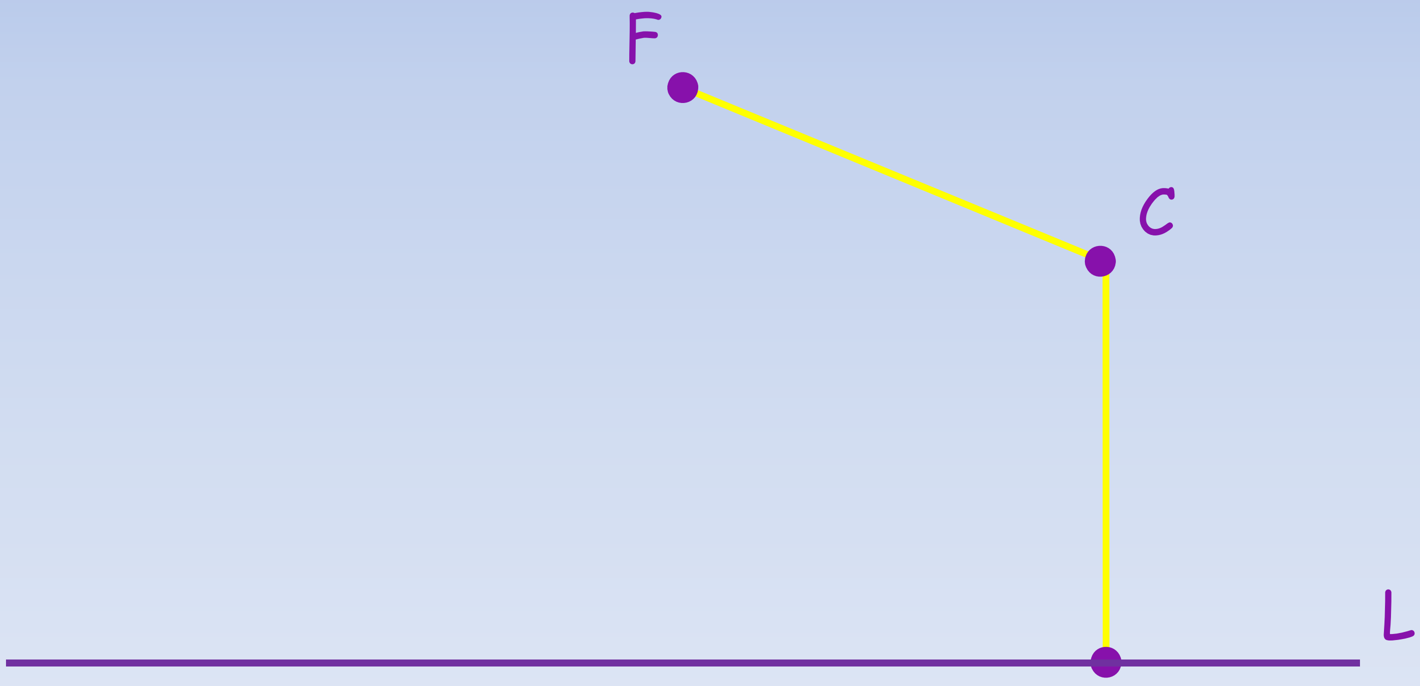
The Terrible Parable of the Parabola

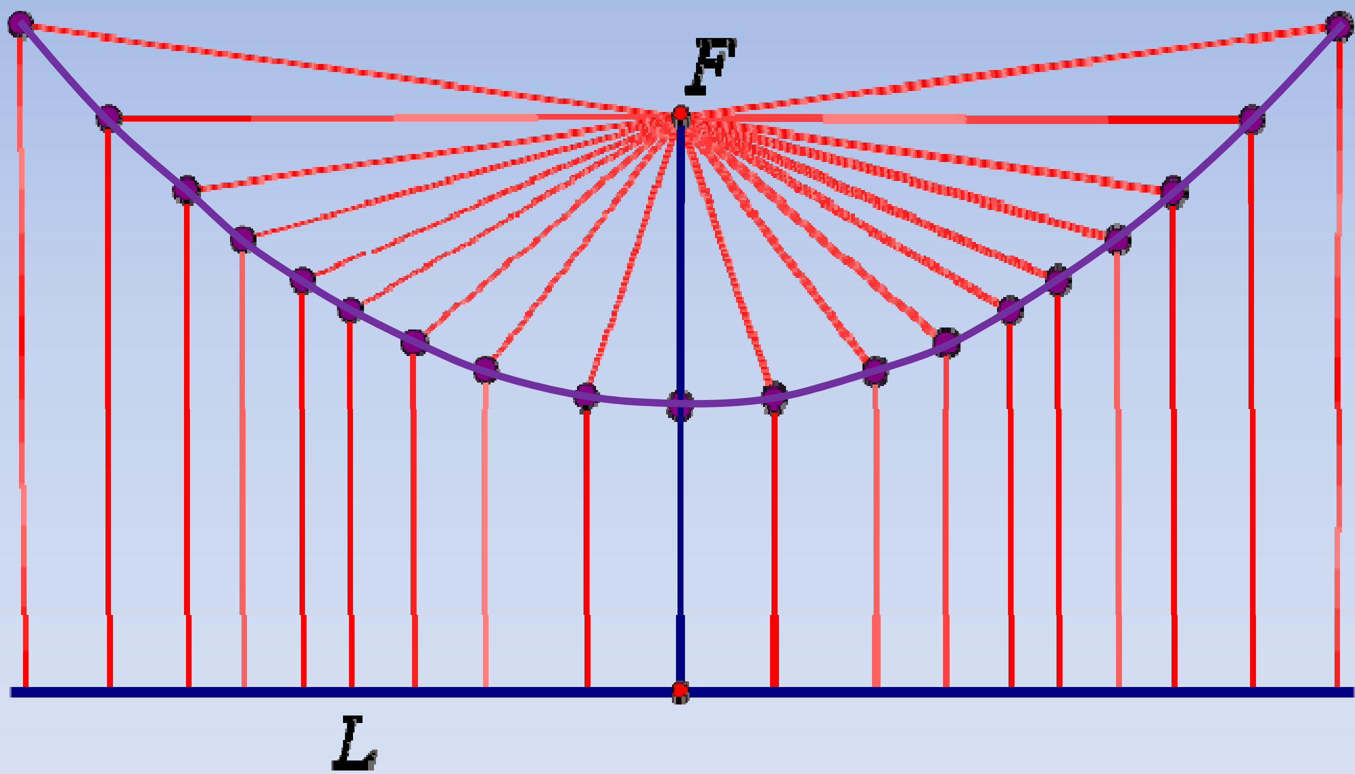
MA 341 – Topics in Geometry
Lecture 28

Parabola

Given a line L and a point F not on L, the locus of points that are equidistant from L and from F is called a parabola



Parabola



Parabola

F = focus

L = directrix

Line perpendicular to L through F = axis

Midpoint of LF = vertex

Chord passing through focus = focal chord

Chord passing through focus perpendicular to axis = latus rectum (right chord)

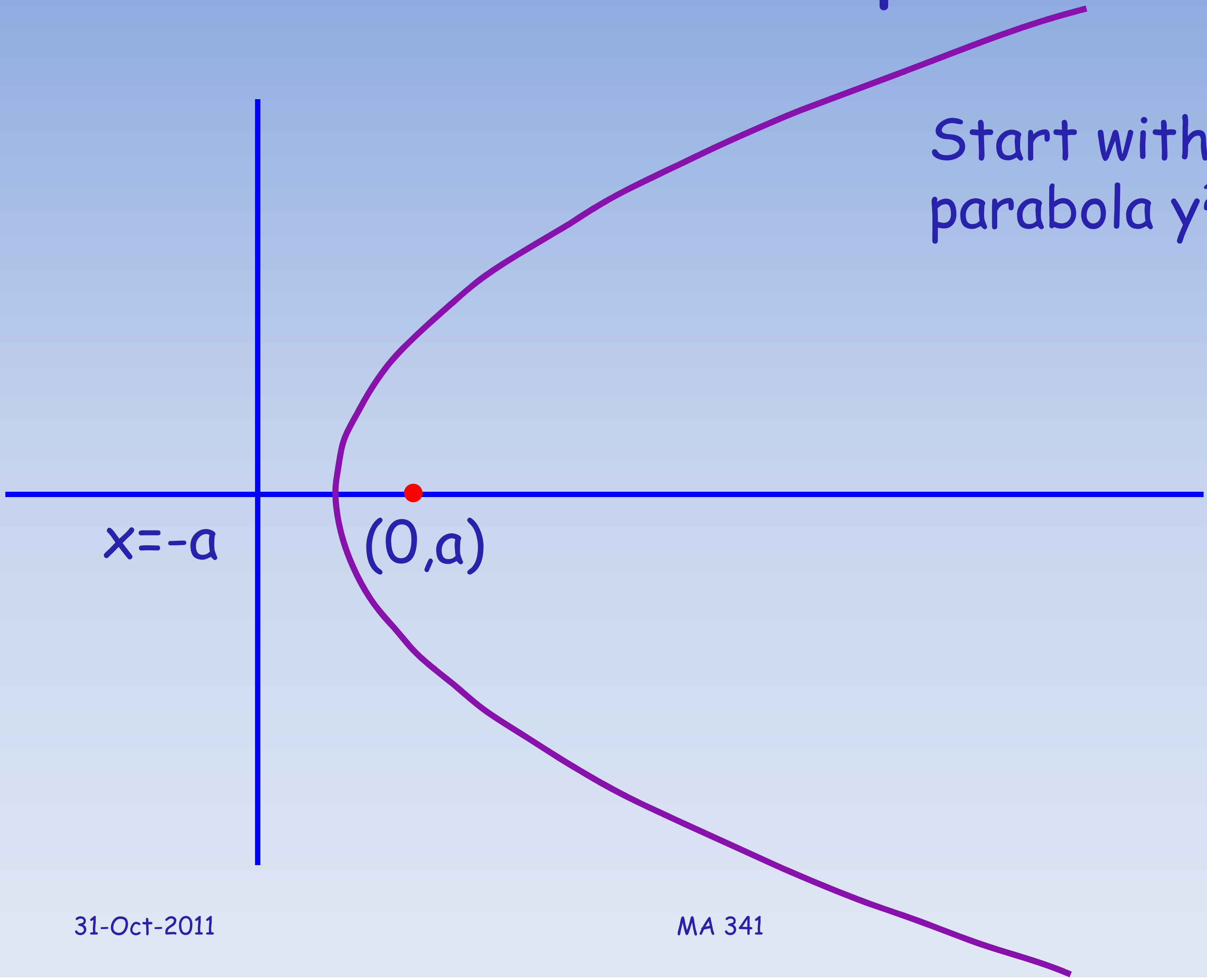
Parabola

Lemma 1: Latus rectum = 2 x distance from focus to vertex.

Lemma 2: The parabola having directrix $y = -a$ and focus $(0,a)$ has equation $x^2 = 4ay$.

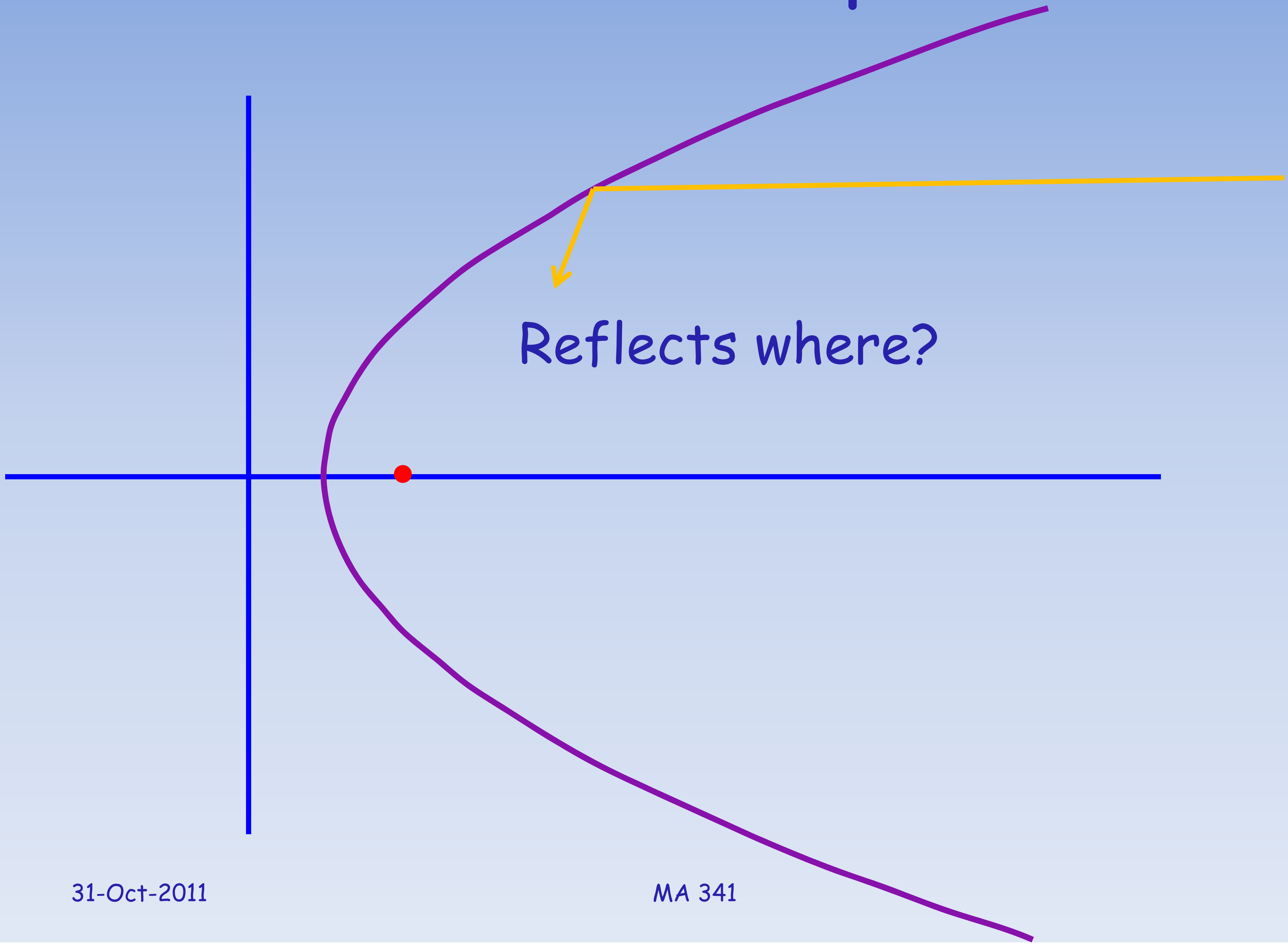
Lemma 3: The parabola having axis parallel to the y-axis and vertex (h,k) has the equation $(x-h)^2 = 4a(y-k)$.

Reflection Properties



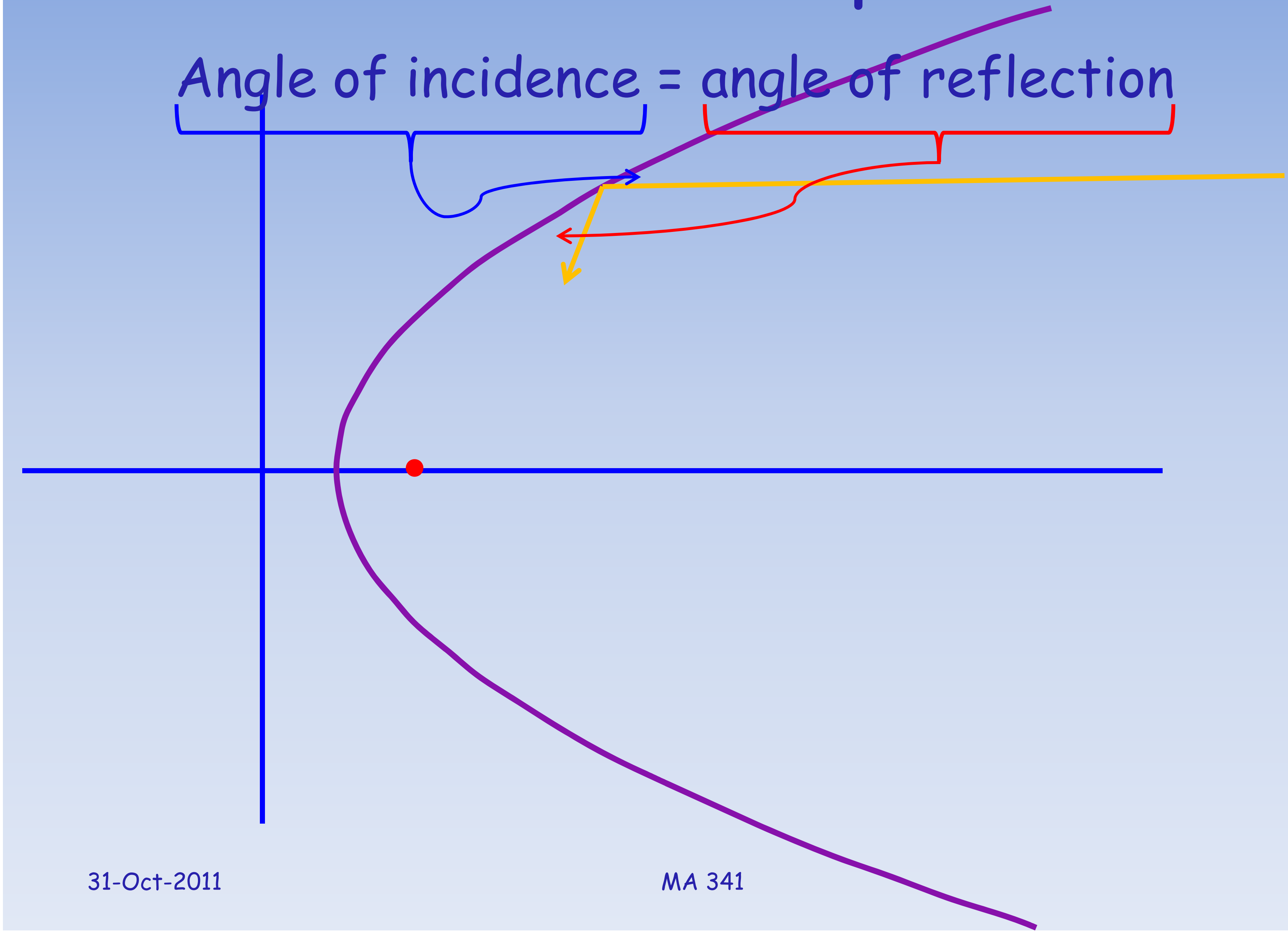
Start with the parabola $y^2 = 4ax$

Reflection Properties

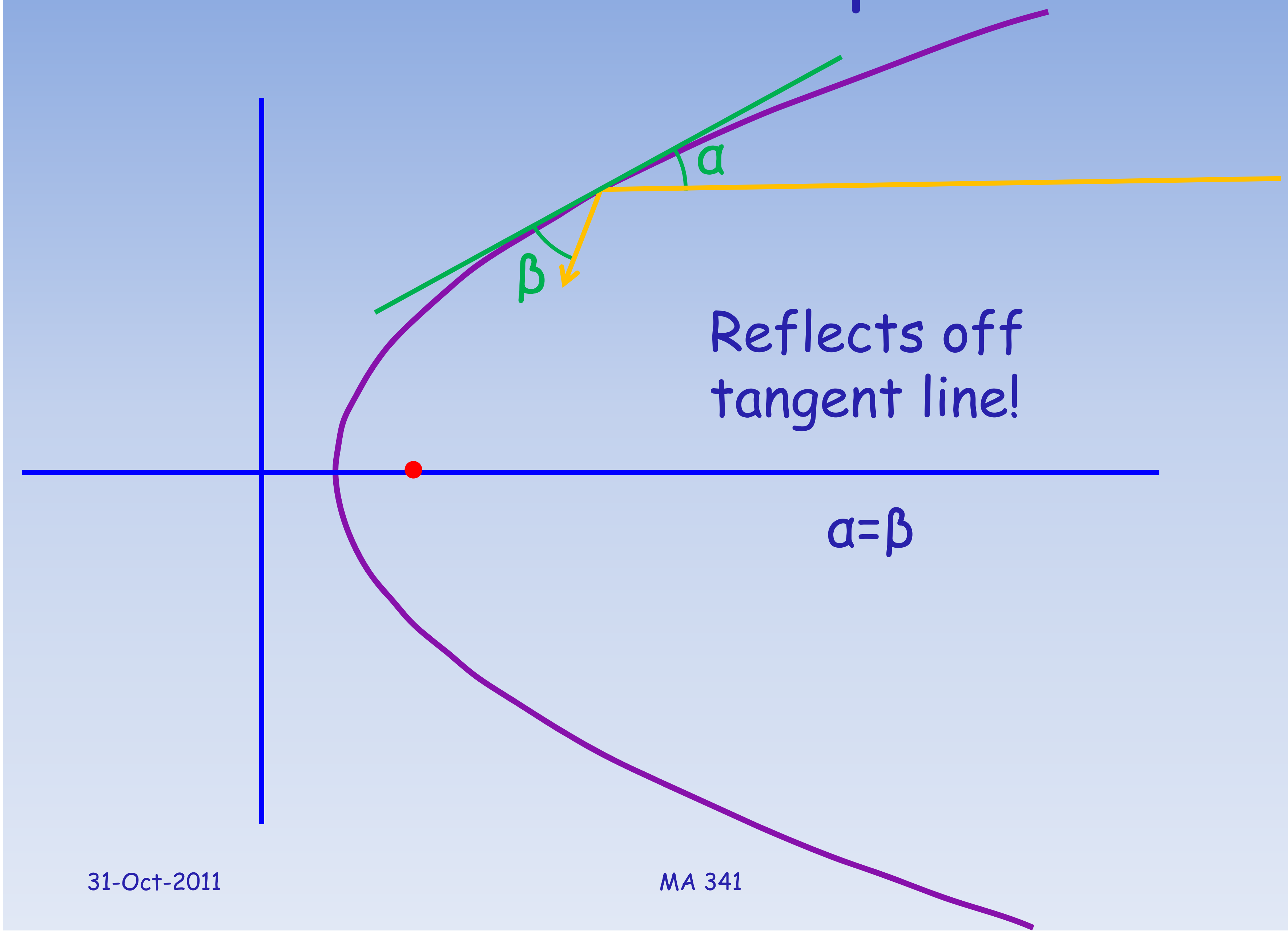


Reflection Properties

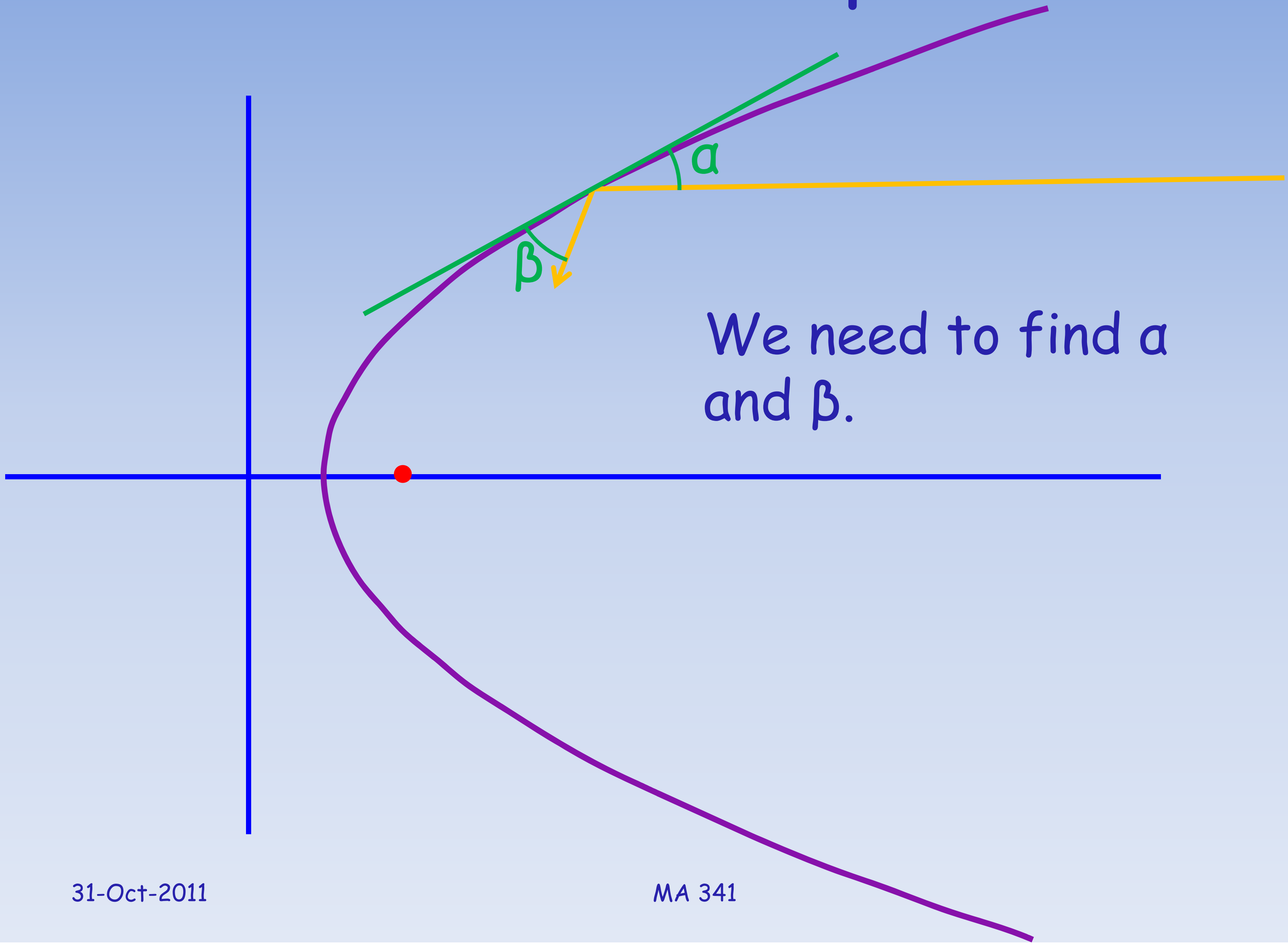
Angle of incidence = angle of reflection



Reflection Properties

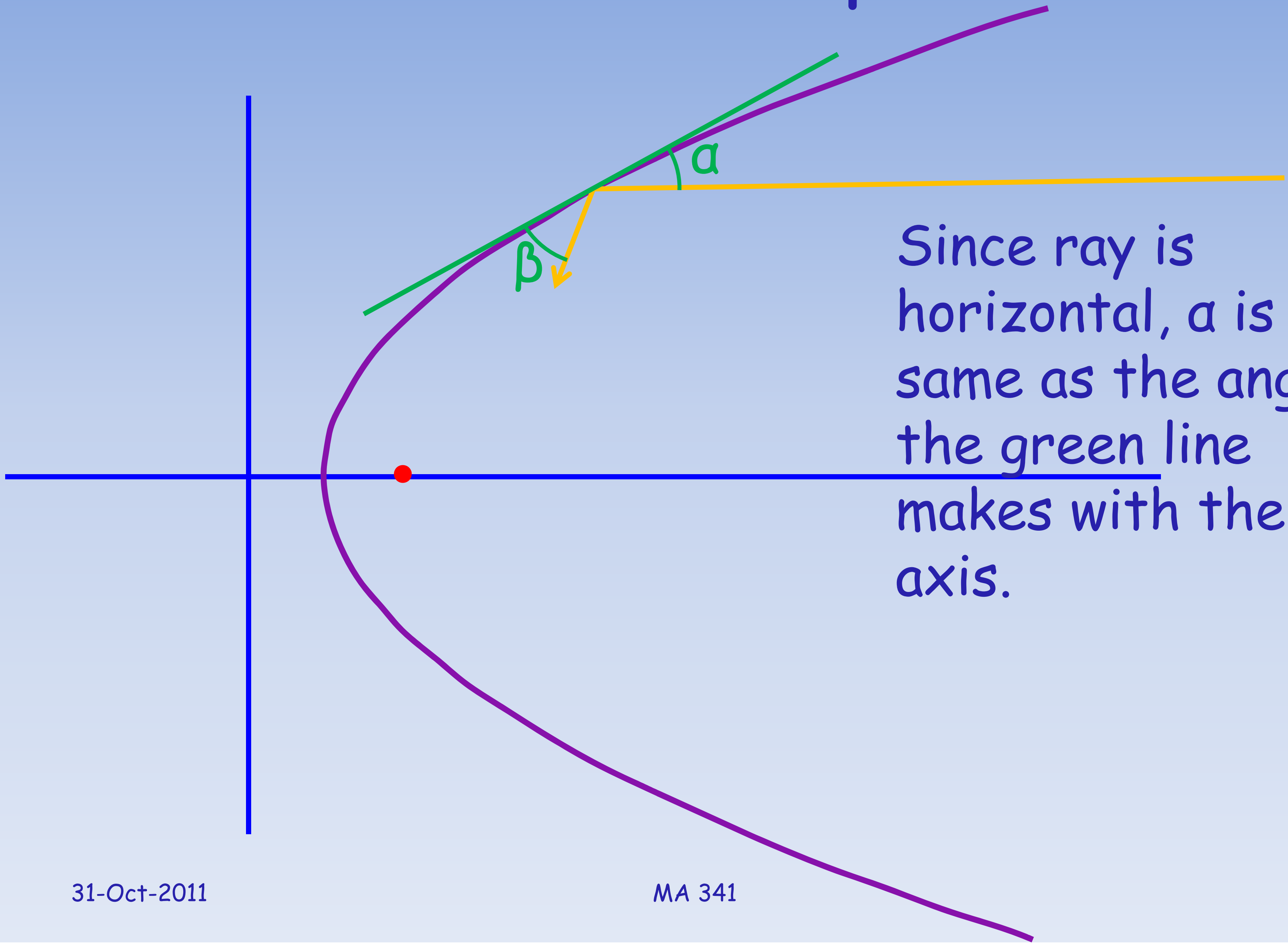


Reflection Properties



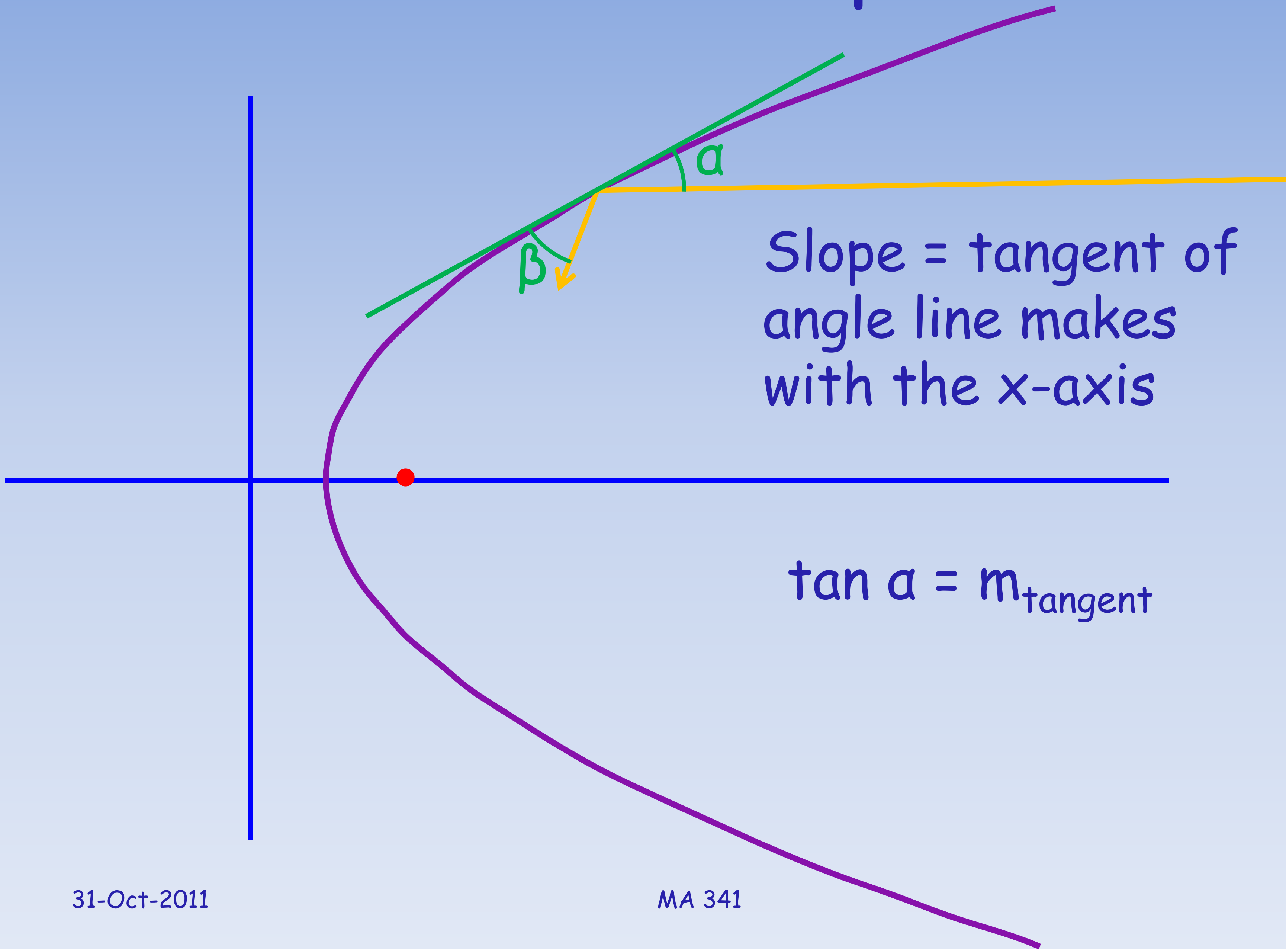
We need to find α and β.

Reflection Properties



Since ray is horizontal, α is same as the angle the green line makes with the x-axis.

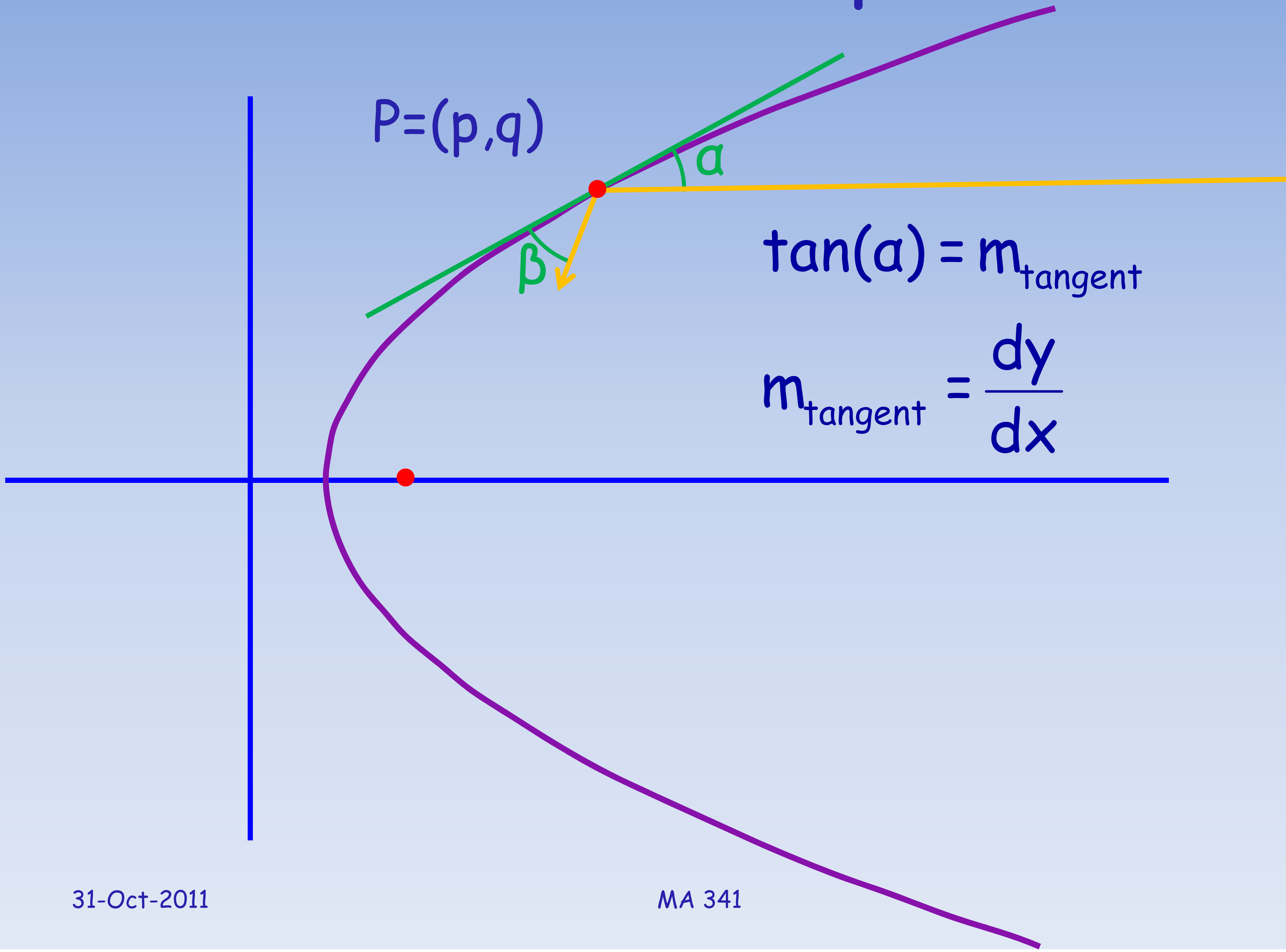
Reflection Properties



Slope = tangent of angle line makes with the x-axis

$\tan \alpha = m_{tangent}$

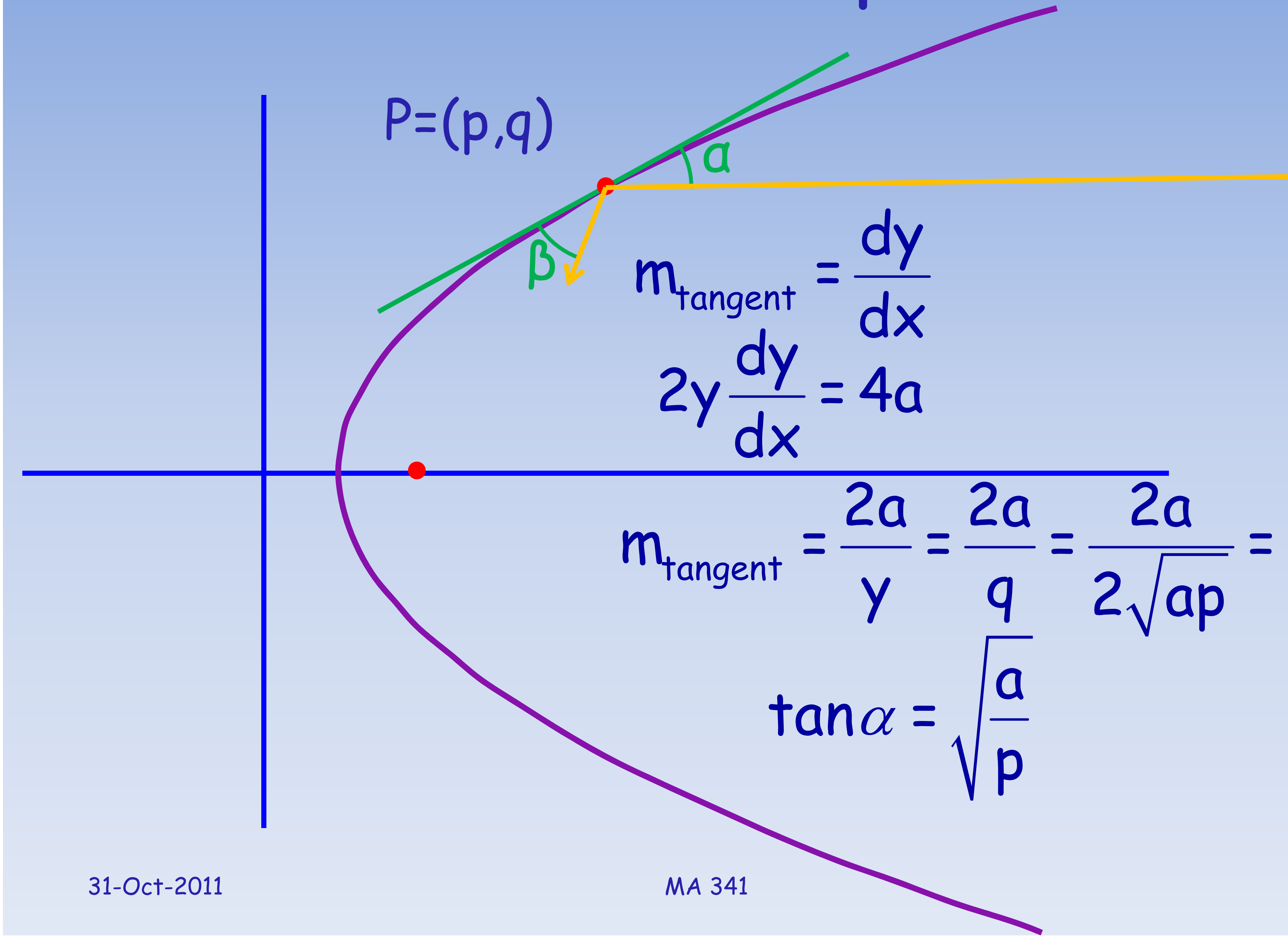
Reflection Properties



$$\tan(\alpha) = m_{tangent}$$

$$m_{tangent} = \frac{dy}{dx}$$

Reflection Properties



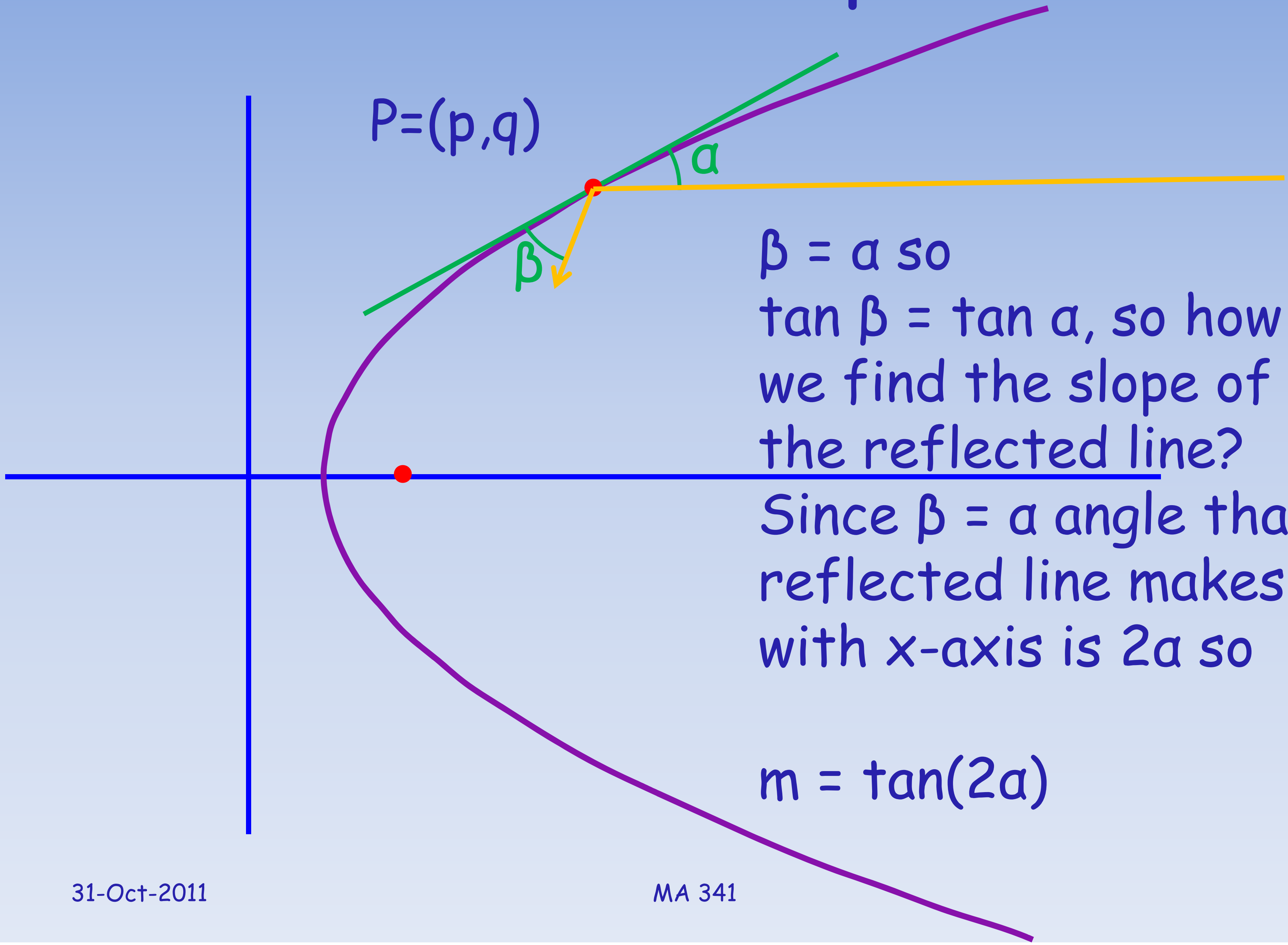
$$m_{tangent} = \frac{dy}{dx}$$

$$2y\frac{dy}{dx} = 4a$$

$$m_{tangent} = \frac{2a}{y} = \frac{2a}{q} = \frac{2a}{2\sqrt{ap}} = \sqrt{\frac{a}{p}}$$

$$\tan\alpha = \sqrt{\frac{a}{p}}$$

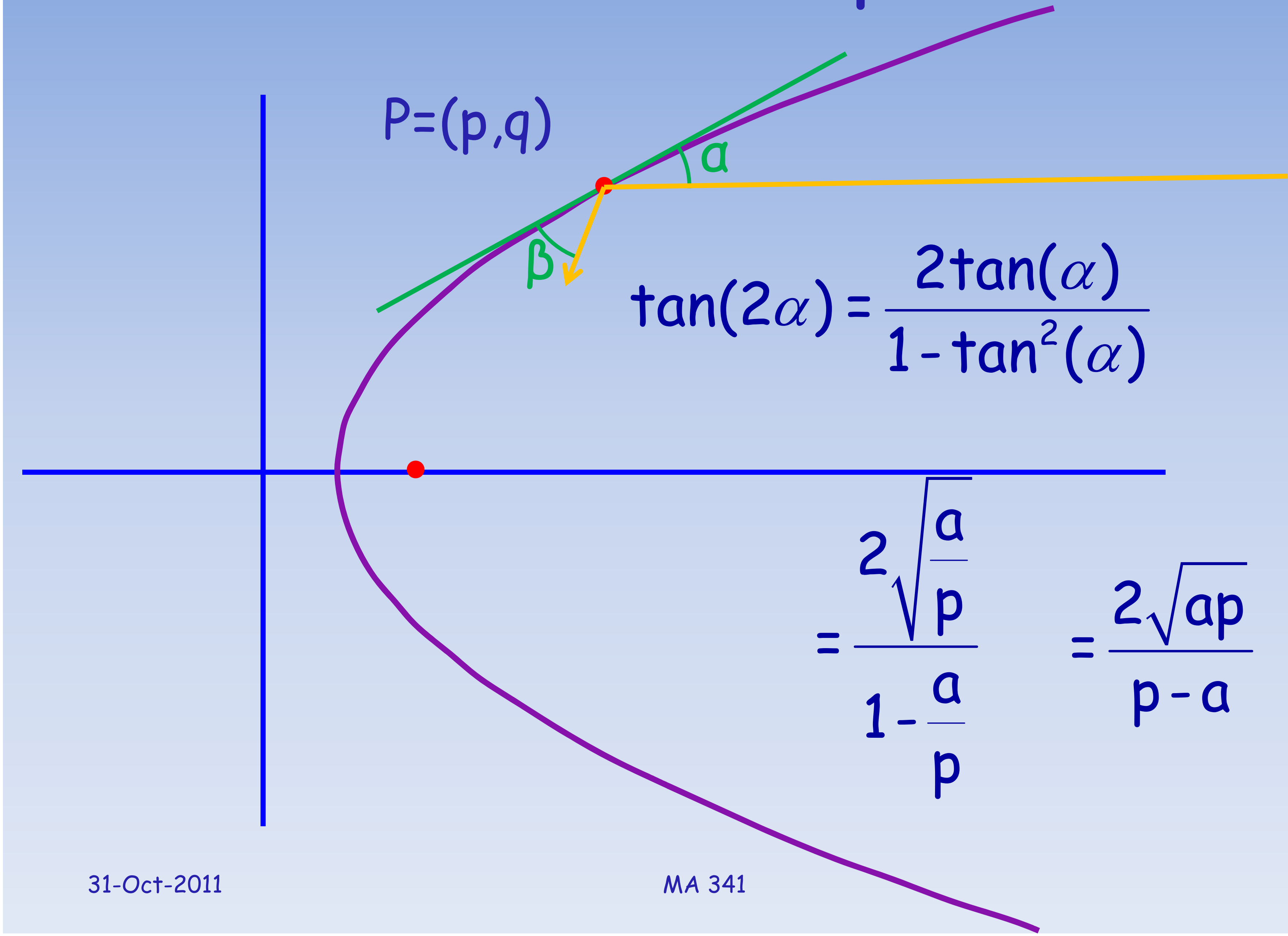
Reflection Properties



β = α so
tan β = tan α, so how do we find the slope of the reflected line?
Since β = α angle that reflected line makes with x-axis is 2α so

m = tan(2α)

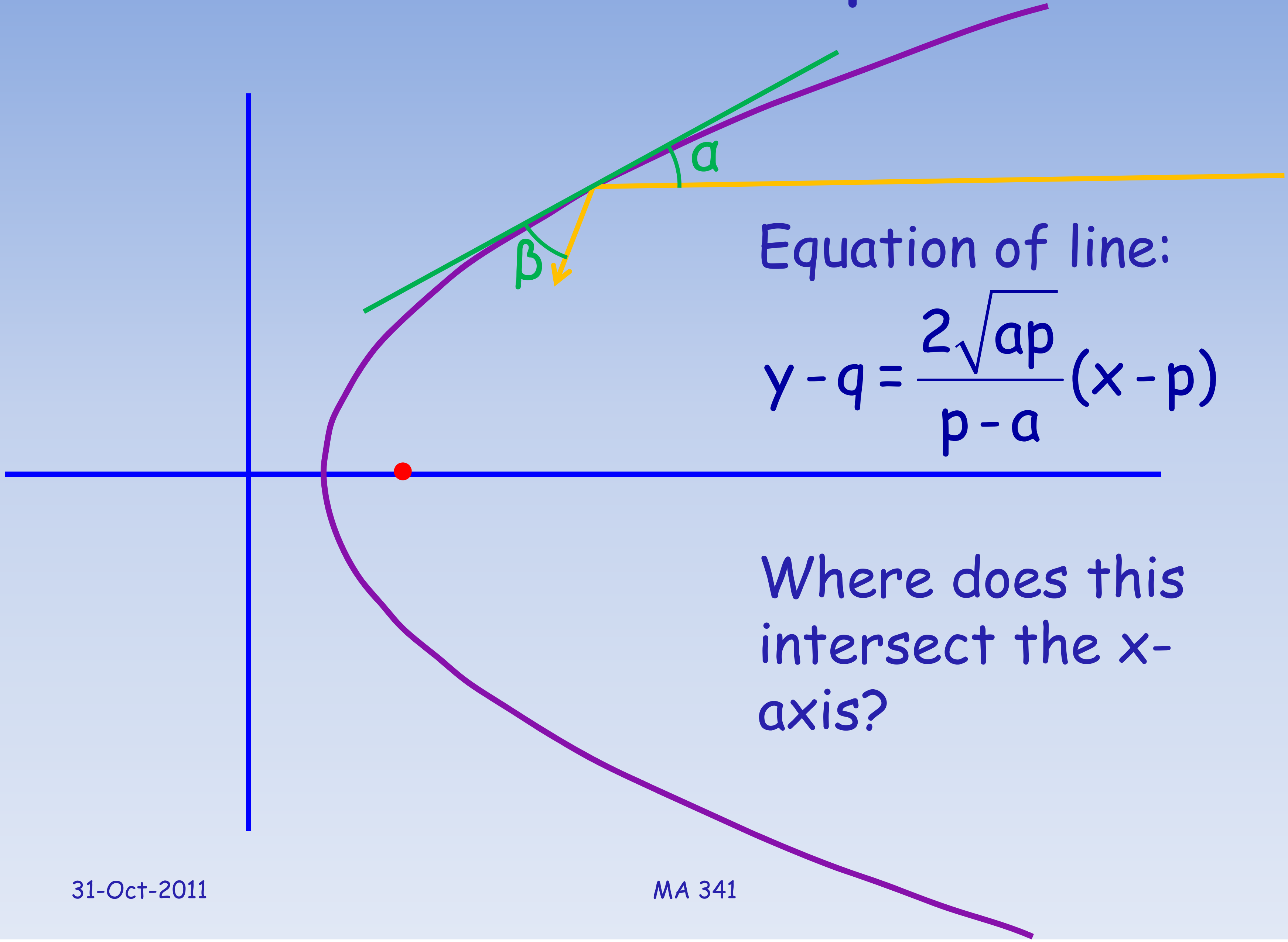
Reflection Properties



$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

$$= \frac{2\sqrt{\frac{a}{p}}}{1-\frac{a}{p}} = \frac{2\sqrt{ap}}{p-a}$$

Reflection Properties

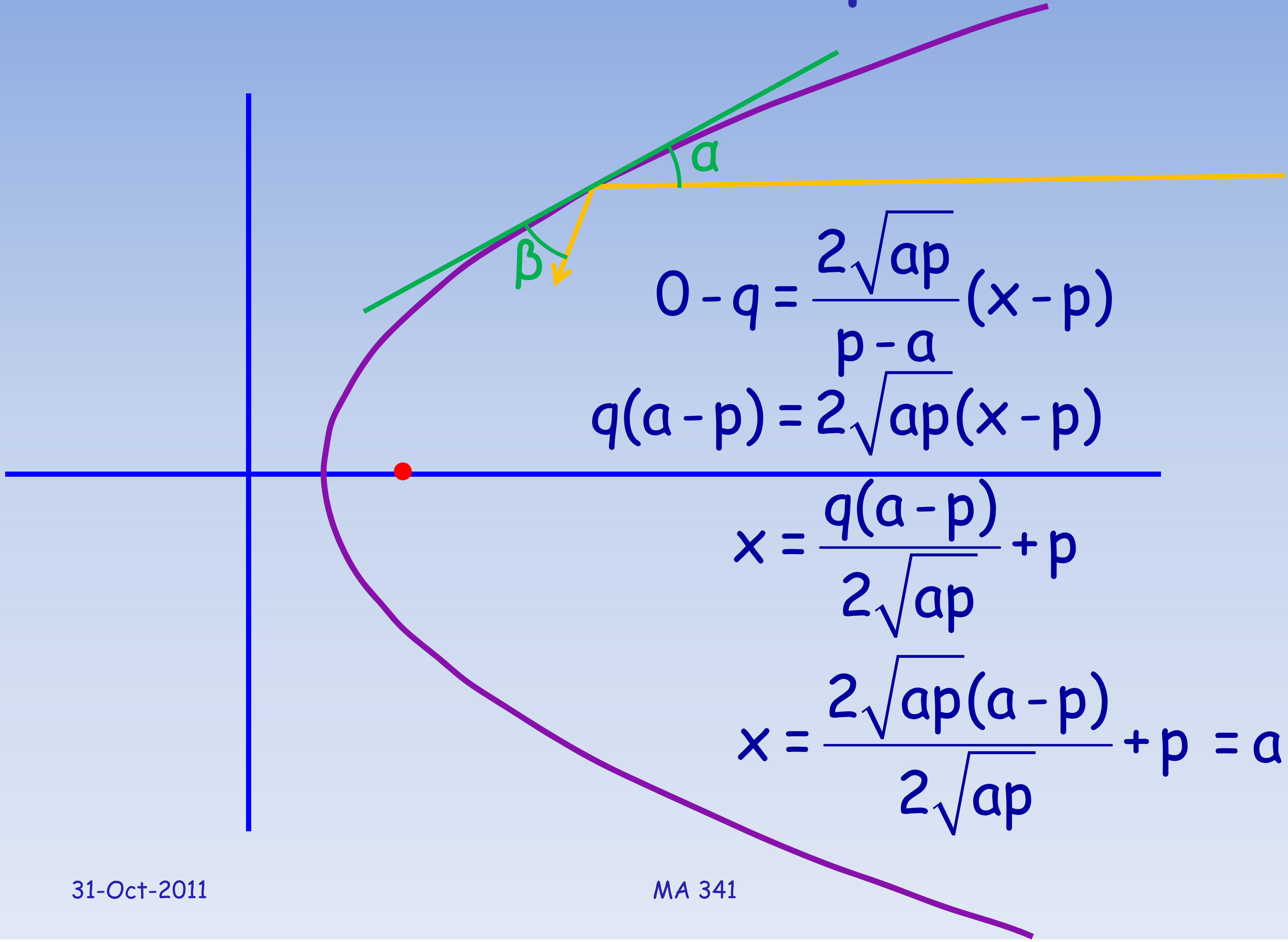


Equation of line:

$$y - q = \frac{2\sqrt{ap}}{p - a}(x - p)$$

Where does this intersect the x-axis?

Reflection Properties



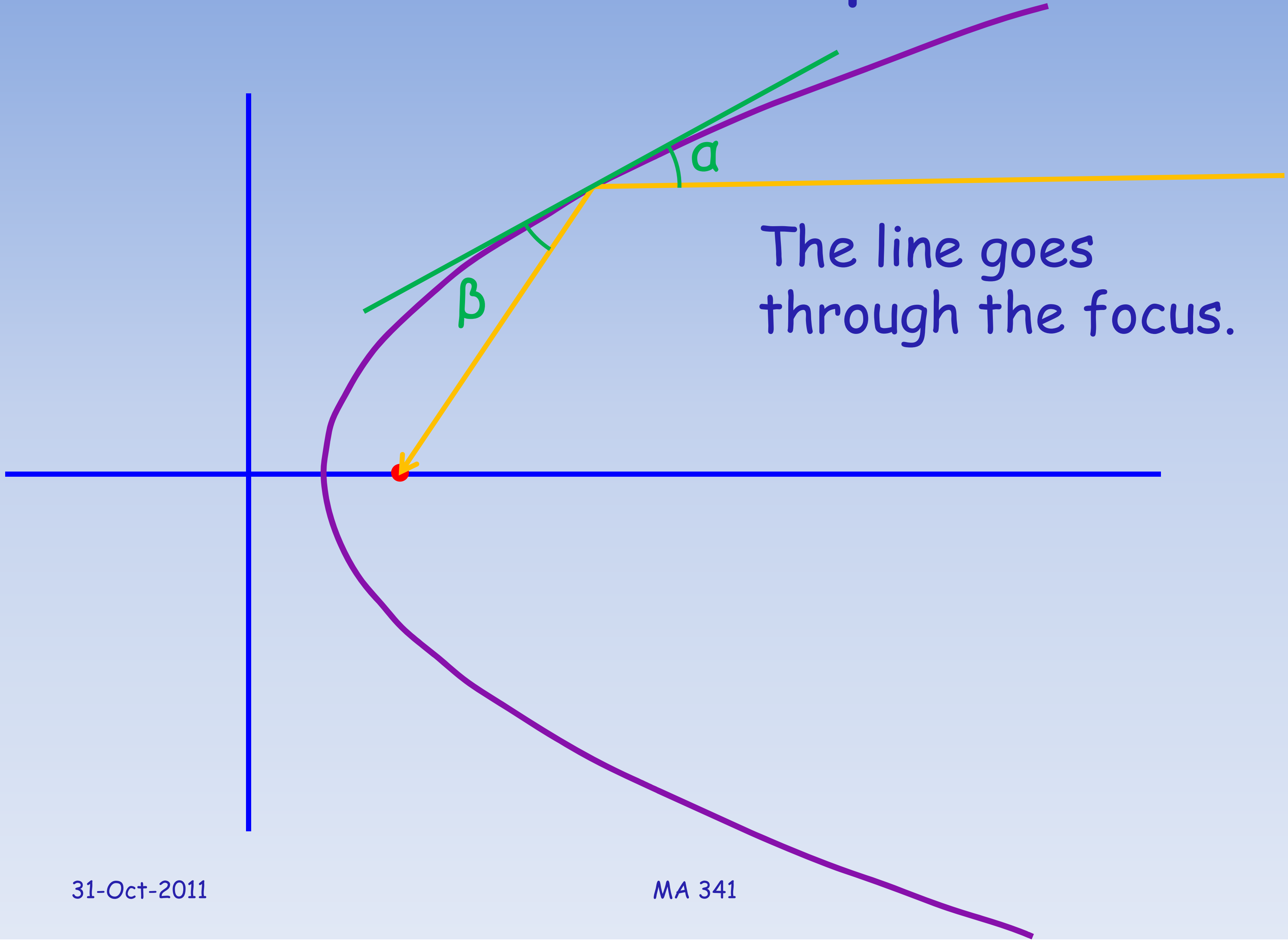
$$0 - q = \frac{2\sqrt{ap}}{p-a}(x-p)$$

$$q(a-p) = 2\sqrt{ap}(x-p)$$

$$x = \frac{q(a-p)}{2\sqrt{ap}} + p$$

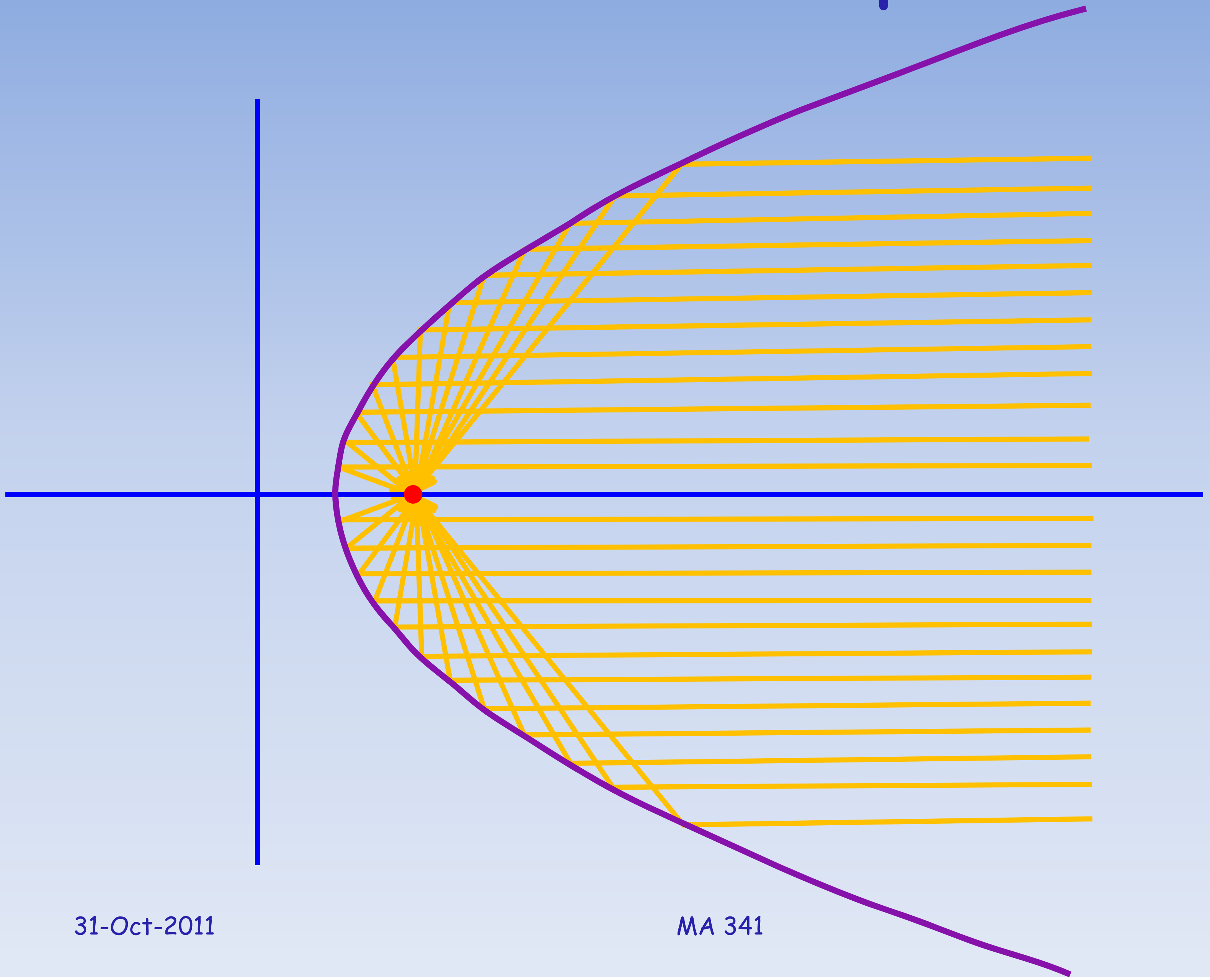
$$x = \frac{2\sqrt{ap}(a-p)}{2\sqrt{ap}} + p = a$$

Reflection Properties

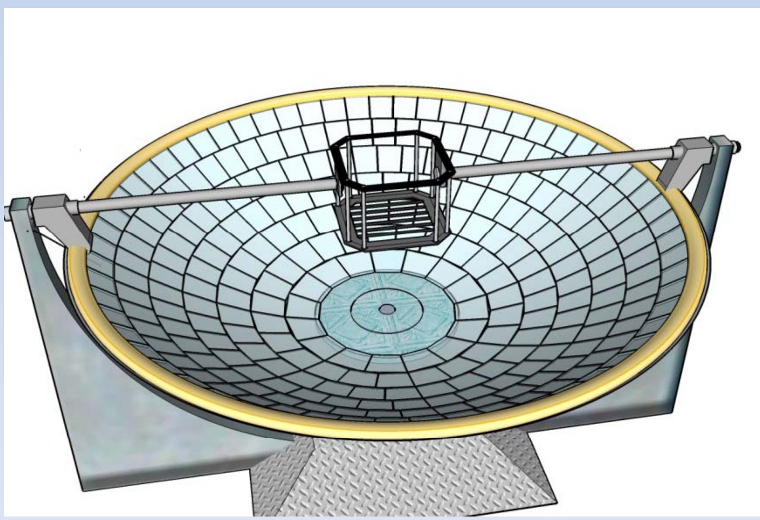


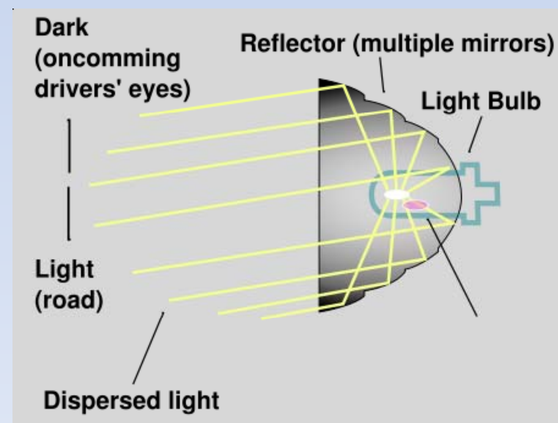
The line goes through the focus.

Reflection Properties

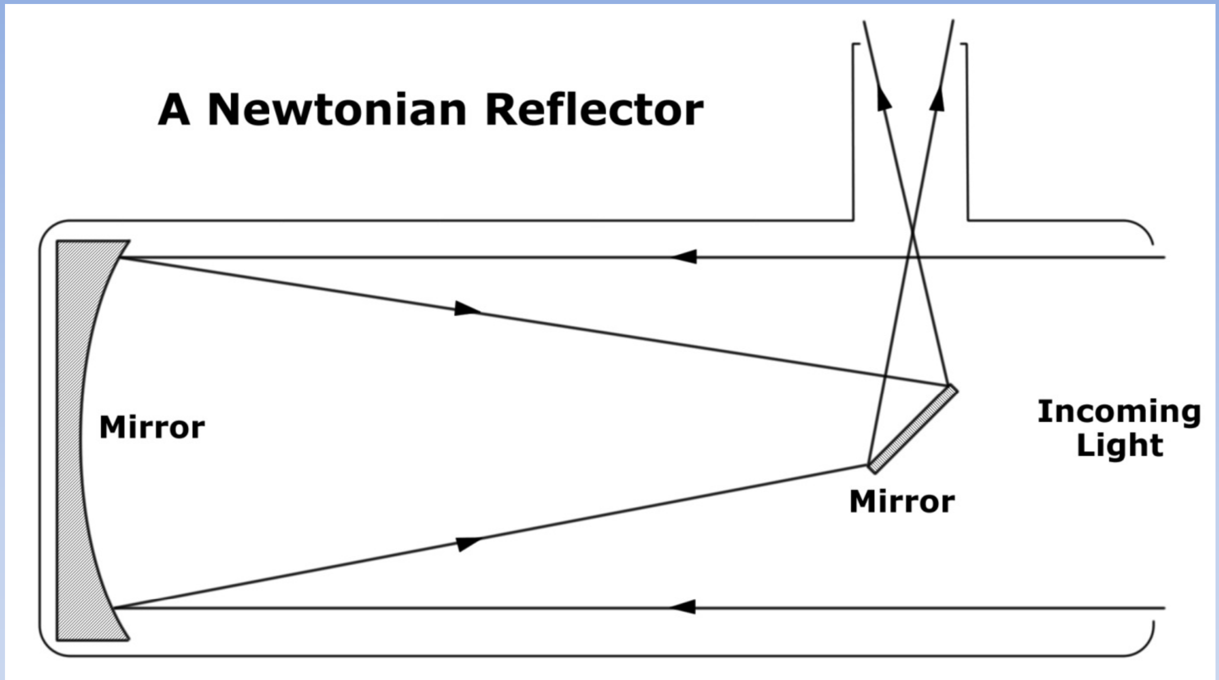


Parabolic Reflectors

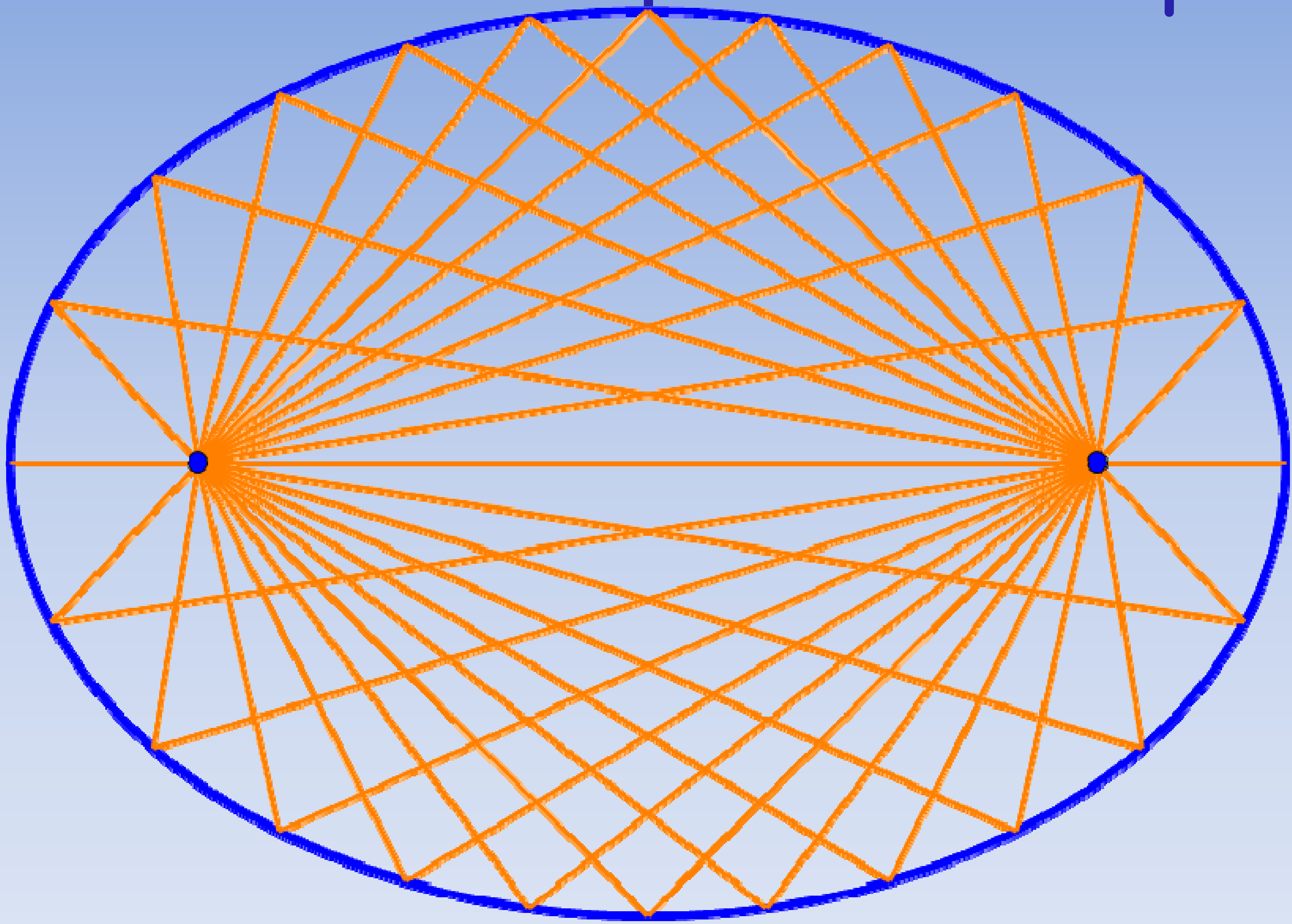




Reflecting Telescope



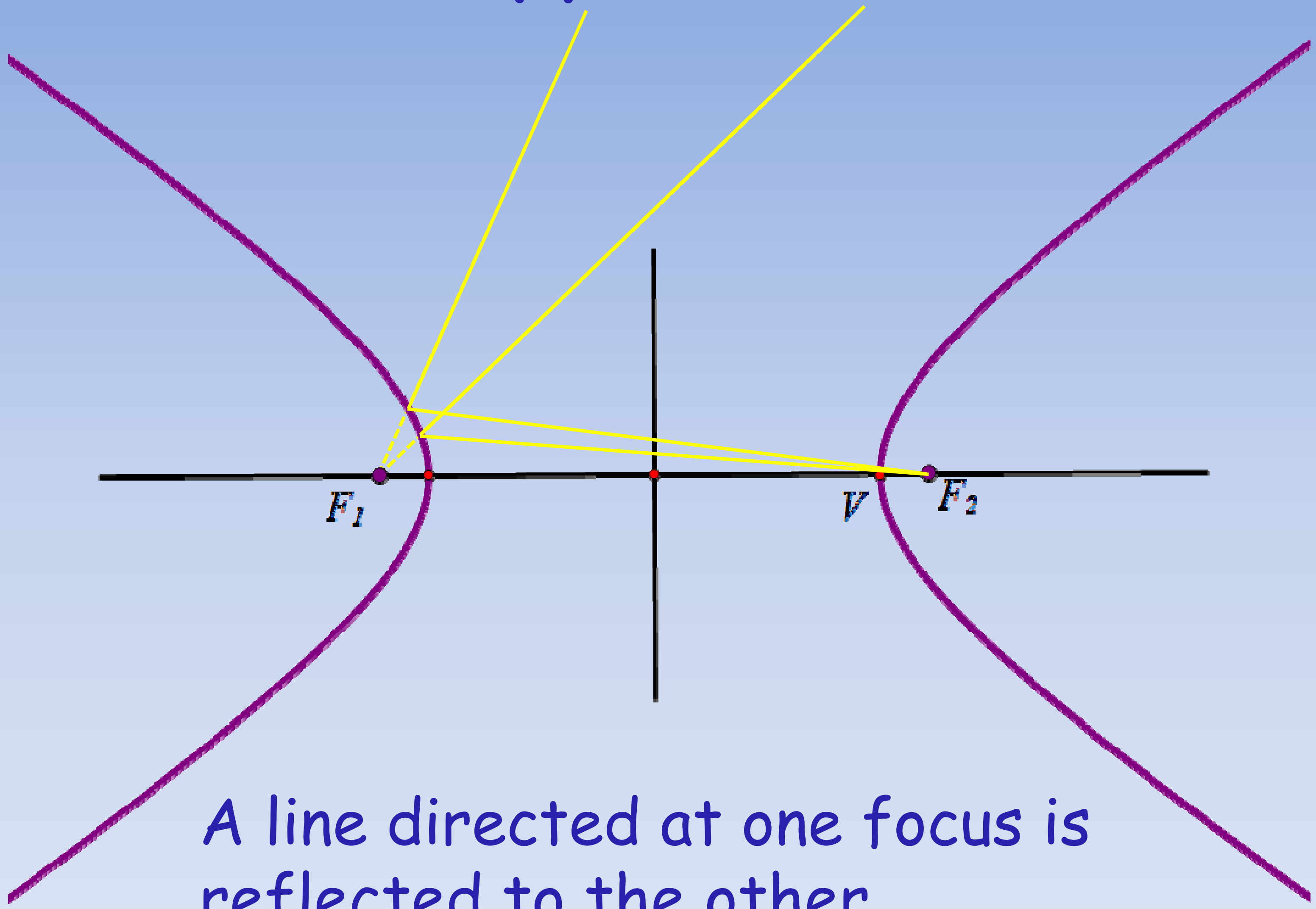
Reflection Properties: Ellipse



Reflection Properties: Ellipse

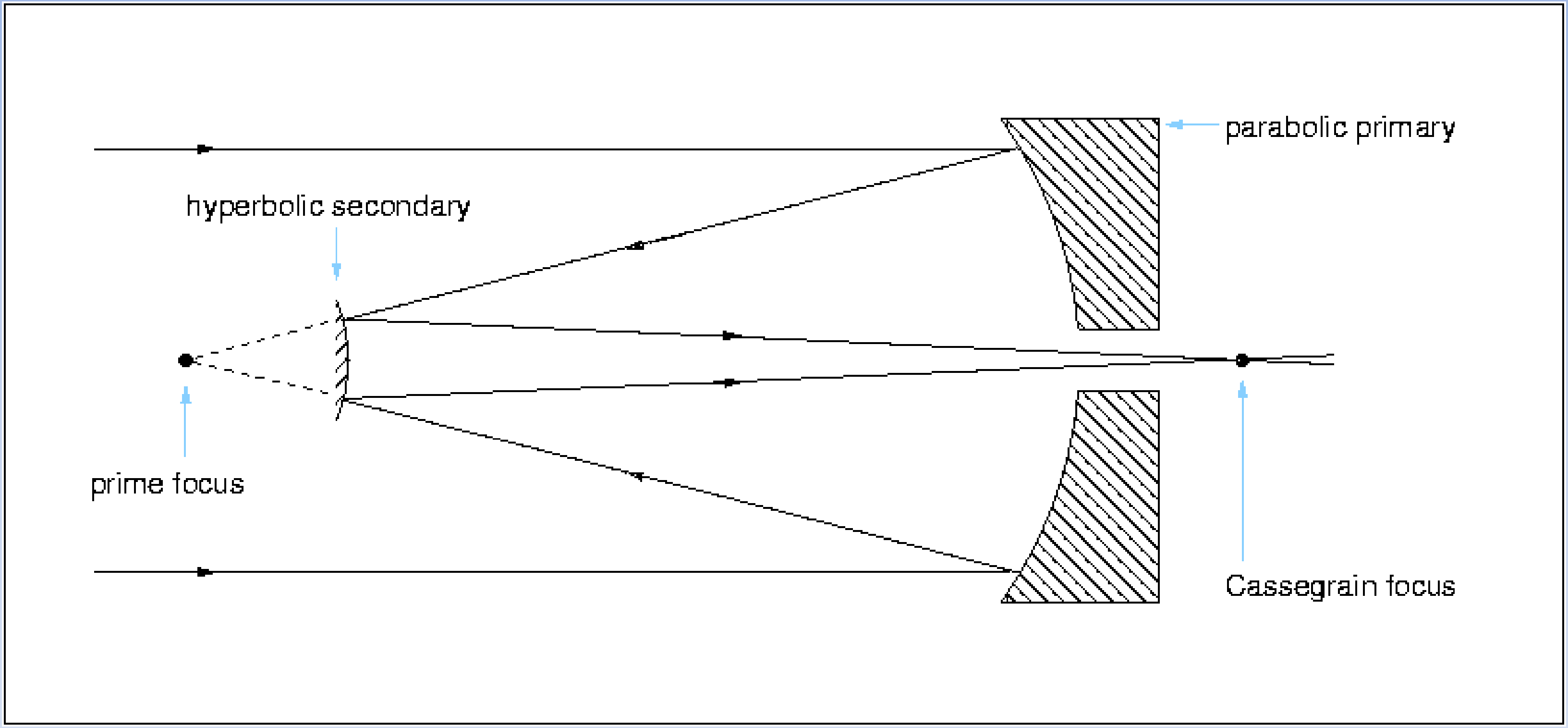
Lookup LITHOTRIPTER
WHISPERING CHAMBER

Hyperbola



A line directed at one focus is reflected to the other

Cassegrain Telescope



Rotation of Axes

General quadratic equation takes the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

How can we tell what conic it represents?

If B = 0, then completing the square will tell us.

Translation of Axes

What conic section is represented by:

$$9x^2 + 16y^2 - 18x + 64y - 71 = 0$$

$$9(x^2 - 2x) + 16(y^2 + 4y) - 71 = 0$$
$$9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9 + 16(4)$$
$$9(x - 1)^2 + 16(y + 2)^2 = 144$$

$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

Rotation of Axes

If $B \neq 0$, then a rotation is required to remove the xy term.

The standard rotation of axes is
$x = x' \cos \alpha - y' \sin \alpha$ and $y = x' \sin \alpha + y' \cos \alpha$

The equation becomes:
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$A(x' \cos \alpha - y' \sin \alpha)^2 + B(x' \cos \alpha - y' \sin \alpha)(x' \sin \alpha + y' \cos \alpha) + C(x' \sin \alpha + y' \cos \alpha)^2 + D(x' \cos \alpha - y' \sin \alpha) + E(x' \sin \alpha + y' \cos \alpha) + F = 0$

Rotation of Axes

$A((x')^2 \cos^2 \alpha - 2x'y' \cos \alpha \sin \alpha + (y')^2 \sin^2\alpha) + B((x')^2 \cos \alpha \sin \alpha - x'y' \sin^2\alpha + x'y' \cos^2\alpha + (y')^2 \cos \alpha \sin \alpha) + C((x')^2 \sin^2 \alpha + 2x'y'\sin \alpha \cos \alpha + (y')^2 \cos^2 \alpha) + D(x' \cos \alpha - y' \sin \alpha) + E(x' \sin \alpha + y' \cos \alpha) + F = 0$

$(A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha)(x')^2 +$
$(B(\cos^2\alpha - \sin^2\alpha) + (2C - 2A)\sin \alpha \cos \alpha)\, x'y' +$
$(C \cos^2 \alpha - B \cos \alpha \sin \alpha + A \sin^2 \alpha)(y')^2 +$
$(D \cos \alpha + E \sin \alpha)x' + (E \cos \alpha - D \sin \alpha)y' +$
$F = 0$

Rotation of Axes

In order to have no "cross term" we need

$(B(\cos^2\alpha - \sin^2\alpha) + (2C - 2A)\sin\alpha\cos\alpha) = 0$

$B\cos 2\alpha + (C - A)\sin 2\alpha = 0$

$(A - C)\sin 2\alpha = B\cos 2\alpha$

$$\tan 2\alpha = \frac{B}{A-C} \quad \text{or} \quad \cot 2\alpha = \frac{A-C}{B}$$

Rotation of Axes

What conic section is represented by:

$x^2 + 4xy + 4y^2 - 30x - 90y - 450 = 0$?

Angle of rotation: $\tan 2a = B/(A-C) = -4/3$

Then $\cos 2a = -3/5$

Then

$$\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} = \sqrt{\frac{1+(-3/5)}{2}} = \frac{1}{\sqrt{5}}$$

$$\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}} = \sqrt{\frac{1-(-3/5)}{2}} = \frac{2}{\sqrt{5}}$$

Rotation of Axes

Now, from before we have:

$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 5$

$B' = B\cos 2\alpha + (C - A)\sin 2\alpha = 0$

$C' = C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha = 0$

$D' = D\cos\alpha + E\sin\alpha = -42\sqrt{5}$

$E' = E\cos\alpha - D\sin\alpha = -6\sqrt{5}$

$F' = 450$

So the equation after rotation becomes:

$$5(x')^2 - 42\sqrt{5}x' - 6\sqrt{5}y' + 450 = 0$$

$$\left(x' - \frac{21}{\sqrt{5}}\right)^2 = \frac{6}{\sqrt{5}}\left(y' - \frac{3\sqrt{5}}{10}\right)$$

Classification of Quadratics

For the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Define the discriminant to be $B^2 - 4AC$

Theorem:

1) if $B^2 - 4AC > 0$, the graph is a hyperbola;
2) if $B^2 - 4AC = 0$, the graph is a parabola;
3) if $B^2 - 4AC < 0$, the graph is an ellipse,

except for degenerate cases