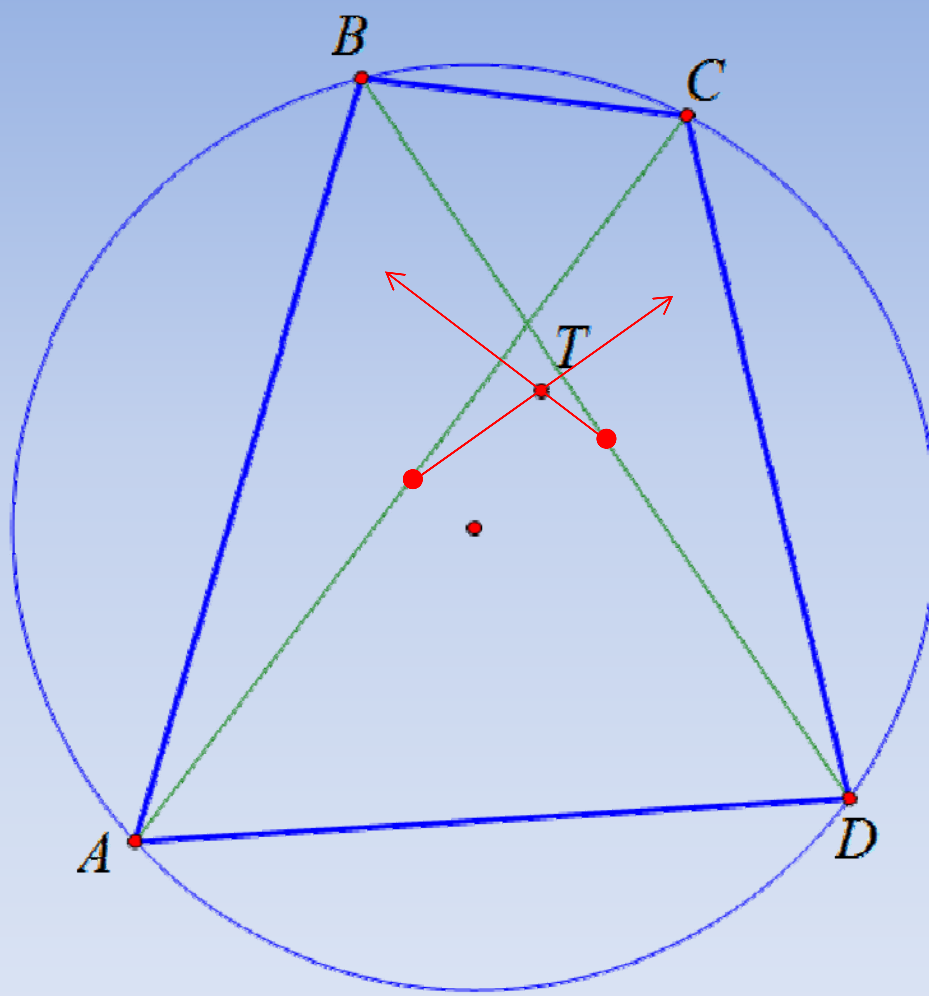
O = circumcenter

OG = GT



Other Anticenter Properties

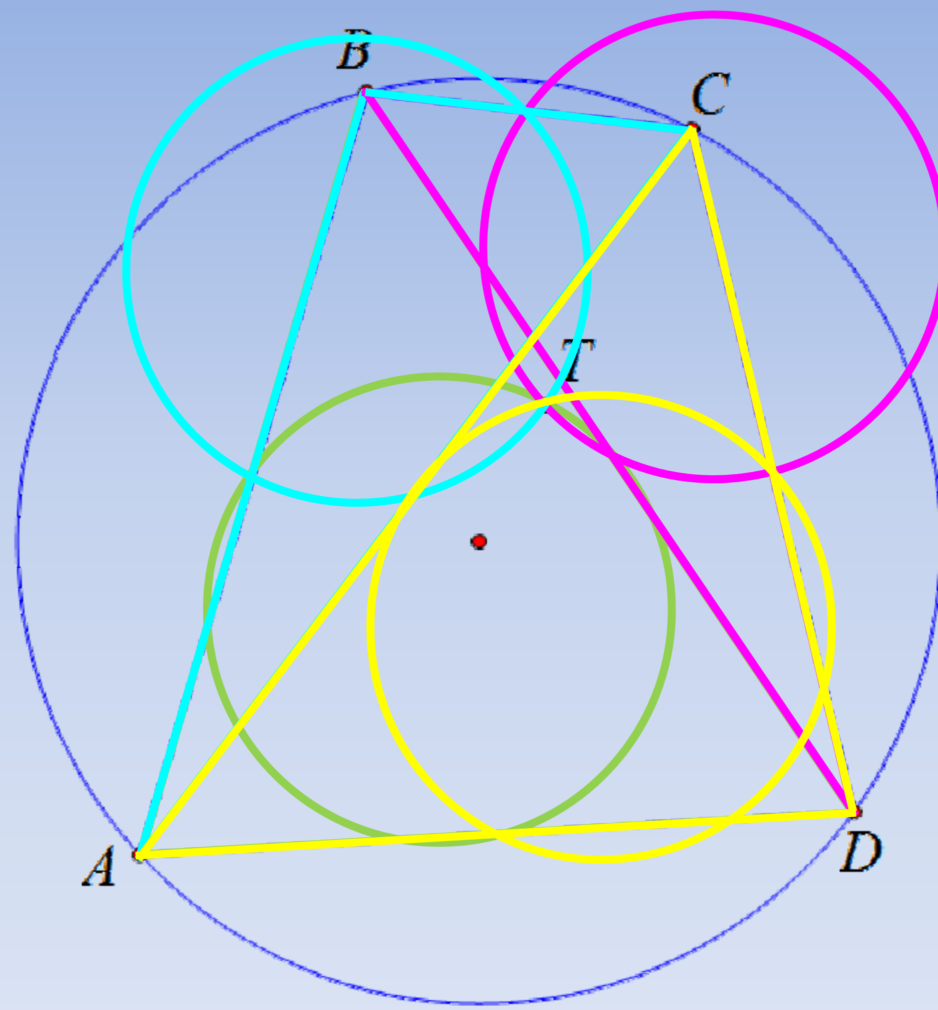
Perpendiculars from midpoint of one diagonal to other intersect at T.



Other Anticenter Properties

Construct 9-point circles of the four triangles ΔABD,
ΔBCD,
ΔABC, and
ΔADC.

The 4 circles intersect at the anticenter.



Other Anticenter Properties

The centers of the 9-point circles are concyclic with center T.