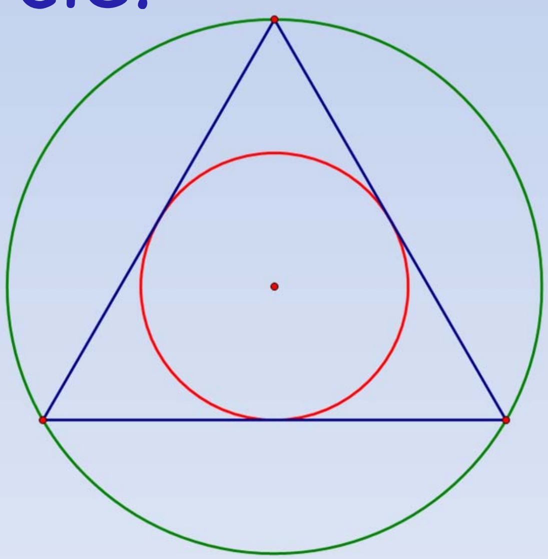
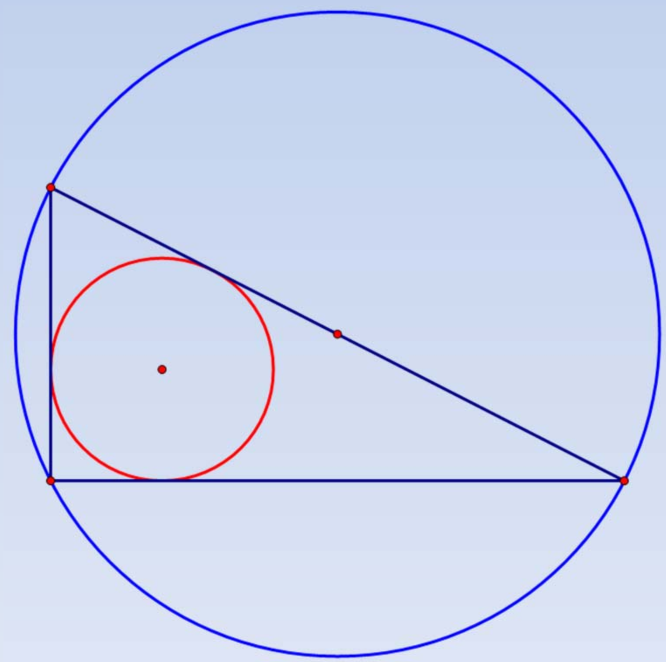
The Incircle and Inradius

MA 341 – Topics in Geometry
Lecture 15

Inscribed Circles

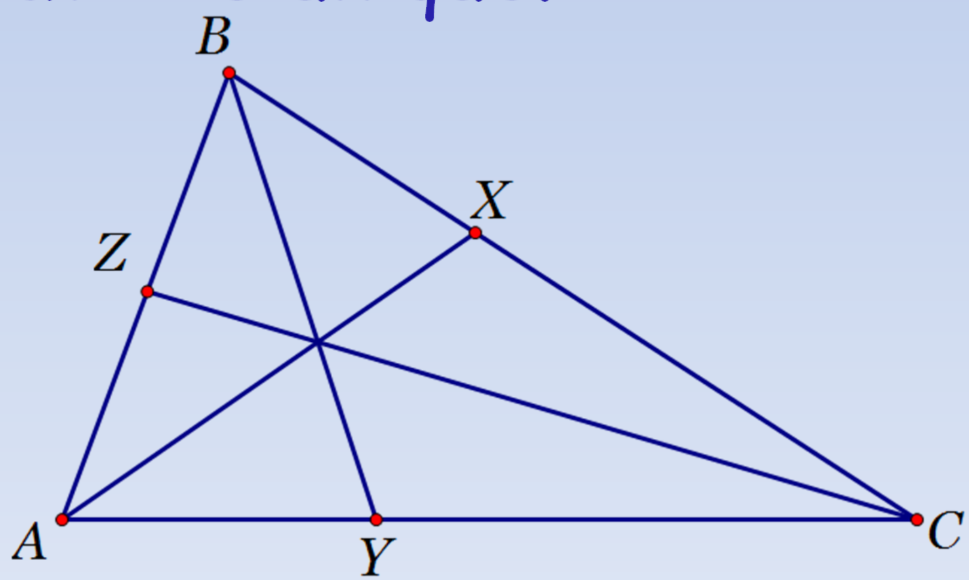
- We know that every triangle has a circumscribing circle.
- Does every triangle have an inscribed circle?





Incircle, Incenter, Inradius

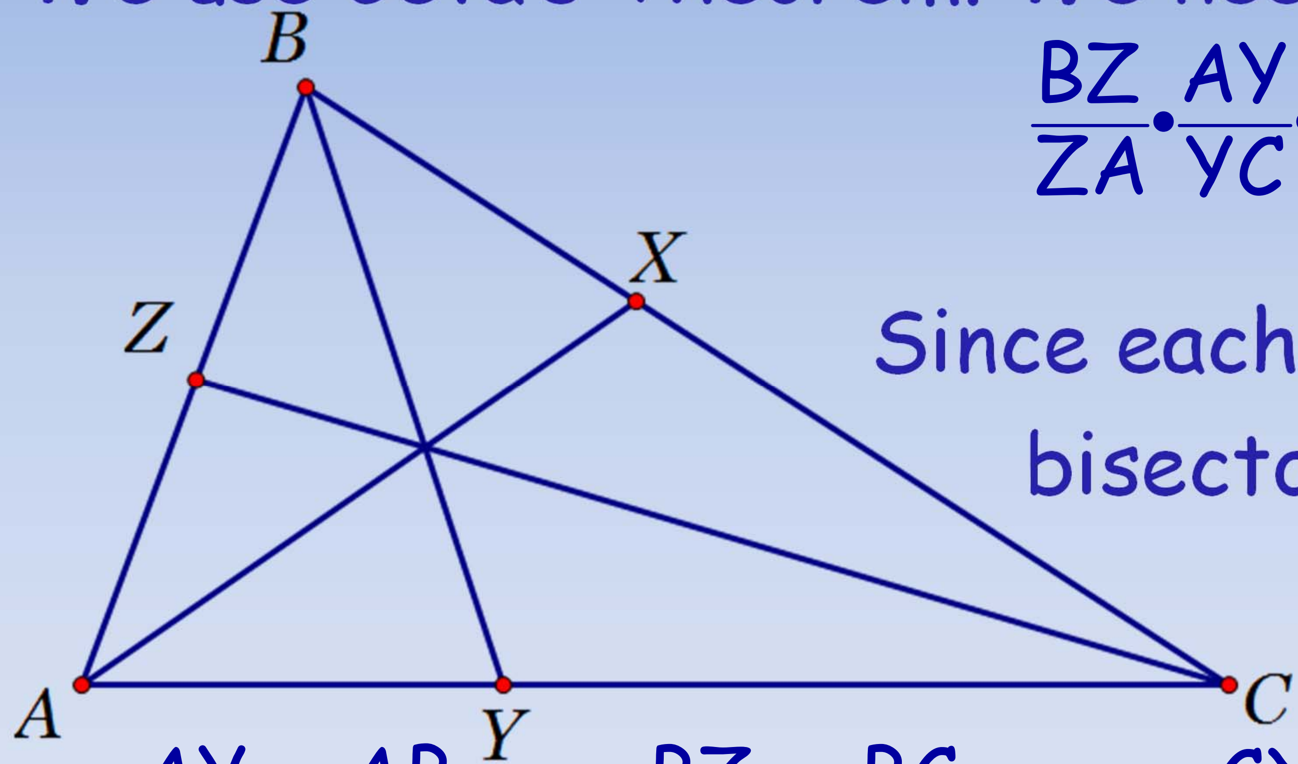
The three angle bisectors of a triangle are concurrent at a point I. This point is equidistant from the sides and the circle centered at I is unique.



Incircle, Incenter, Inradius

We use Ceva's Theorem. We need to show:

$$\frac{BZ}{ZA}\bullet\frac{AY}{YC}\bullet\frac{CX}{XB}=1$$



Since each is an angle bisector, we know

$$\frac{AY}{YC}=\frac{AB}{BC} \qquad \frac{BZ}{ZA}=\frac{BC}{AC} \qquad \frac{CX}{XB}=\frac{AC}{AB}$$

Incircle, Incenter, Inradius

$$\frac{BZ}{ZA}\bullet\frac{AY}{YC}\bullet\frac{CX}{XB}=\frac{BC}{AC}\bullet\frac{AB}{BC}\bullet\frac{AC}{AB}=1$$

By Ceva's Theorem the angle bisectors are concurrent at a point, I.

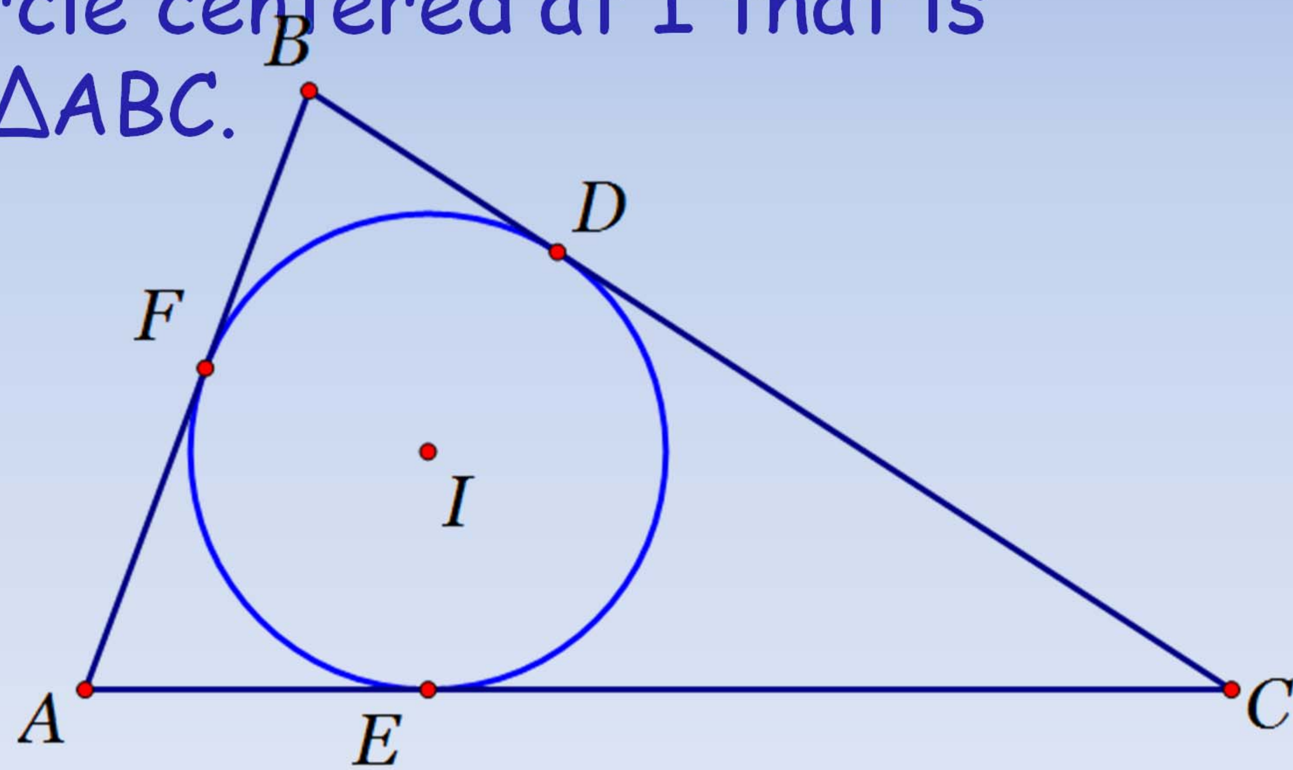
Drop a perpendicular from I to AC and from I to BC, intersecting at points E and D.

$\angle$ECI = $\angle$DCI, $\angle$IEC = $\angle$IDC and IC=IC. By AAS ΔIEC=ΔIDC $\Rightarrow$ IE=ID.

Incircle, Incenter, Inradius

Similarly we can show that IE=IF.

Hence I is equidistant from the sides and there is a circle centered at I that is inscribed in ΔABC.



Inradius

Theorem: If K is the area, s the semiperimeter, and r the inradius of ΔABC, then $K = rs$.

Consider ΔAIC.

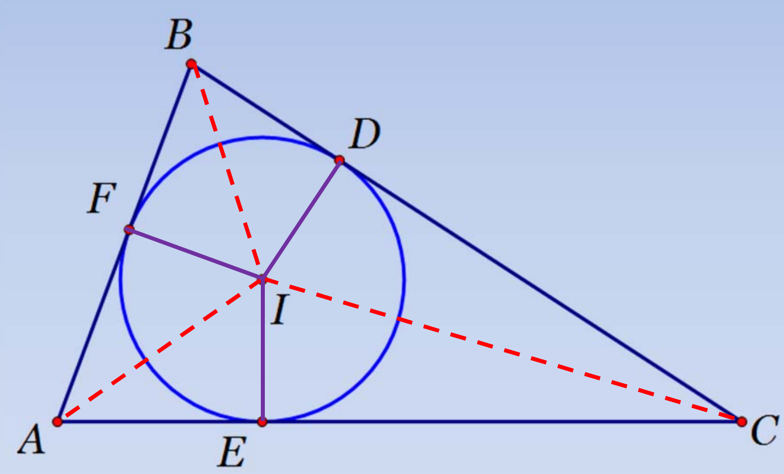
$K_{AIC} = \frac{1}{2}(IE)(AC) = \frac{1}{2}rb$

Also,

$K_{AIB} = \frac{1}{2}(IF)(AB) = \frac{1}{2}rc$

$K_{BIC} = \frac{1}{2}(ID)(BC) = \frac{1}{2}ra$

$K = K_{AIC} + K_{AIB} + K_{BIC} = \frac{1}{2}(a+b+c)r = rs$



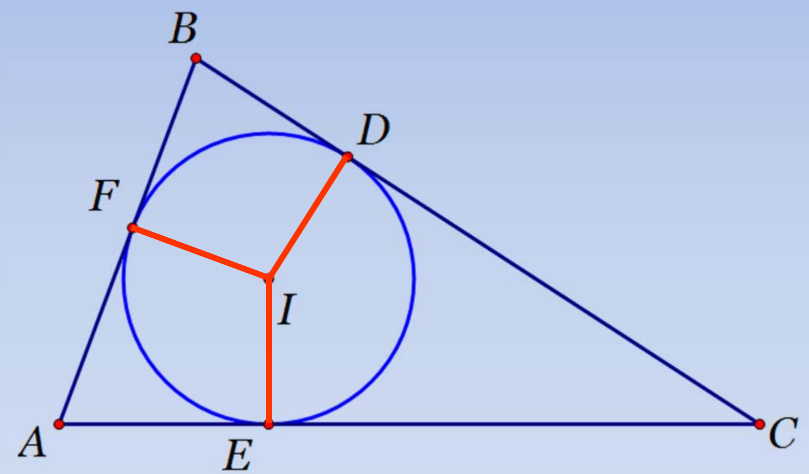
Inradius

K = rs.

$$r = \frac{K}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Points of Tangency

Can we find the points of tangency, D, E, and F?



We know CE=CD, AF=AE and BD=BF. Let

x=CE=CD

y=AE=AF

z=BD=BF

$$x + y = b$$

$$x + z = a$$

$$y + z = c$$

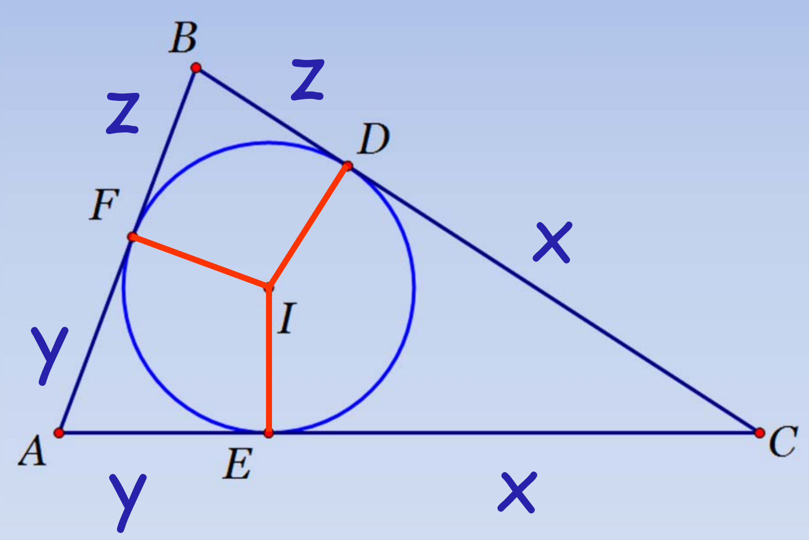
Points of Tangency

Solving, we get:

$$x = \frac{a+b-c}{2} = s-c$$

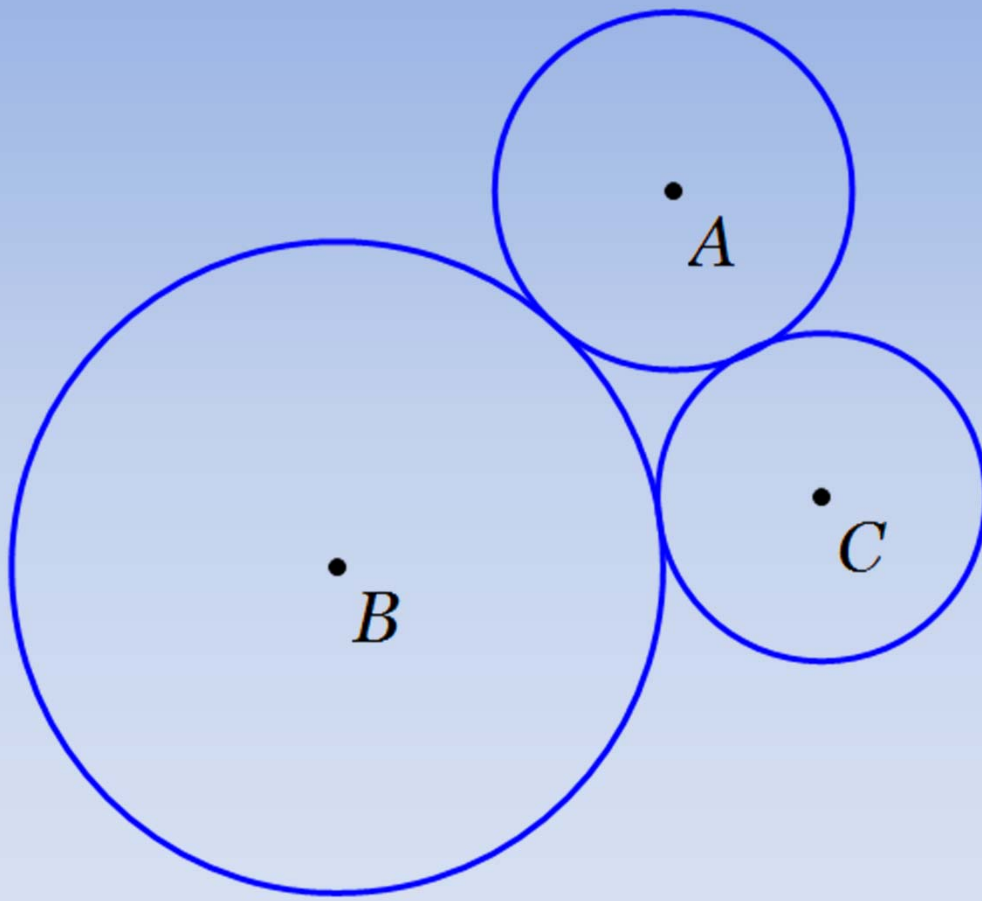
$$y = \frac{b+c-a}{2} = s-a$$

$$z = \frac{a+c-b}{2} = s-b$$



Extension: Tangent Circles

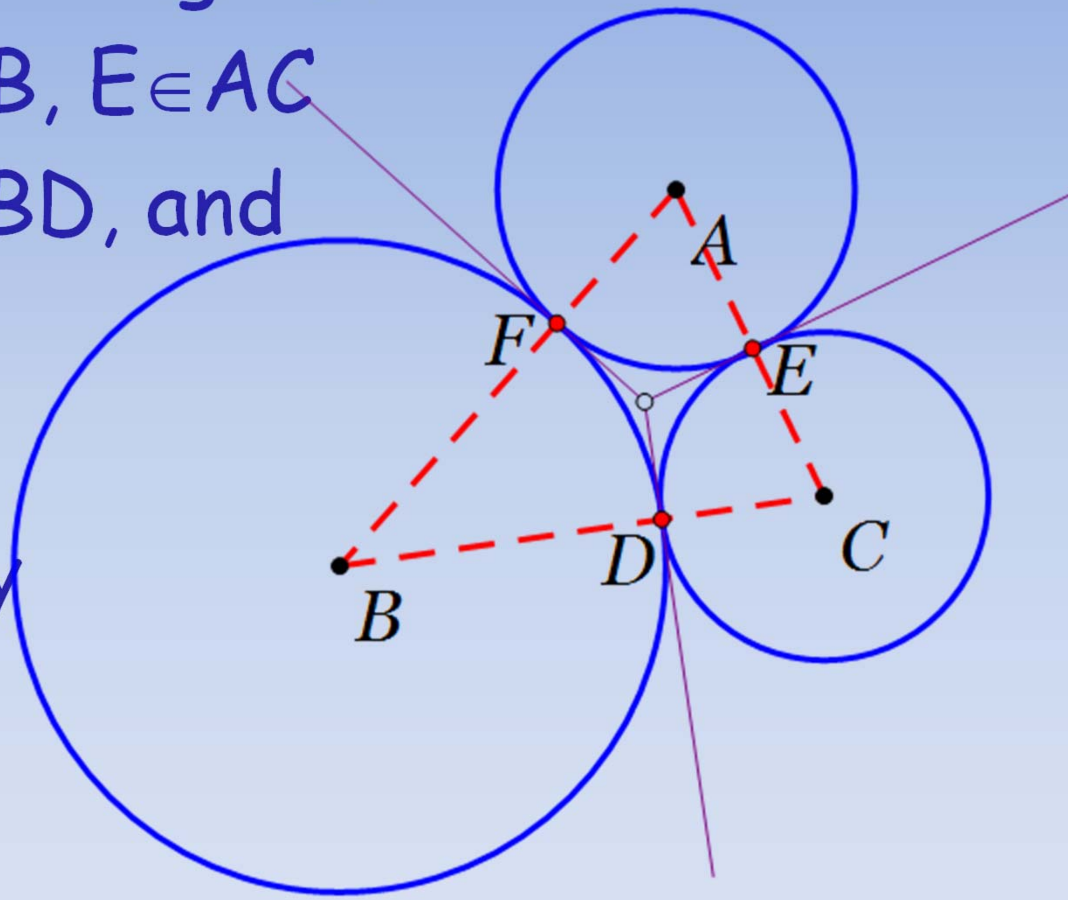
If we have three mutually, externally, pairwise tangent circles, then their common tangent lines are concurrent.



BD and CD are perpendicular to the common tangent through D.

Thus, D∈BC, F∈AB, E∈AC

Now, AF=AE, BF=BD, and CE=CD. We see That D,E,F are points of tangency of the incircle of ΔABC



More from Incircle

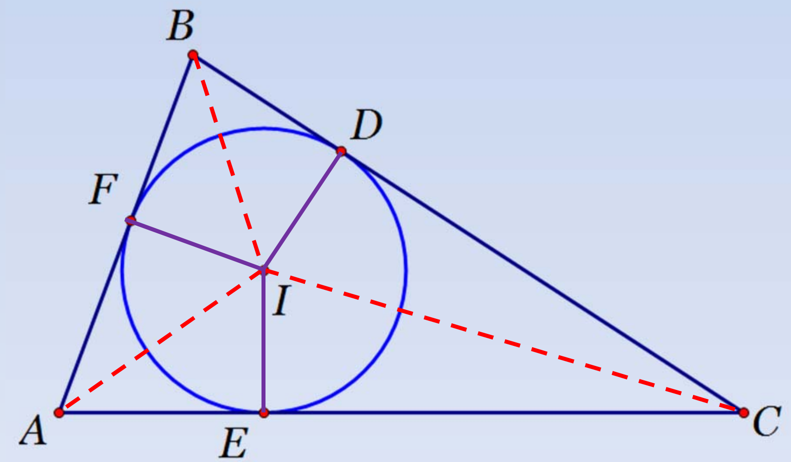
Law of Tangents: If a, b, c, and s are as usual, then

$$\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

In ΔCIE,

$$\tan\left(\frac{C}{2}\right) = \frac{IE}{CE} = \frac{r}{s-c}$$

$$= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$



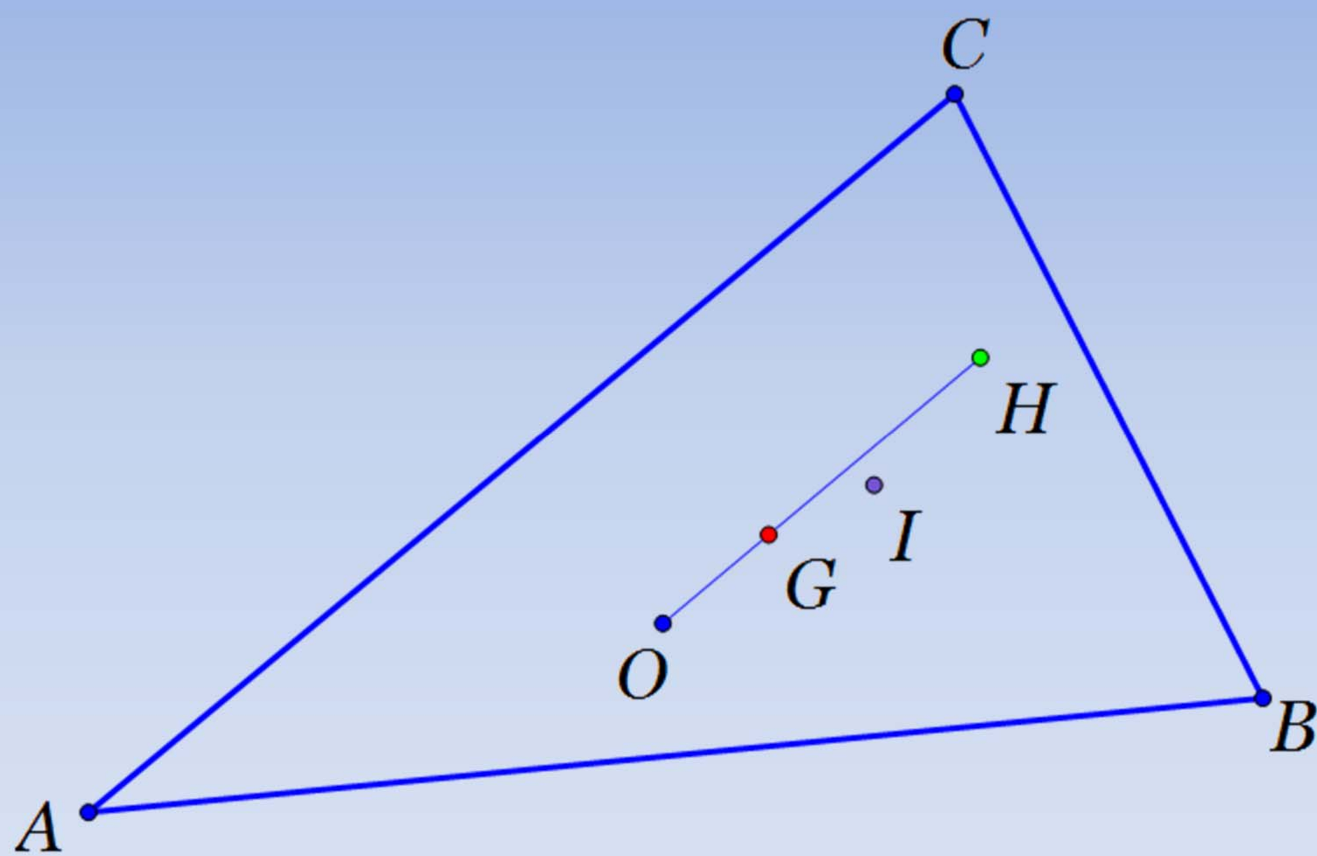
Incenter Coordinates

Cartesian coordinates

If $A=(x_a,y_a)$, $B=(x_b,y_b)$, $C=(x_c,y_c)$ and sides are a,b,c then

$$I=\left(\frac{ax_a+bx_b+cx_c}{a+b+c},\frac{ay_a+by_b+cy_c}{a+b+c}\right)$$

Incenter and Euler Segment



Euler's Theorem

Let d=OI, the distance from the circumcenter to the incenter.

$$d^2 = R(R-2r)$$

where R=circumradius and r=inradius.

Circumcenter Coordinates

Cartesian coordinates

If $A=(x_a,y_a)$, $B=(x_b,y_b)$, $C=(x_c,y_c)$ and sides are a,b,c then

$$C=\left(\frac{(x_a^2+y_a^2)(y_b-y_c)+(x_b^2+y_b^2)(y_c-y_a)+(x_c^2+y_c^2)(y_a-y_b)}{2(x_a(y_b-y_c)+x_b(y_c-y_a)+x_c(y_a-y_b))},\right.$$

$$\left.\frac{(x_a^2+y_a^2)(x_c-x_b)+(x_b^2+y_b^2)(x_a-x_c)+(x_c^2+y_c^2)(x_b-x_a)}{2(x_a(y_b-y_c)+x_b(y_c-y_a)+x_c(y_a-y_b))}\right)$$

Centroid Coordinates

Cartesian coordinates

If $A=(x_a,y_a)$, $B=(x_b,y_b)$, $C=(x_c,y_c)$ then

$$G=\left(\frac{x_a+x_b+x_c}{3},\frac{y_a+y_b+y_c}{3}\right)$$

Orthocenter Coordinates

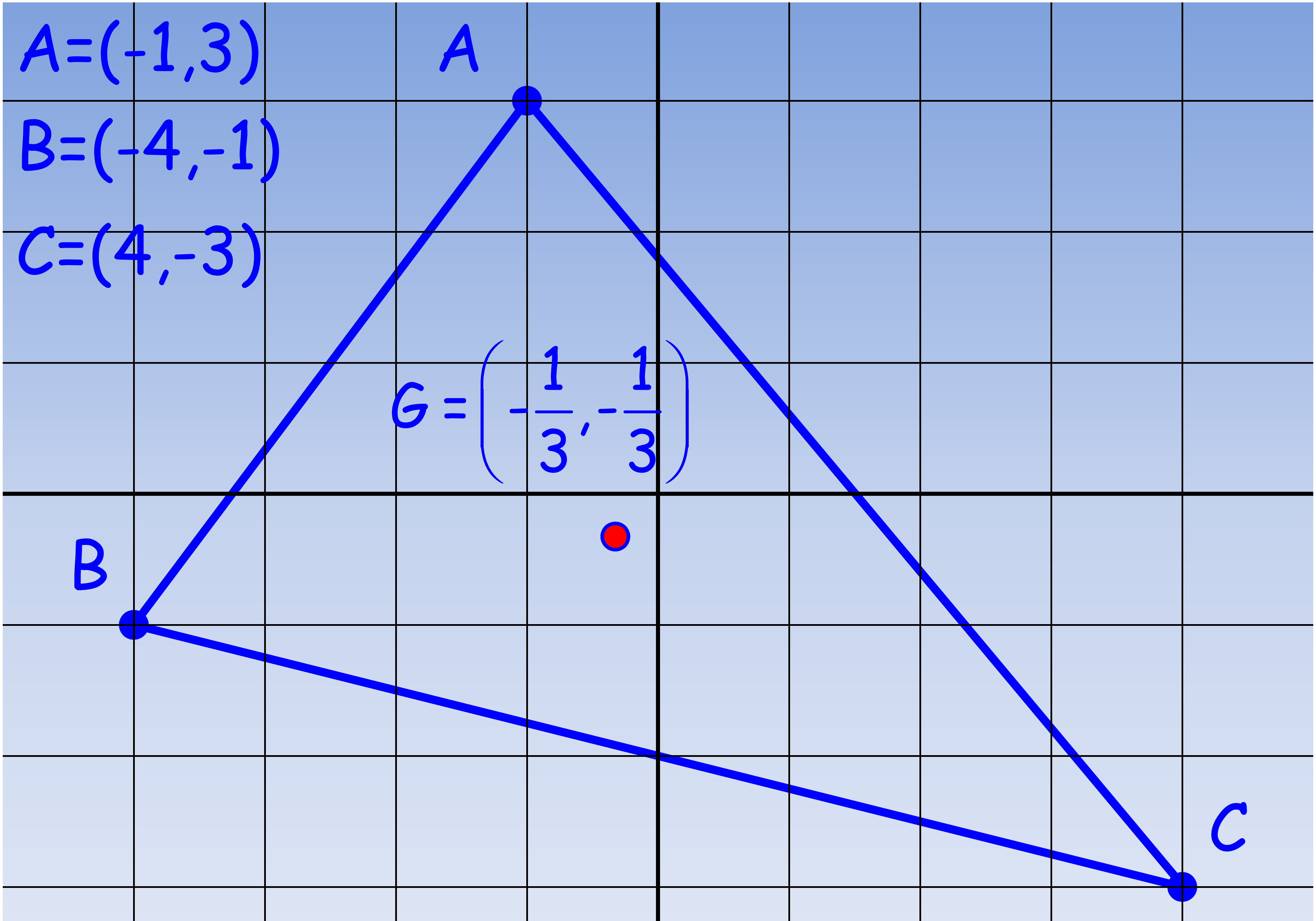
Cartesian coordinates

If $A=(x_a,y_a)$, $B=(x_b,y_b)$, $C=(x_c,y_c)$ then

$$H=(x_H,y_H)$$

$$x_H=\frac{(y_b-y_c)y_a^2+(y_c-y_a)y_b^2+(y_a-y_b)y_c^2+x_ay_a(x_b-x_c)+x_by_b(x_c-x_a)+x_cy_c(x_a-x_b)}{y_ax_b+y_bx_c+y_cx_a-x_by_c-x_cy_a-x_ay_b}$$

$$y_H=\frac{(x_c-x_b)x_a^2+(x_a-x_c)x_b^2+(x_b-x_a)x_c^2+x_ay_a(y_c-y_b)+x_by_b(y_a-y_c)+x_cy_c(y_b-y_a)}{y_ax_b+y_bx_c+y_cx_a-x_by_c-x_cy_a-x_ay_b}$$

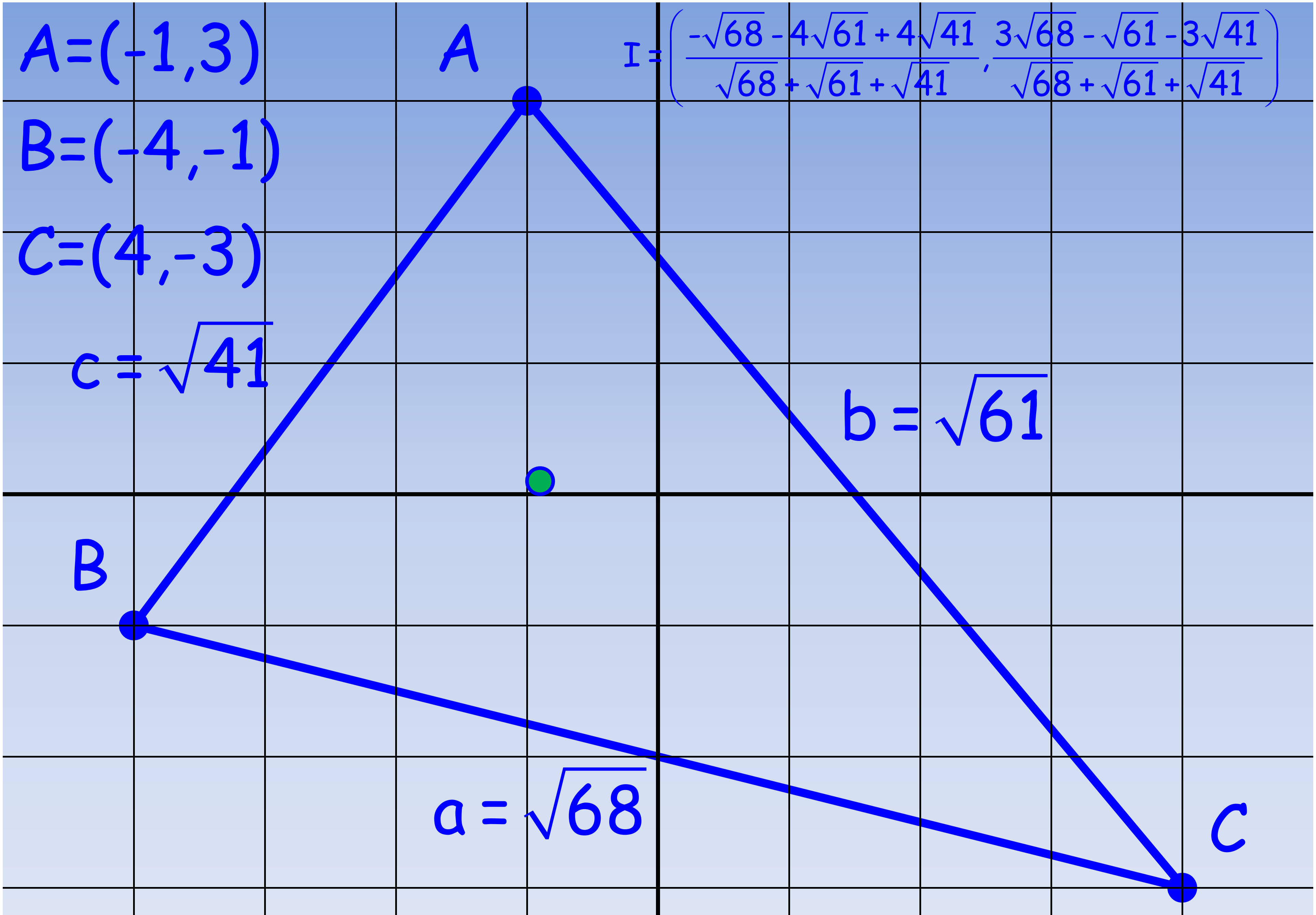

A=(-1,3)
B=(-4,-1)
C=(4,-3)
A
B
C
$G=\left(-\frac{1}{3},-\frac{1}{3}\right)$

$A=(-1,3)$

$B=(-4,-1)$

$C=(4,-3)$

$$I=\left(\frac{-\sqrt{68}-4\sqrt{61}+4\sqrt{41}}{\sqrt{68}+\sqrt{61}+\sqrt{41}},\frac{3\sqrt{68}-\sqrt{61}-3\sqrt{41}}{\sqrt{68}+\sqrt{61}+\sqrt{41}}\right)$$



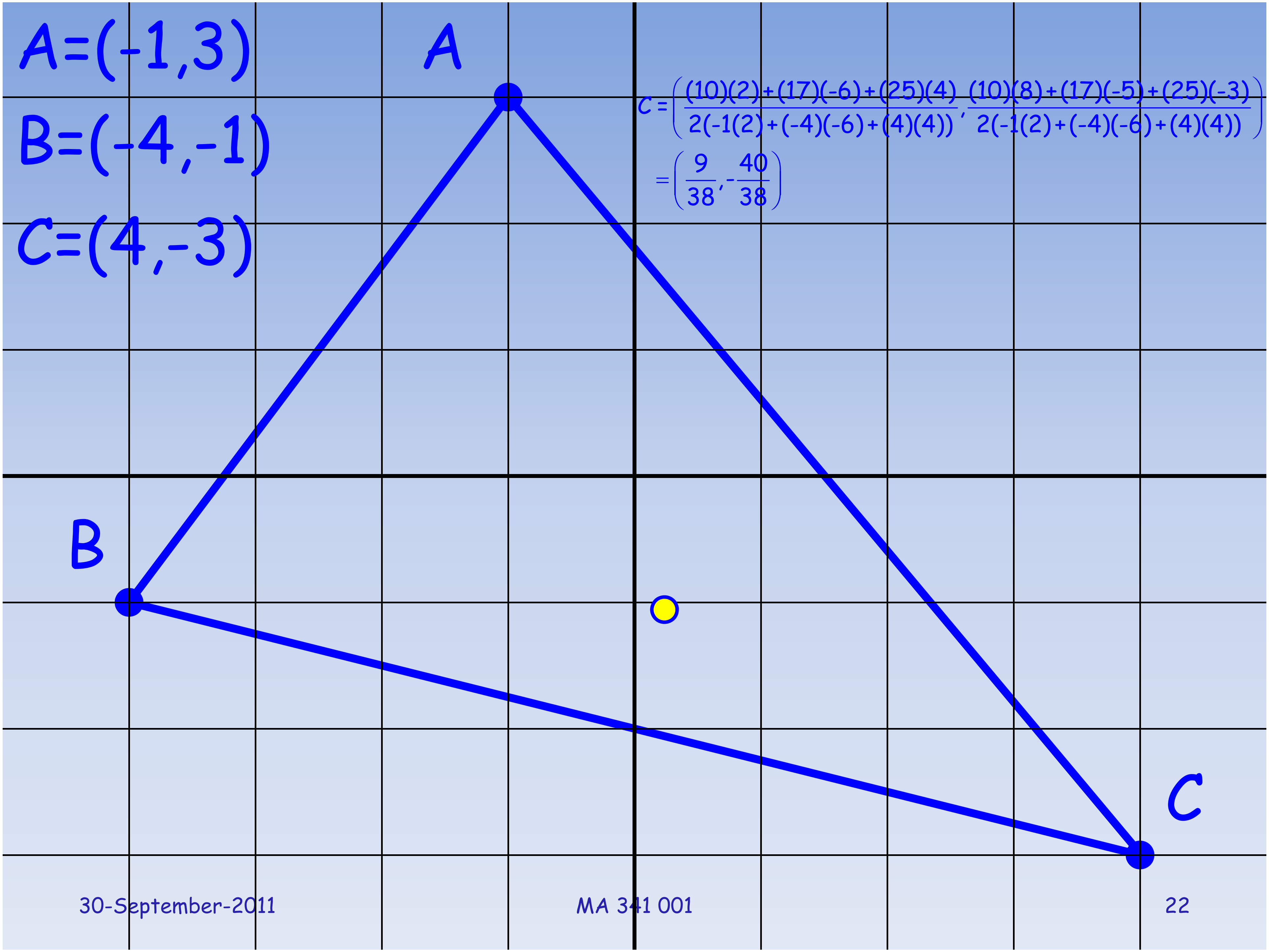
A=(-1,3)

B=(-4,-1)

C=(4,-3)

$$C=\left(\frac{(10)(2)+(17)(-6)+(25)(4)}{2(-1(2)+(-4)(-6)+(4)(4))},\frac{(10)(8)+(17)(-5)+(25)(-3)}{2(-1(2)+(-4)(-6)+(4)(4))}\right)$$

$$=\left(\frac{9}{38},-\frac{40}{38}\right)$$



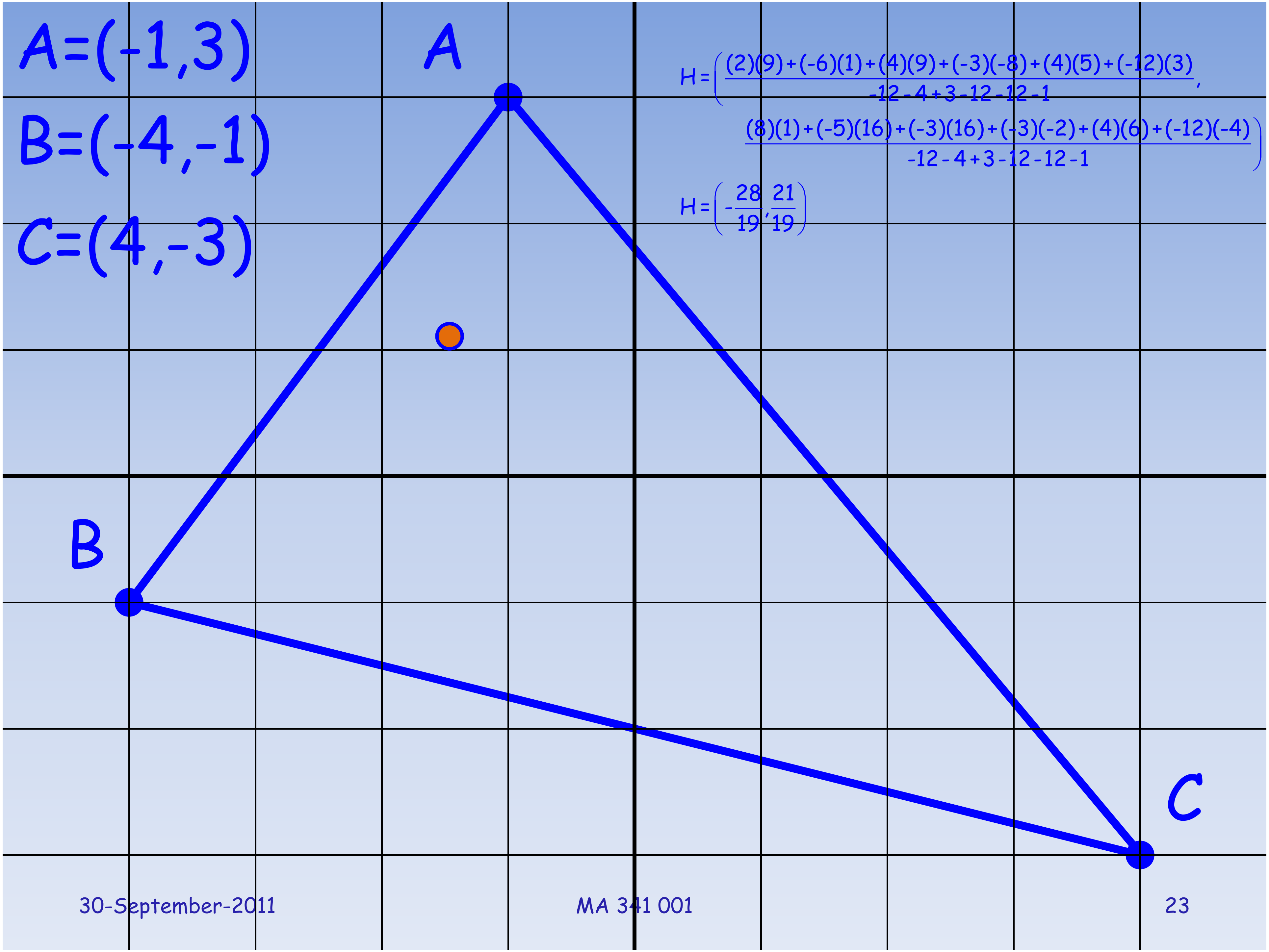
A=(-1,3)

B=(-4,-1)

C=(4,-3)

$$H=\left(\frac{(2)(9)+(-6)(1)+(4)(9)+(-3)(-8)+(4)(5)+(-12)(3)}{-12-4+3-12-12-1},\frac{(8)(1)+(-5)(16)+(-3)(16)+(-3)(-2)+(4)(6)+(-12)(-4)}{-12-4+3-12-12-1}\right)$$

$$H=\left(-\frac{28}{19},\frac{21}{19}\right)$$



$$I=\left(\frac{-\sqrt{68}-4\sqrt{61}+4\sqrt{41}}{\sqrt{68}+\sqrt{61}+\sqrt{41}},\frac{3\sqrt{68}-\sqrt{61}-3\sqrt{41}}{\sqrt{68}+\sqrt{61}+\sqrt{41}}\right)$$