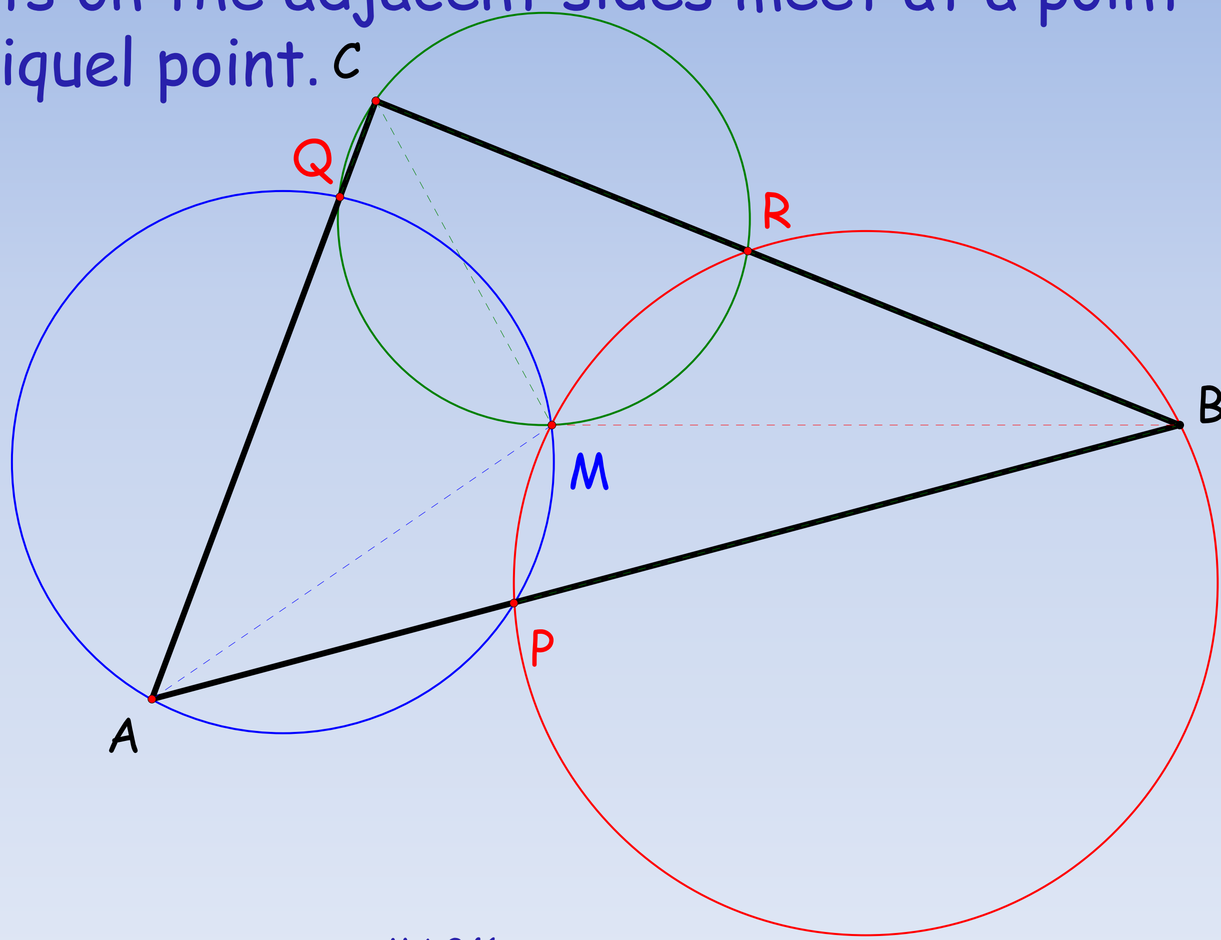
# Pedal Triangles and the Simson Line

MA 341 – Topics in Geometry
Lecture 18

# Miquel's Theorem

If P, Q, and R are on BC, AC, and AB respectively, then the three circles determined by a vertex and the two points on the adjacent sides meet at a point called the Miquel point.



# Miquel's Theorem

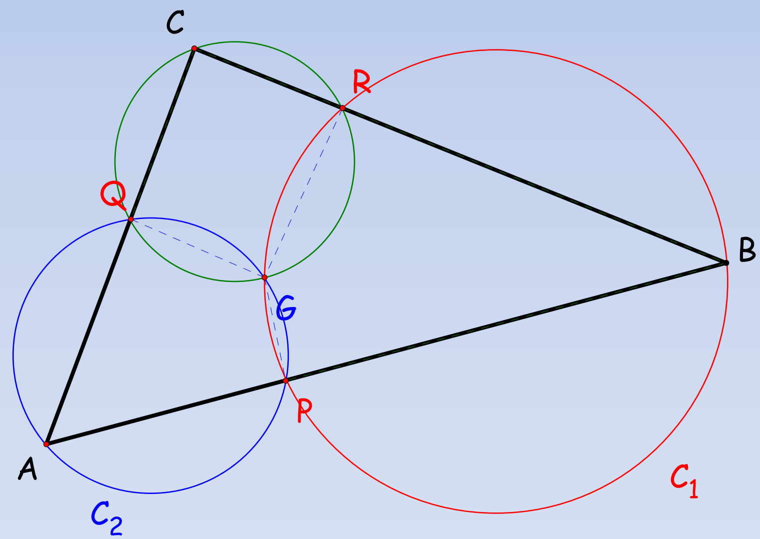
Let $\triangle ABC$ be our triangle and let P,Q, and R be the points on the sides of the triangle. Construct the circles of the theorem. Consider two of the circles, $C_1$ and $C_2$, that pass through P. They intersect at P, so they must intersect at a second point, call it G.

In circle $C_2$

$\angle QGP + \angle QAP = 180$

In circle $C_1$

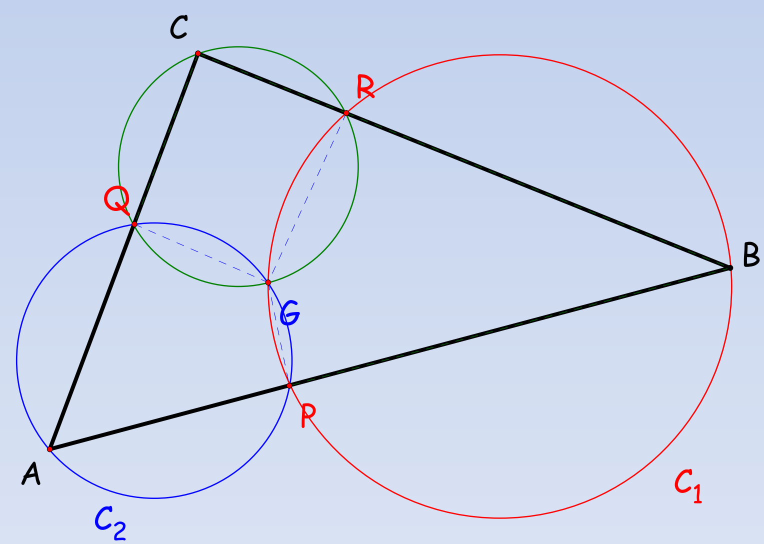
$\angle RGP + \angle RBP = 180$



# Miquel's Theorem

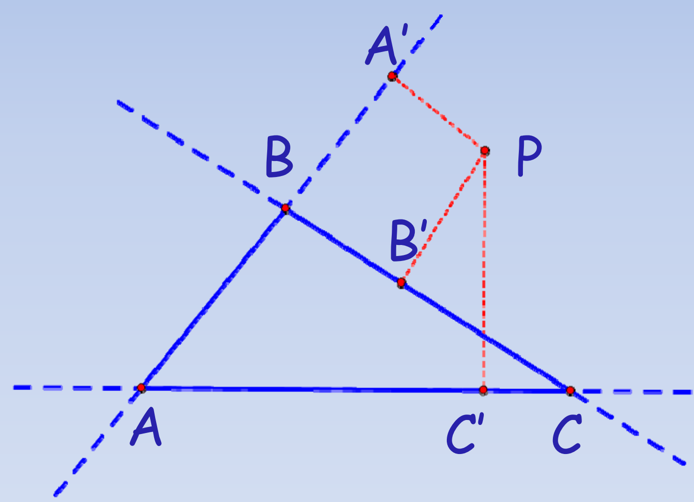
$$\measuredangle QGP + \measuredangle QGR + \measuredangle RGP = 360$$
$$(180 - \measuredangle A) + \measuredangle QGR + (180 - \measuredangle B) = 360$$
$$\measuredangle QGR = \measuredangle A + \measuredangle B$$
$$= 180 - \measuredangle C$$

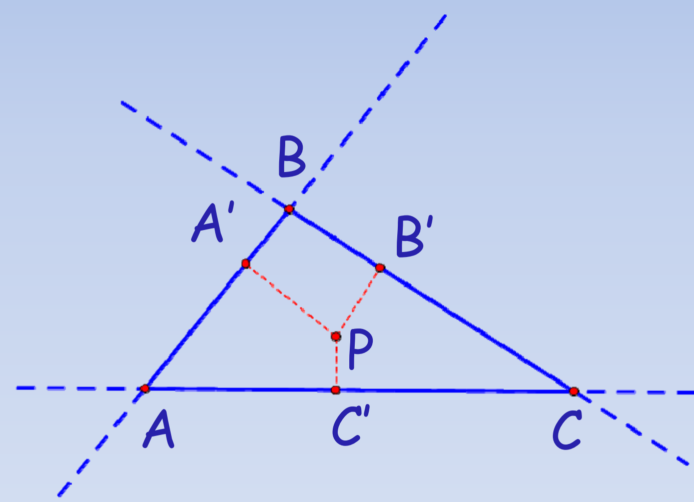
Thus, $\angle QGR$ and $\angle C$ are supplementary and so Q, G, R, and C are concyclic. These circle then intersect in one point.



# Pedal Triangle

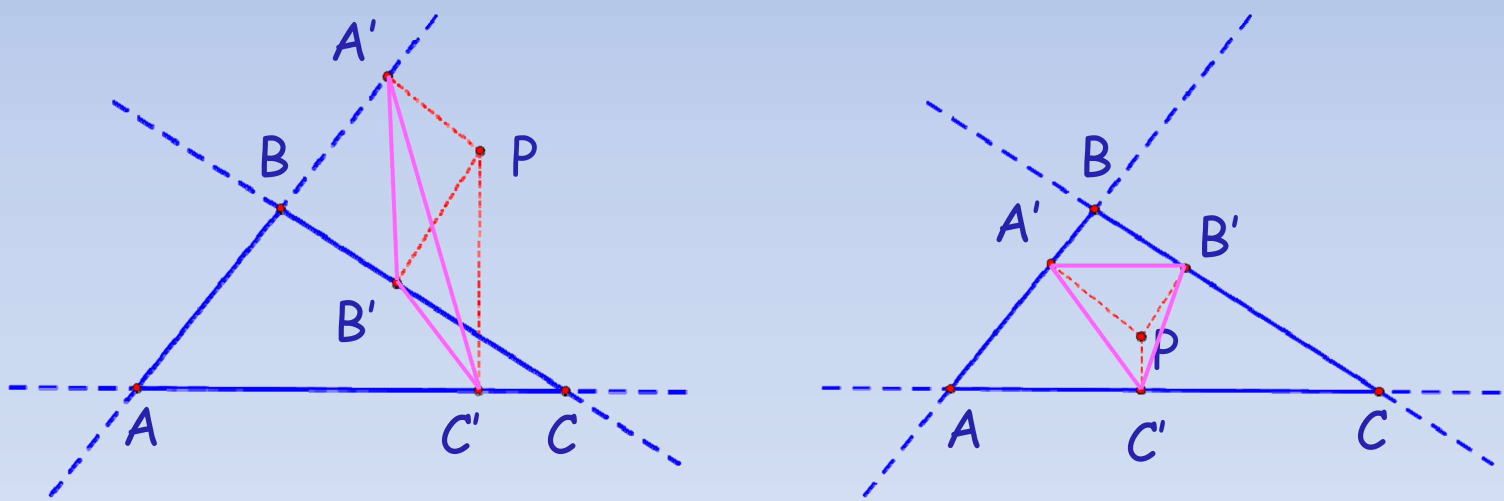
For any triangle $\Delta ABC$ and any point P, let $A'$, $B'$, $C'$ be the feet of the perpendiculars from P to the (extended) sides of $\Delta ABC$.



# Pedal Triangle

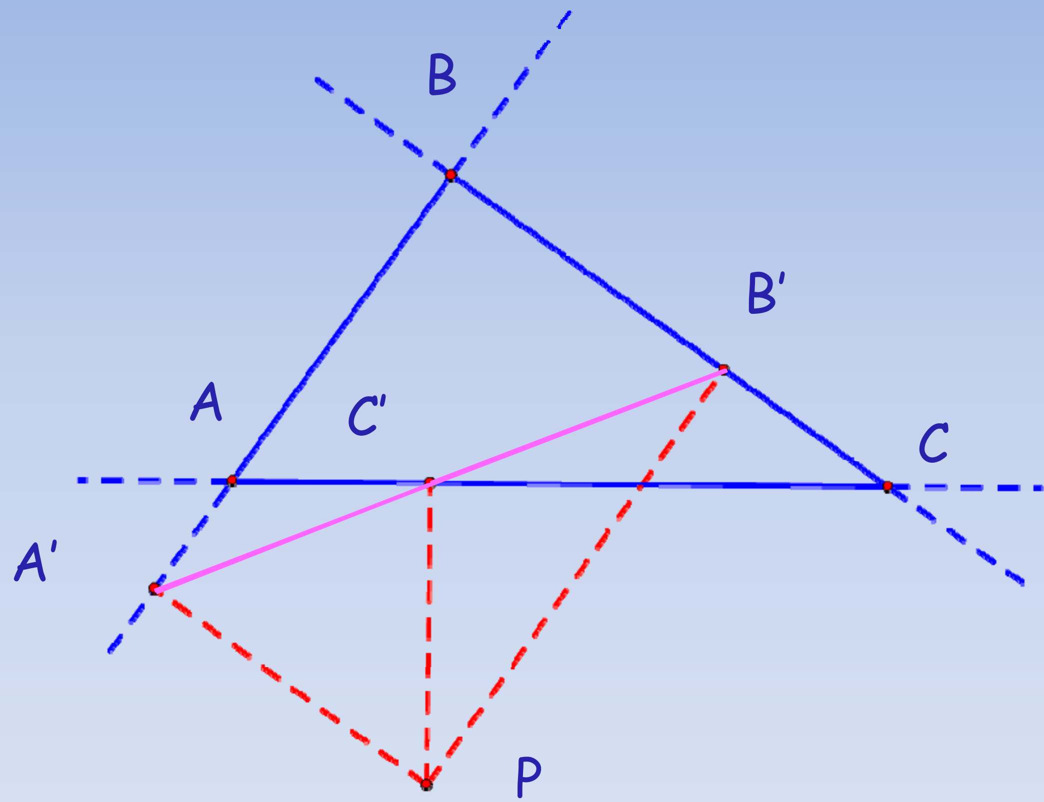
Form the triangle $\Delta A'B'C'$.




Do we always get a triangle?

# Pedal Triangle

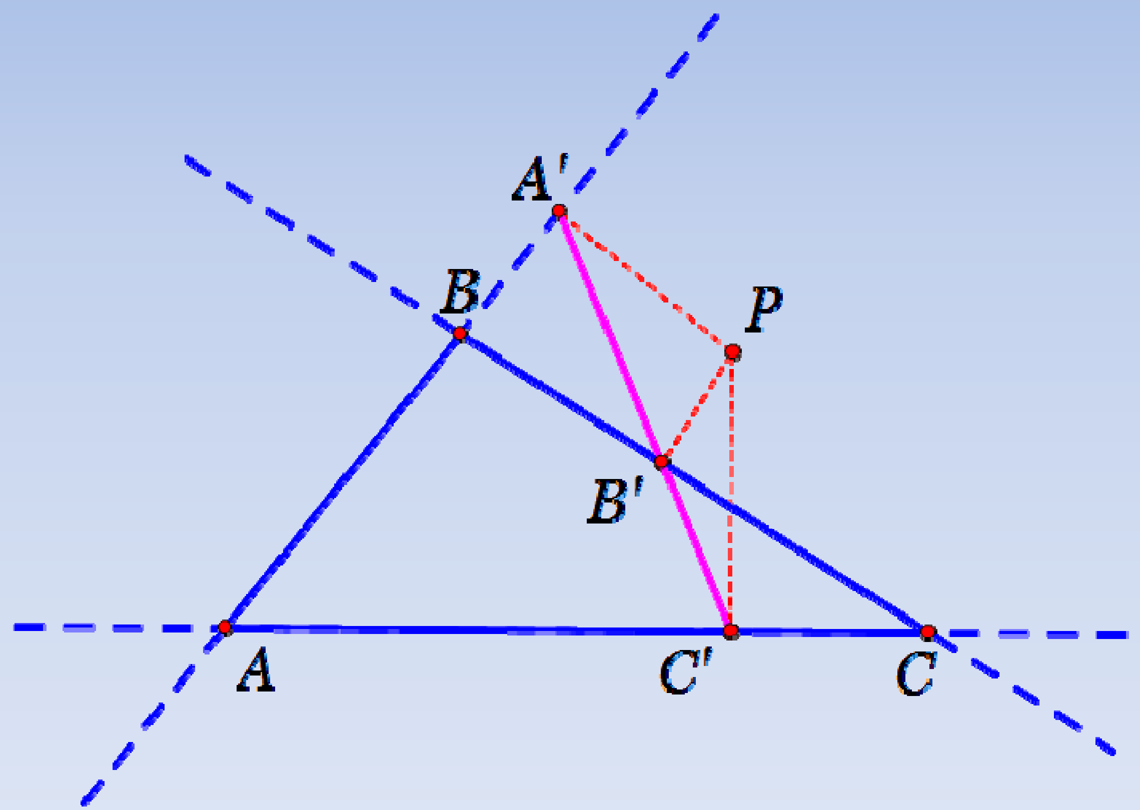
Form the triangle $\Delta A'B'C'$.




What is it with P?

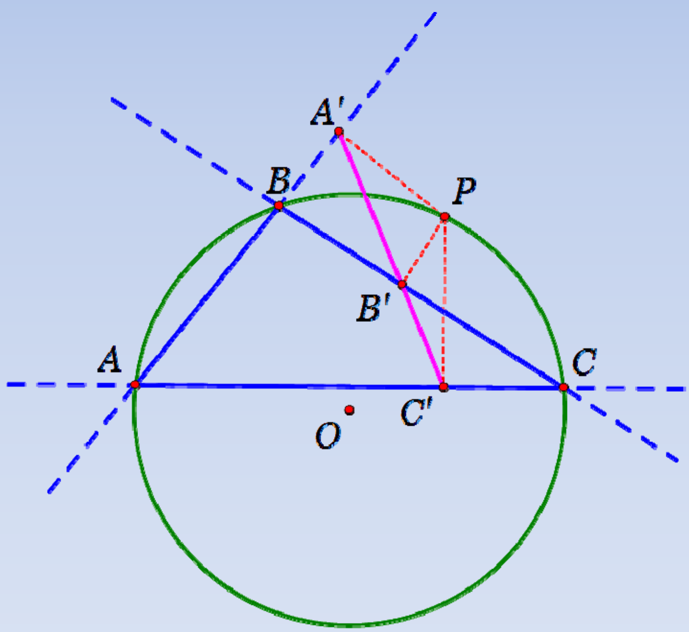
# Pedal Triangle

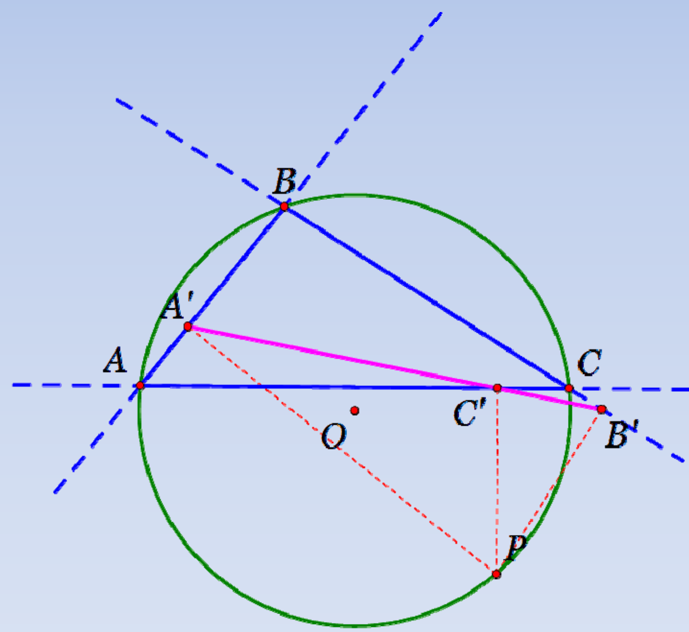
Can we characterize the points where the pedal triangle is a "degenerate triangle"?



# Simson-Wallace Line

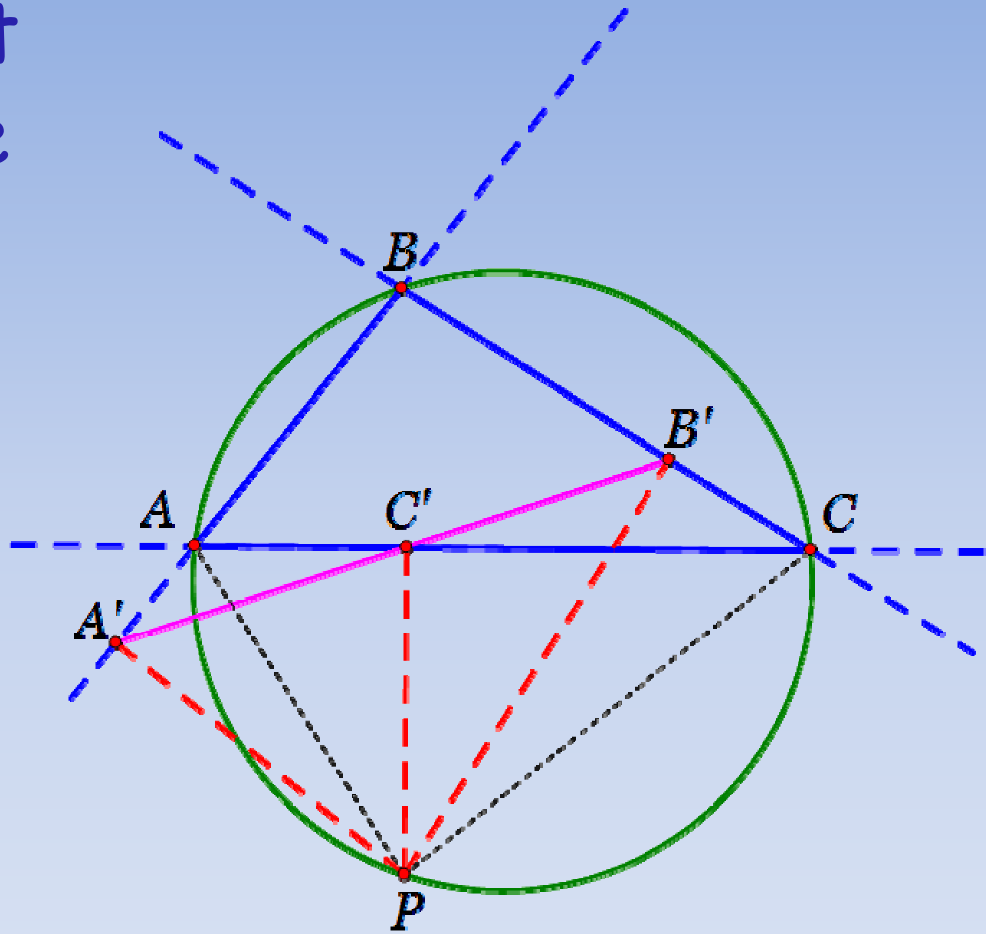
**Theorem (Wallace, Simson):** Given a reference triangle $\Delta ABC$, if P lies on the circumcircle of $\Delta ABC$ then the pedal triangle is degenerate.



# The Simson Line

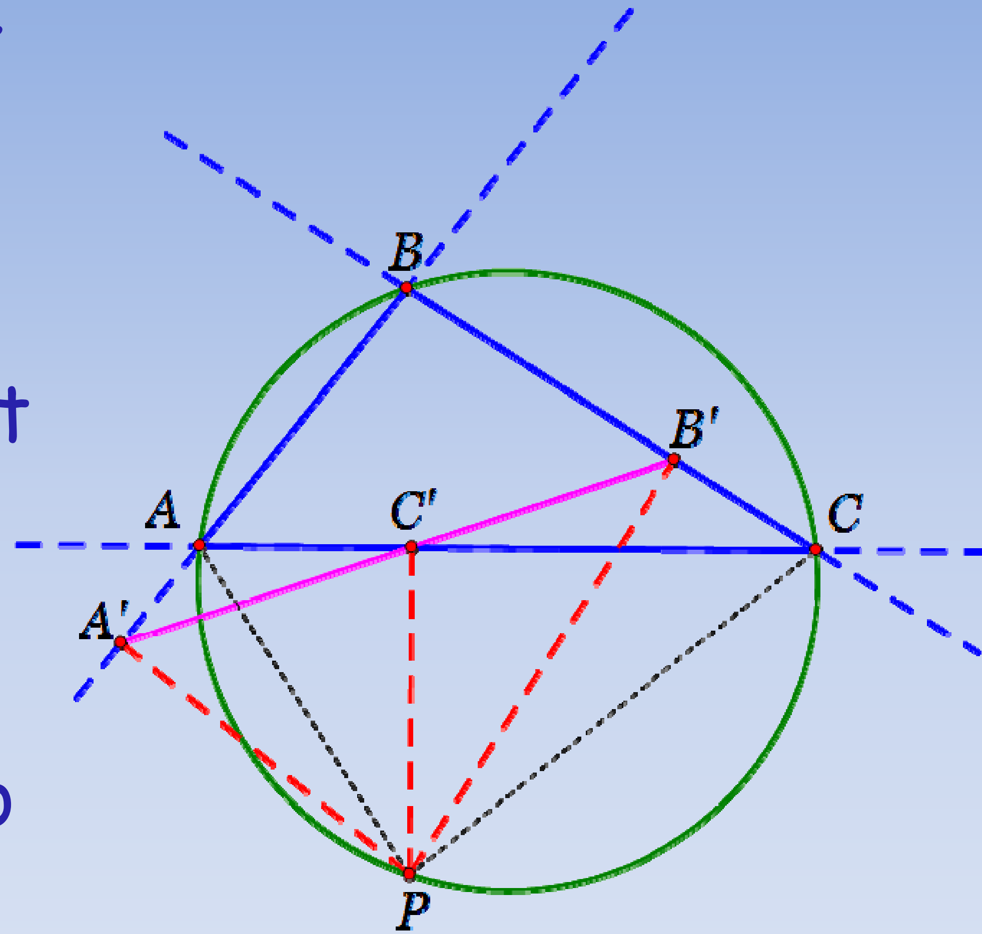
**Proof**: Assume that P is on circumcircle of $\Delta ABC$



# The Simson Line

**Proof**: First, assume that $P$ is on the circumcircle.

WLOG we can assume that $P$ is on arc $AC$ that does not contain $B$ and $P$ is at least as far from $C$ as it is from $A$. If necessary you can relabel the points to make this so.

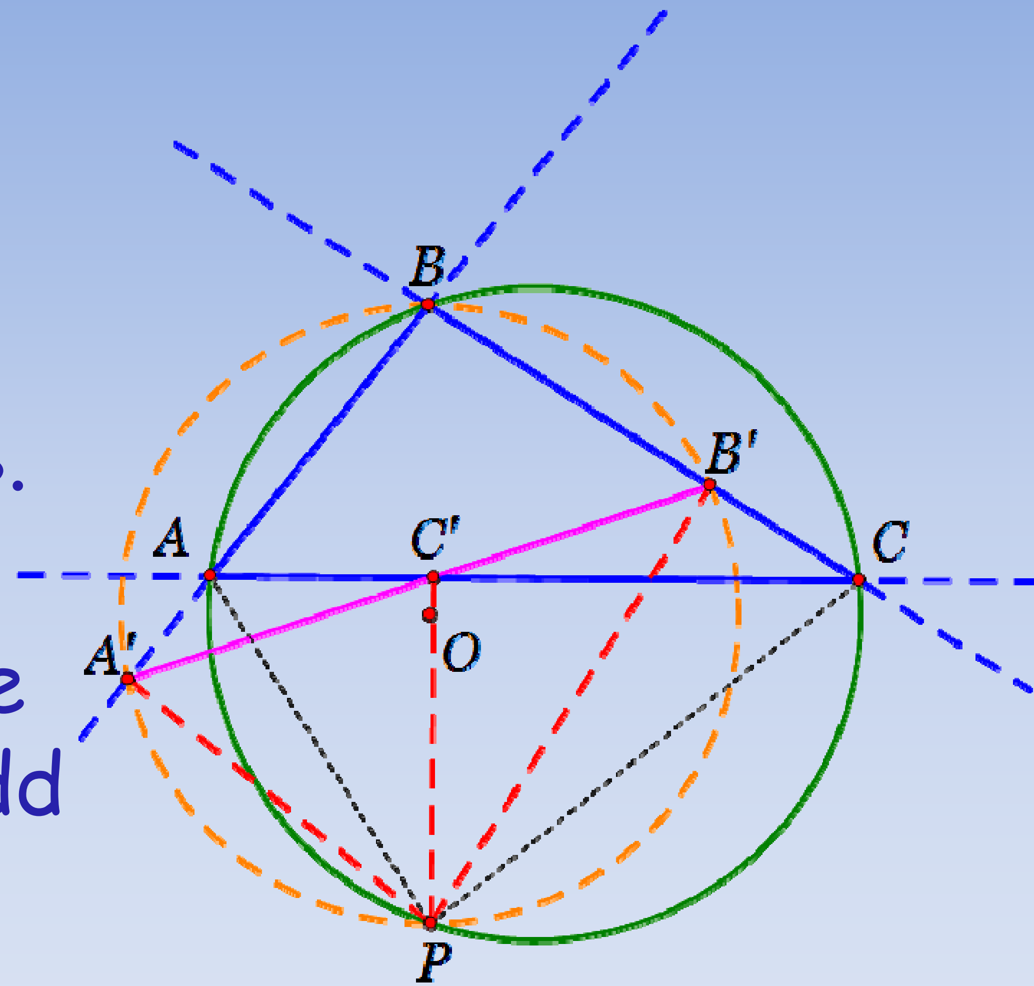


# The Simson Line

P also lies on the circumcircle of triangle ΔB'BA'. Why?

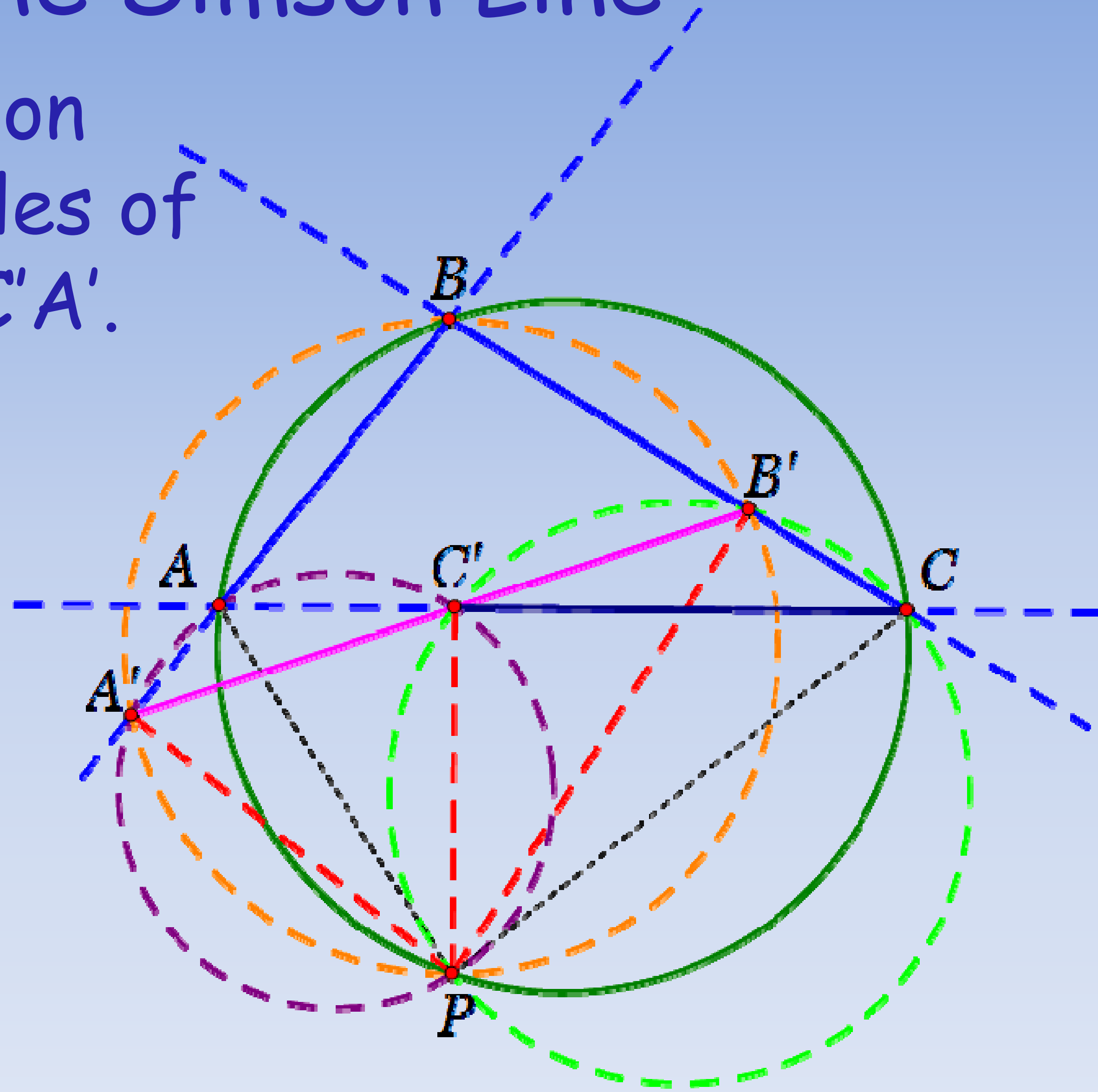
∠PB'B =90= ∠PA'B.

⇒ PA'BB' cyclic quadrilateral since opposite angles add up to 180.



# The Simson Line

Likewise P lies on the circumcircles of $\Delta B'C'C$ and $\Delta AC'A'$.



# The Simson Line

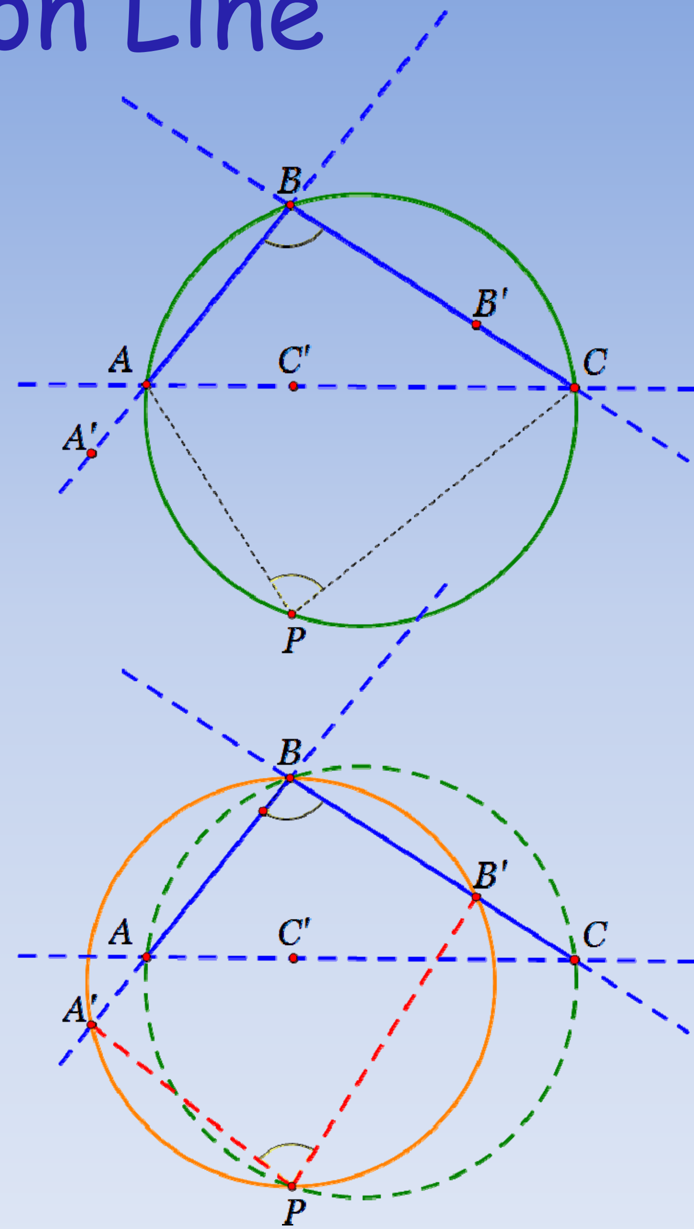
$\angle APC + \angle B = 180$

and

$\angle A'PB' + \angle B = 180$

So

$\angle APC = \angle A'PB'$



# The Simson Line

$\angle APC - \angle APB' = \angle A'PB' - \angle APB'$

$\angle B'PC = \angle A'PA.$

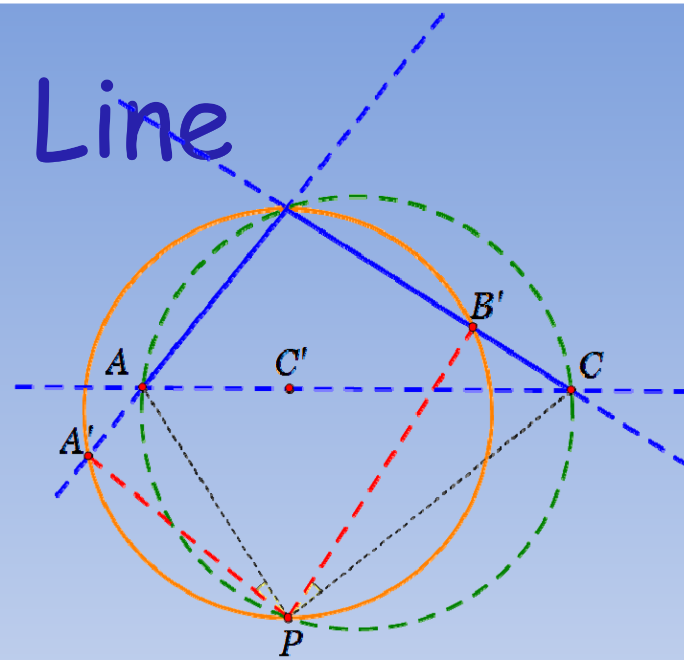
Now, B', C, P and C' are concyclic so by Star Trek Lemma

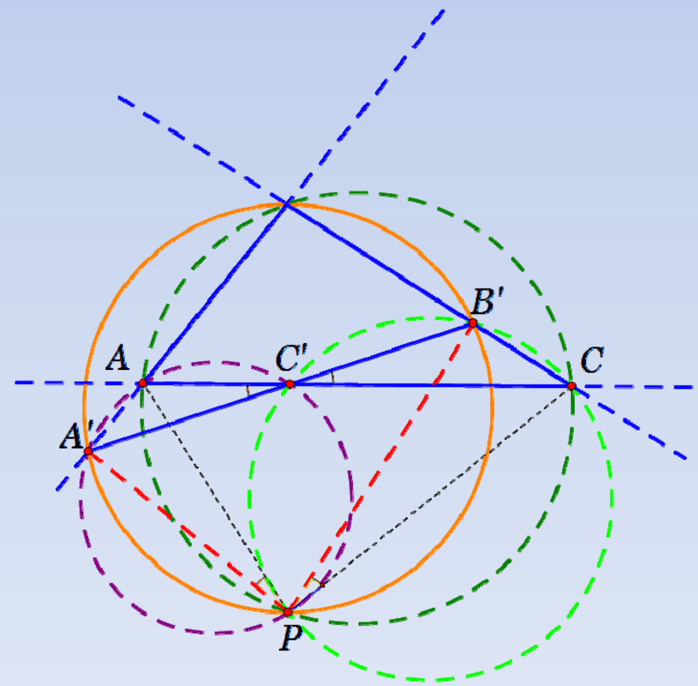
$\angle B'PC = \angle B'C'C.$

Similarly,

$\angle A'PA = \angle A'C'A.$

making A', B', C' collinear.



# The Simson Line

The converse of this theorem is also true.

That is if $\Delta A'B'C'$ is degenerate then $P$ must lie on the circumcircle of $\Delta ABC$.

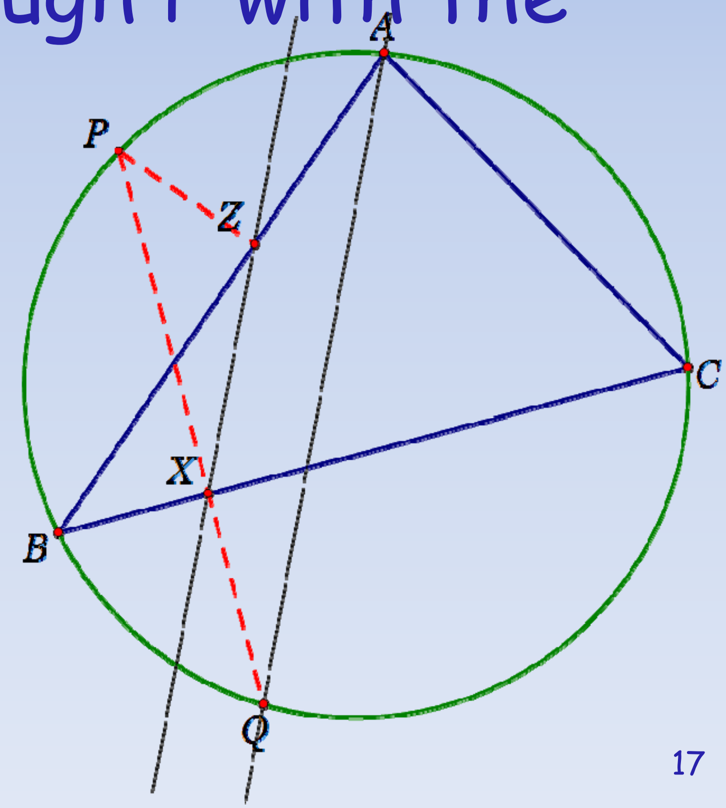
# Lemma 1

Choose P on the circumcircle of ΔABC. Let Q be the intersection of the perpendicular to BC through P with the circumcircle (Q≠P).

Let X be foot of P in BC.

Let Z be foot of P in AB.

If Q≠A, then ZX||QA.



# Proof

Assume X≠Z. If P=B, then P=B=X=Z, so P≠B. So, consider the unique circle with diameter PB.

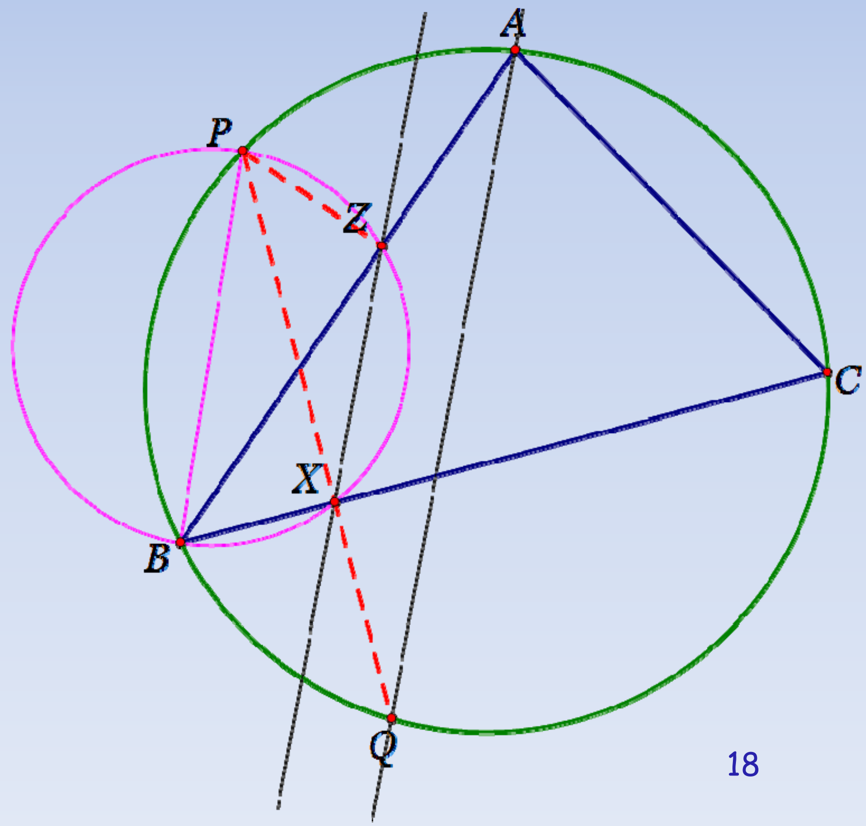
$\angle PXB = 90 = \angle PZB$

$\Rightarrow$ X,Z are concyclic with P & B.

$\Rightarrow \angle PXZ = \angle PBZ$

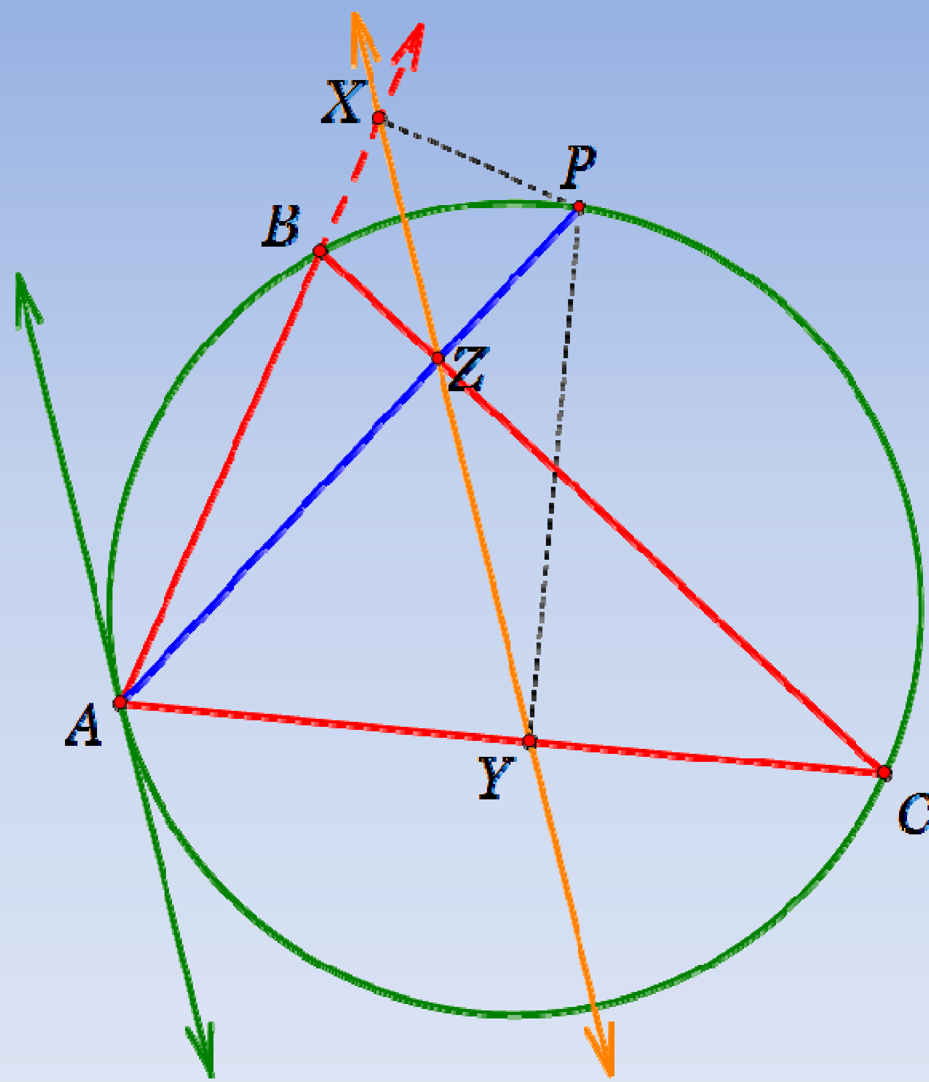
$\angle PBZ = \angle PBA = \angle PQA$

$\Rightarrow XZ || QA$



# Lemma 2

If the altitude AD of ΔABC meets the circumcircle at P, then the Simson line of P is parallel to the line tangent to the circle at A.



# Proof

XYZ is the Simson line of P.

$\angle PXB = 90 = \angle PZB$

$\Rightarrow$ P,Z,B,X concyclic

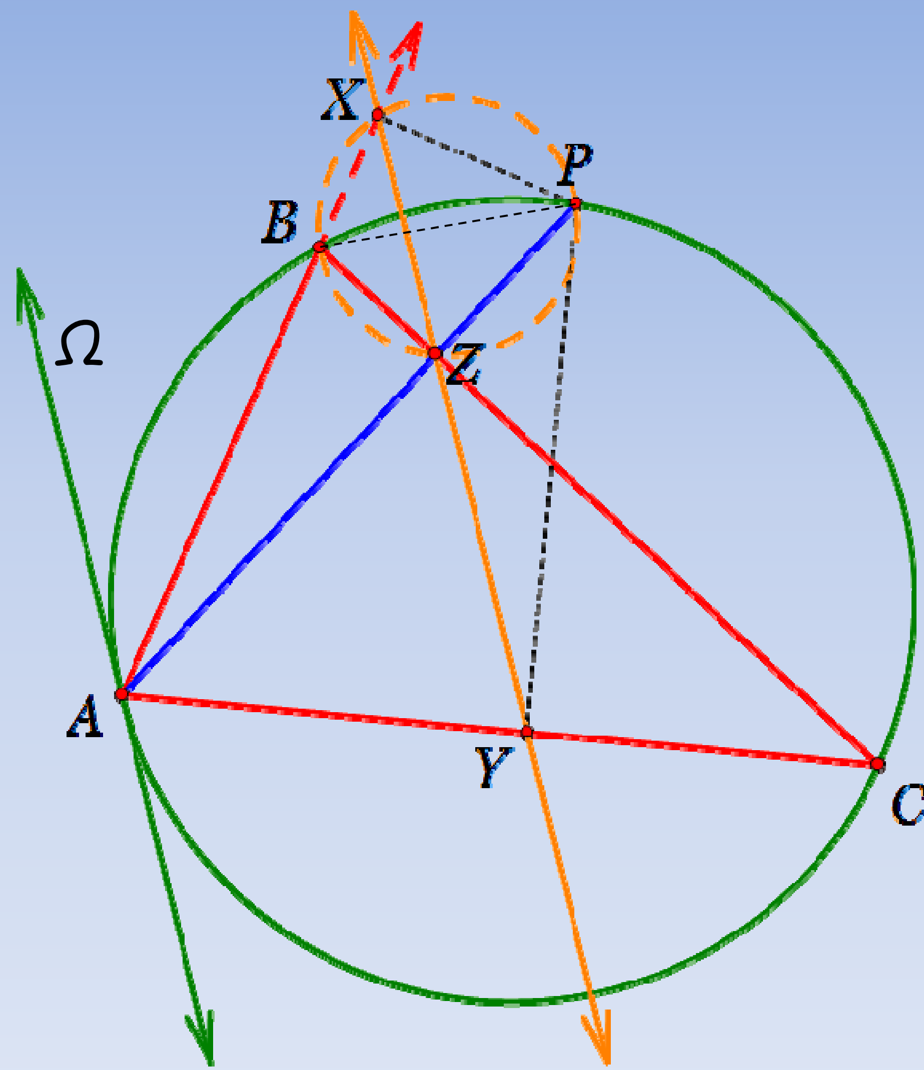
$\angle BXZ = \angle BPZ$

$\angle BPZ = \frac{1}{2}\overset{\frown}{AB}$

$\frac{1}{2}\overset{\frown}{AB} = \angle \Omega AB$

$\angle \Omega AB = \angle AXZ$

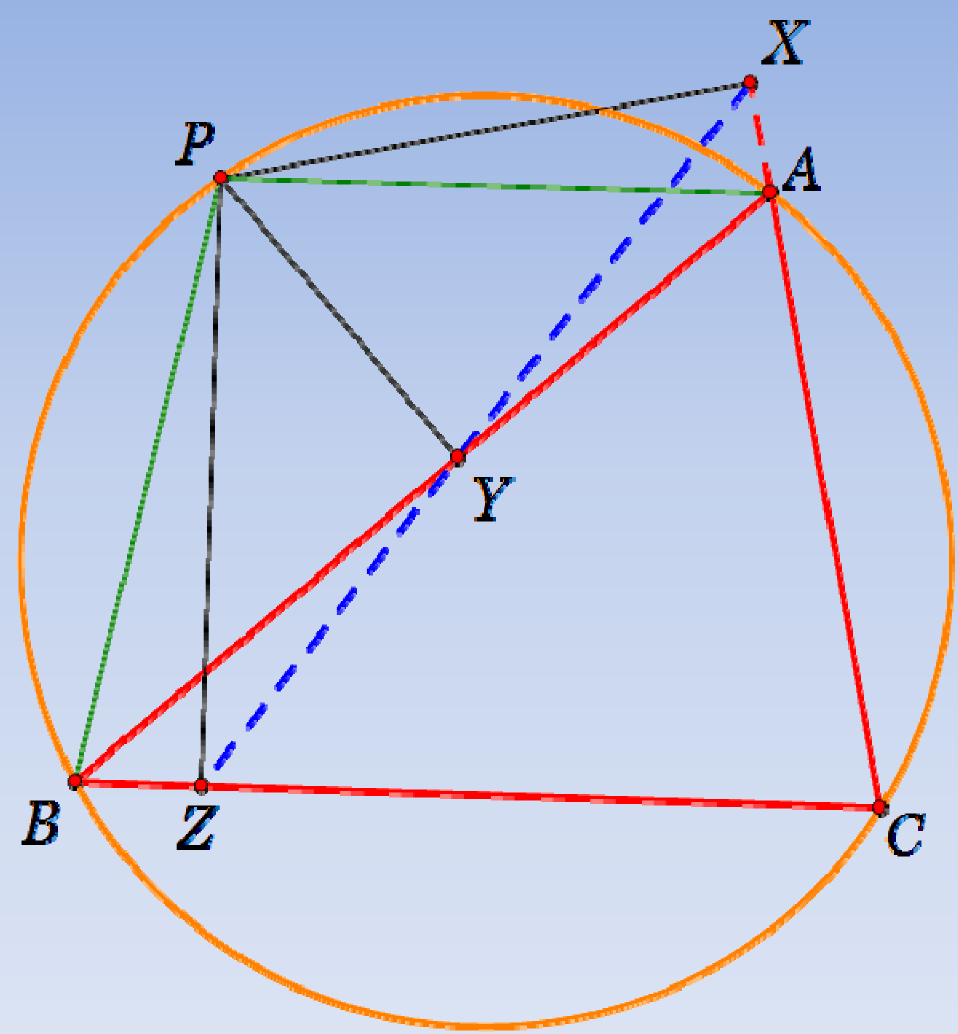
$\Rightarrow \Omega A \,||\, XY$



# Lemma 3

From P on the circumcircle of $\Delta ABC$ if perpendiculars PX, PY, PZ are drawn to *AC*, *AB*, and *AC*, then

(P*A*)(PZ)=(PB)(PX).



# Proof

$\angle PYB=90$ and $\angle PZB=90$

Thus, P,Y,Z,B concyclic

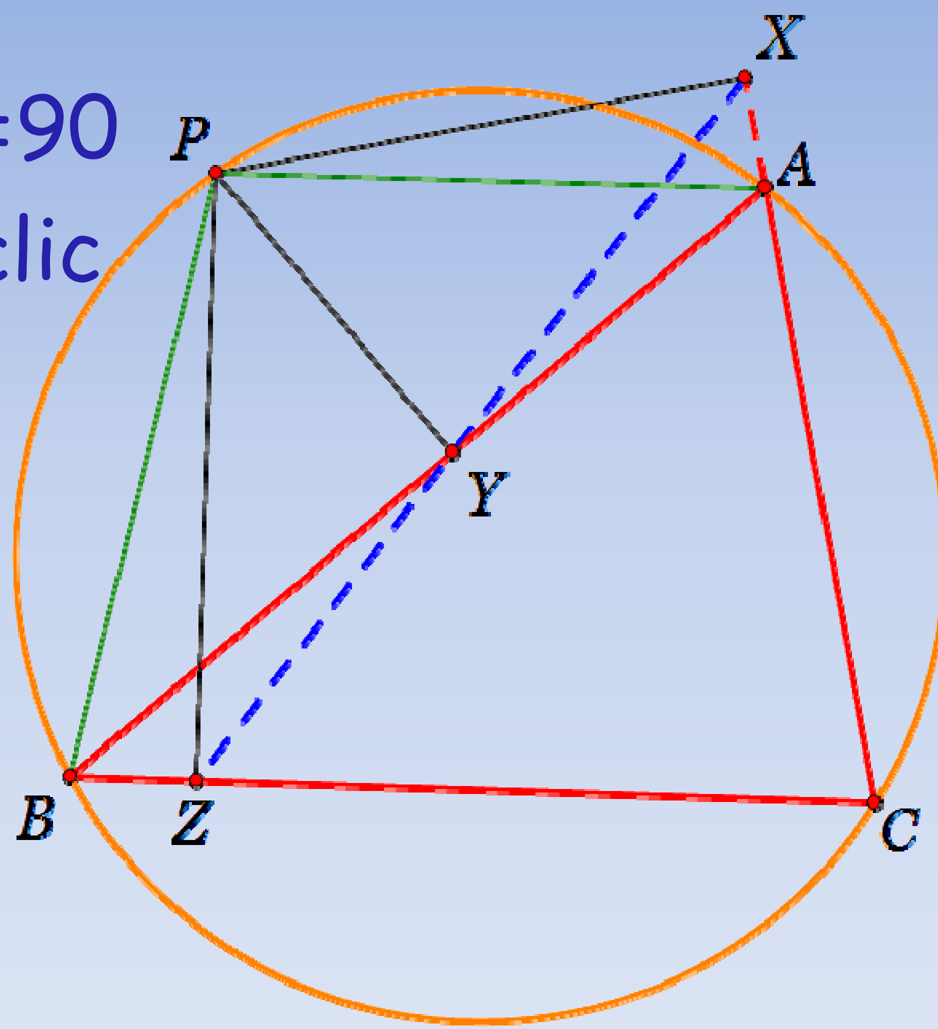
Thus, $\angle PBY=\angle PZY$

Likewise P,X,A,Y concyclic

Thus, $\angle PXY=\angle PAY$

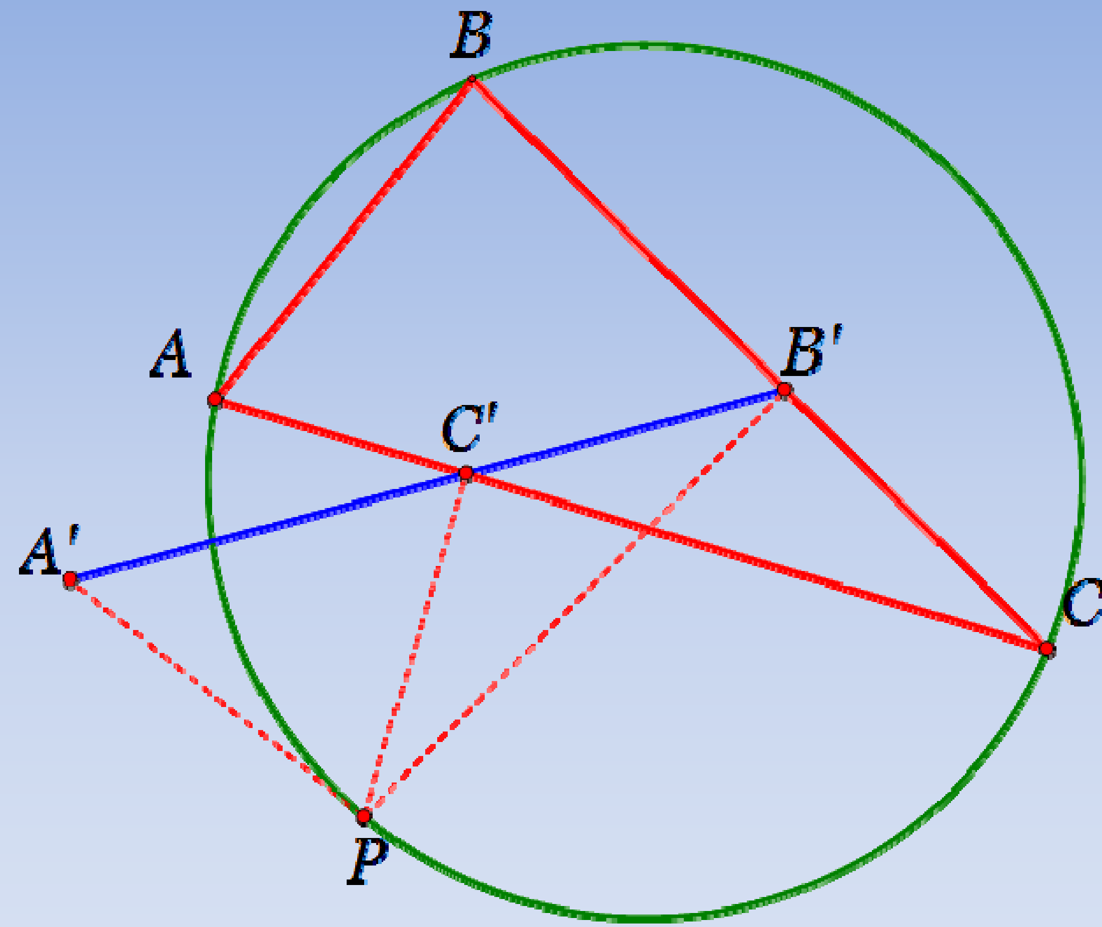
$\Delta PAB \sim \Delta PXZ$

(PA)(PZ)=(PB)(PX)



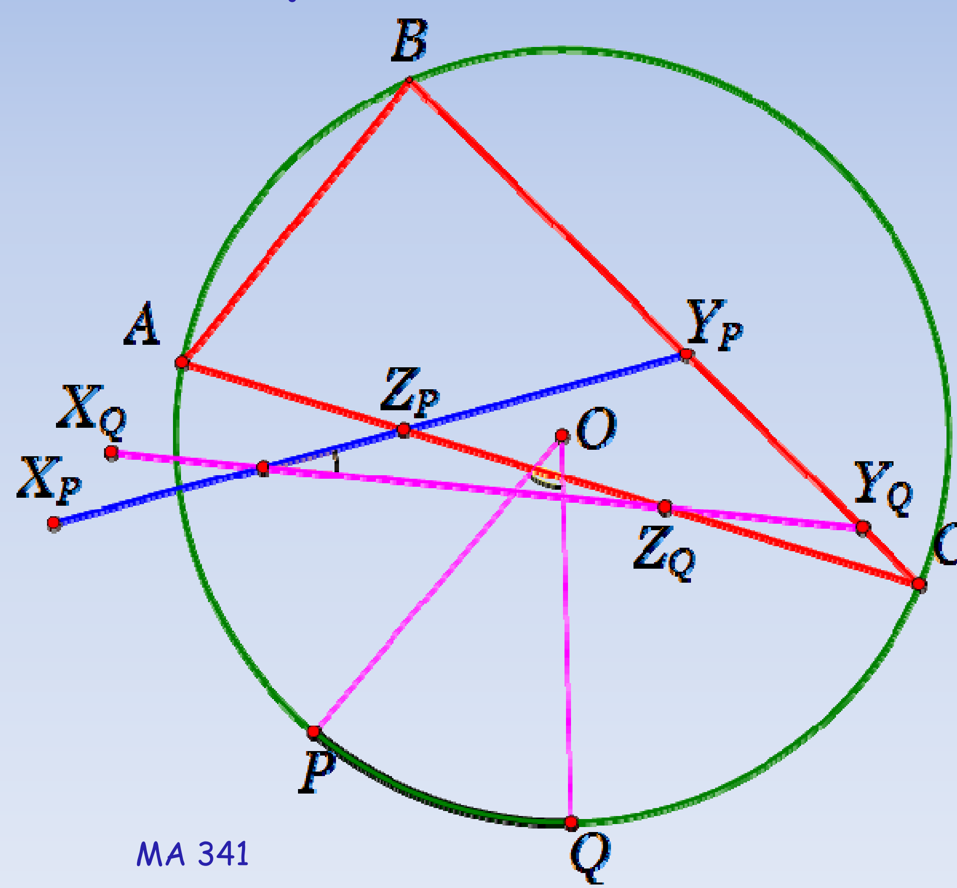
# Properties of Simson Line

P is called the pole of the line A'B'.



# Lemma 4

Let P and Q be points on the circumcircle of ABC. The angle between the Simson lines having P and Q as poles is half of the arc, $\overset{\frown}{PQ}$.



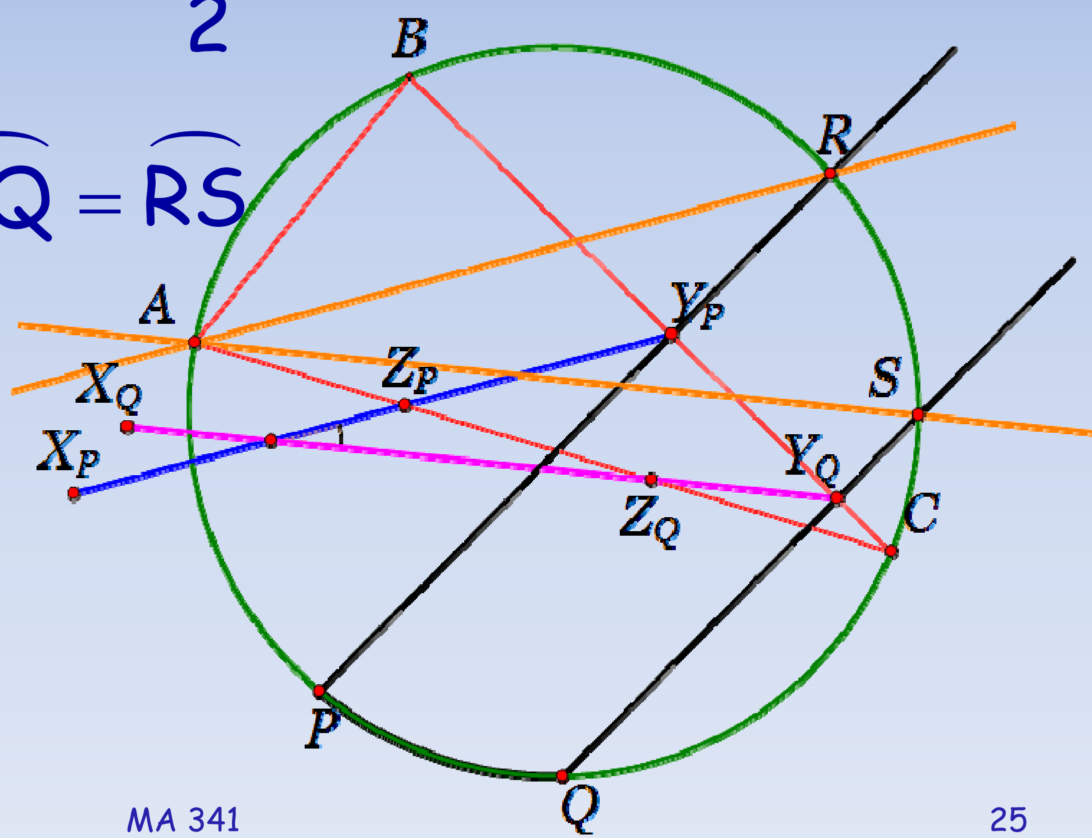
# Proof

**Proof:** Extend $PY_P$ to R and $QY_Q$ to S.

AS || $X_QY_Q$ and AR|| $X_PY_P$

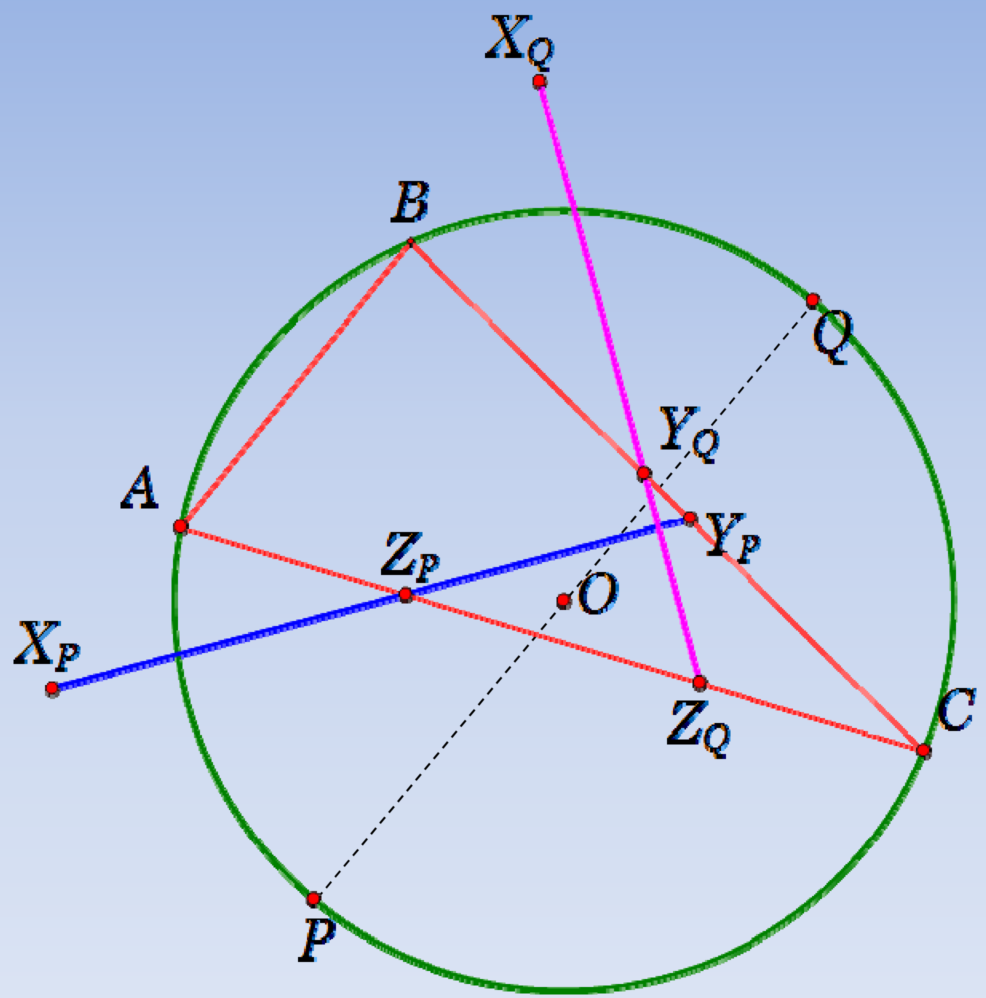
$\Rightarrow \angle Y_P\Omega Y_Q = \angle RAS = \frac{1}{2}\overset{\frown}{RS}$

Since PR||QS, $\overset{\frown}{PQ} = \overset{\frown}{RS}$



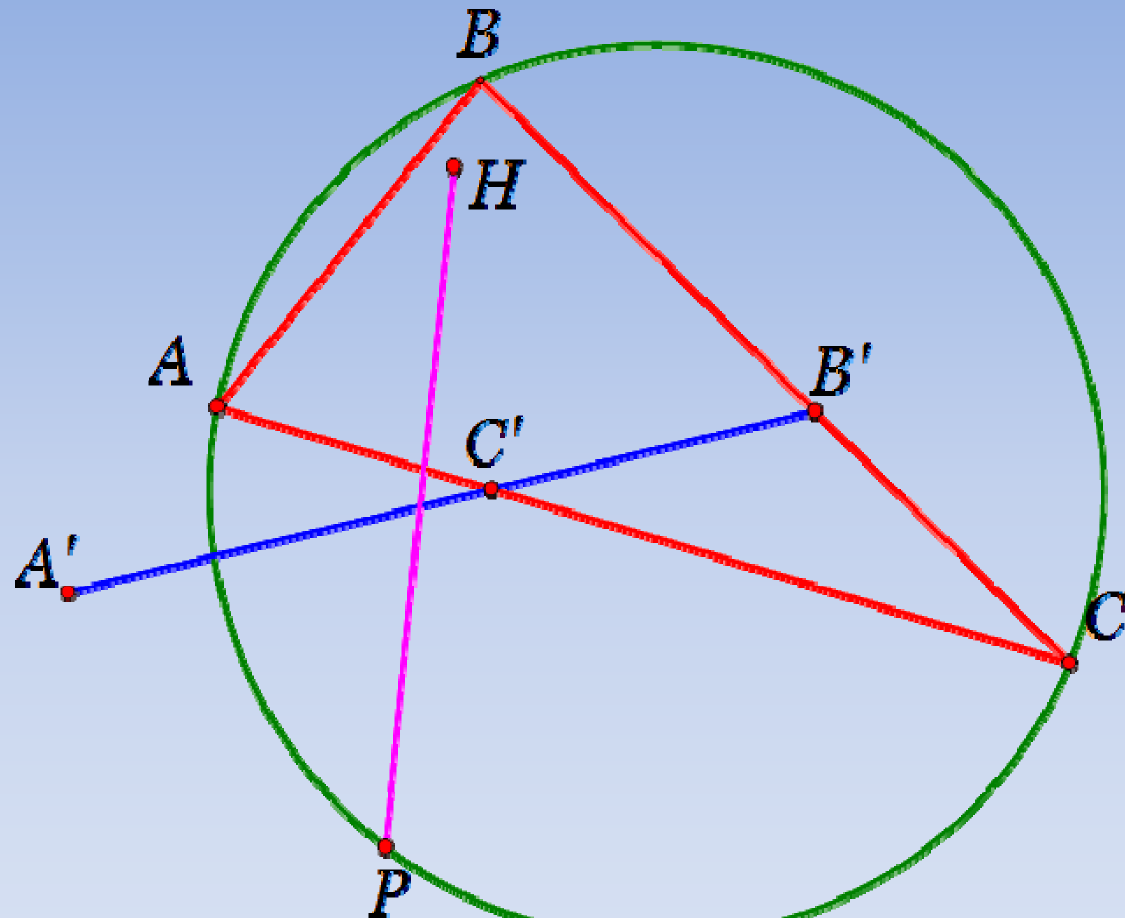
# Lemma 5

Two Simson lines are perpen-dicular iff their poles are on opposite ends of a diameter.



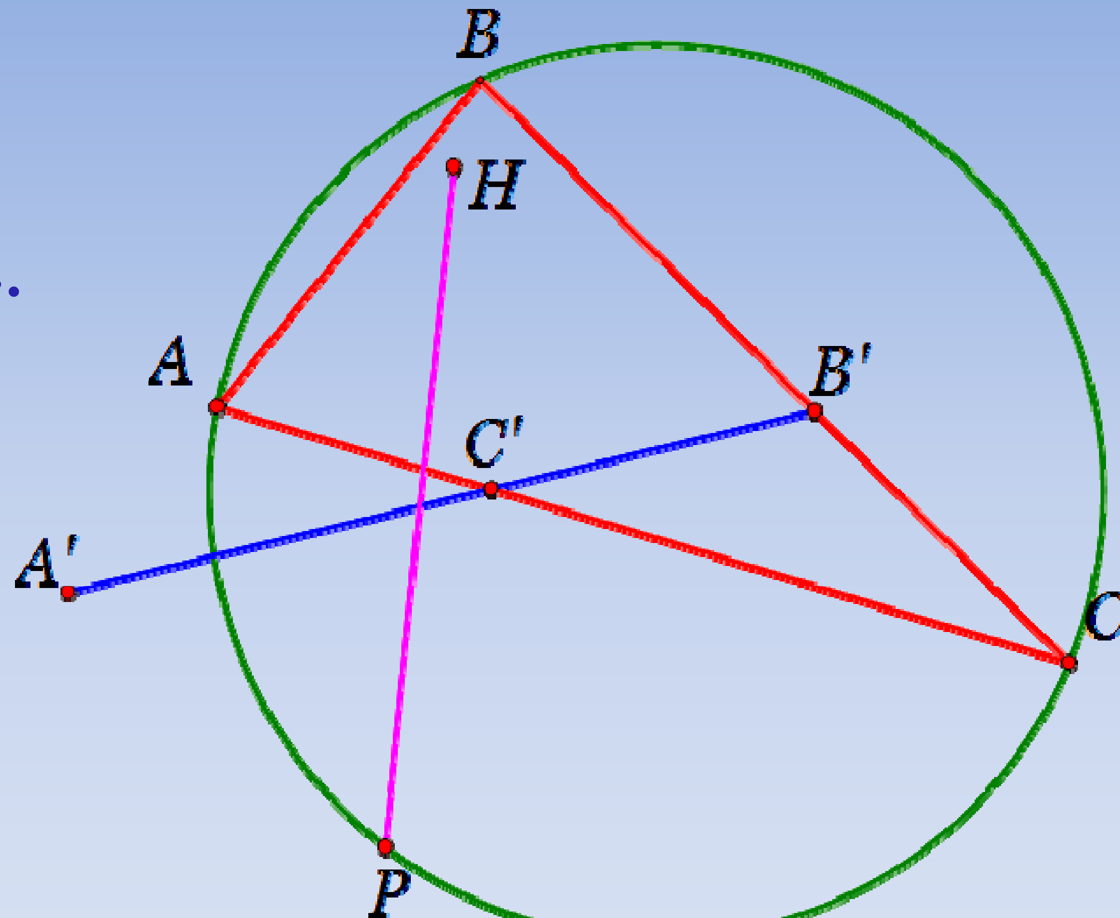
# Properties of Simson Line

Find the orthocenter of $\Delta ABC$ and construct HP.



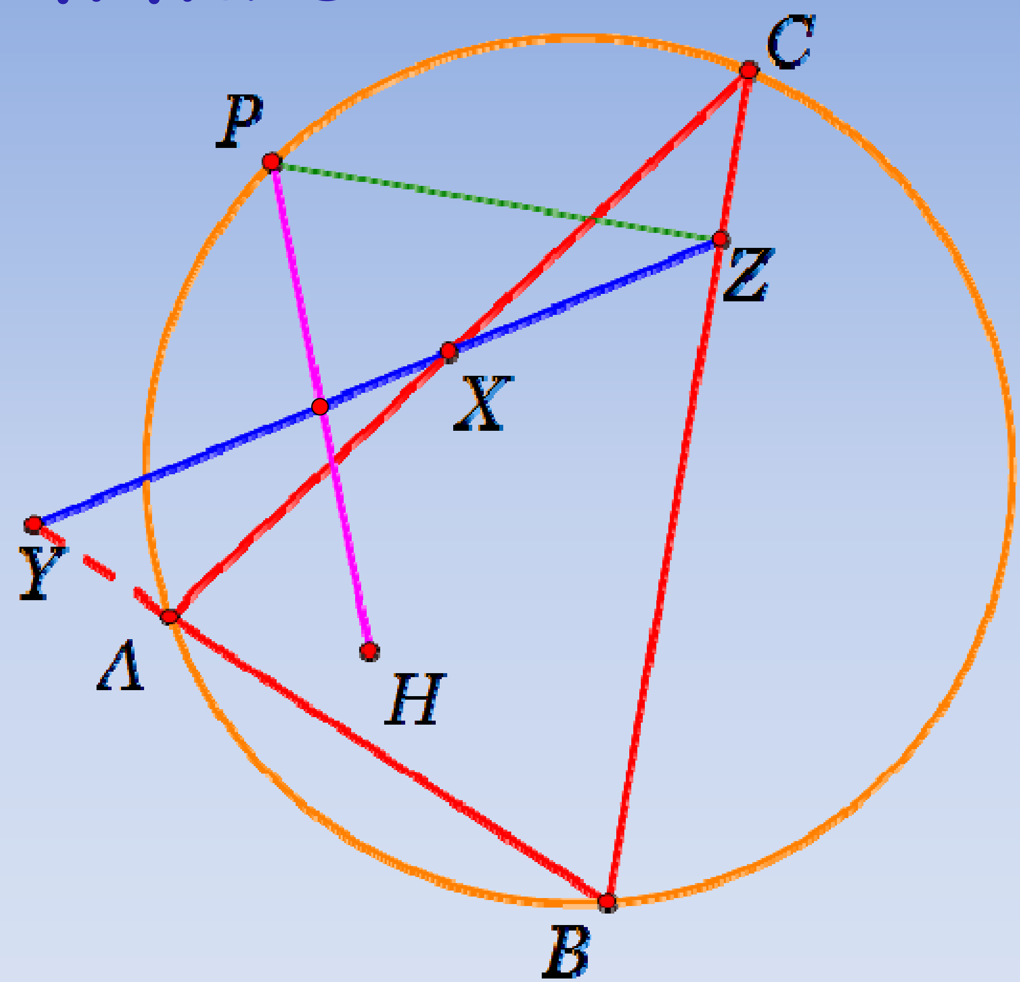
# Properties of Simson Line

HP intersects the Simson line.



# Lemma 6

The point of intersection is the midpoint of HP.



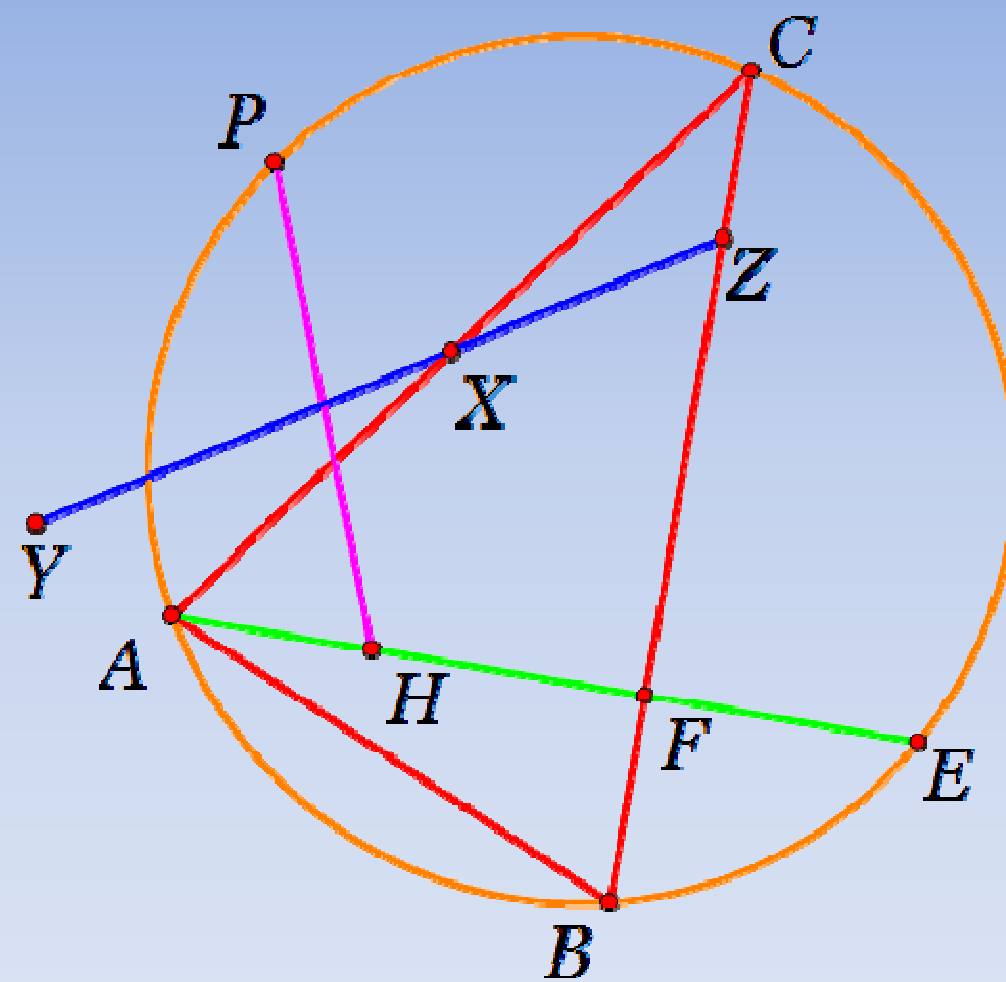
# Proof

Construct AF.

Extend to E.

Mark H on AF.

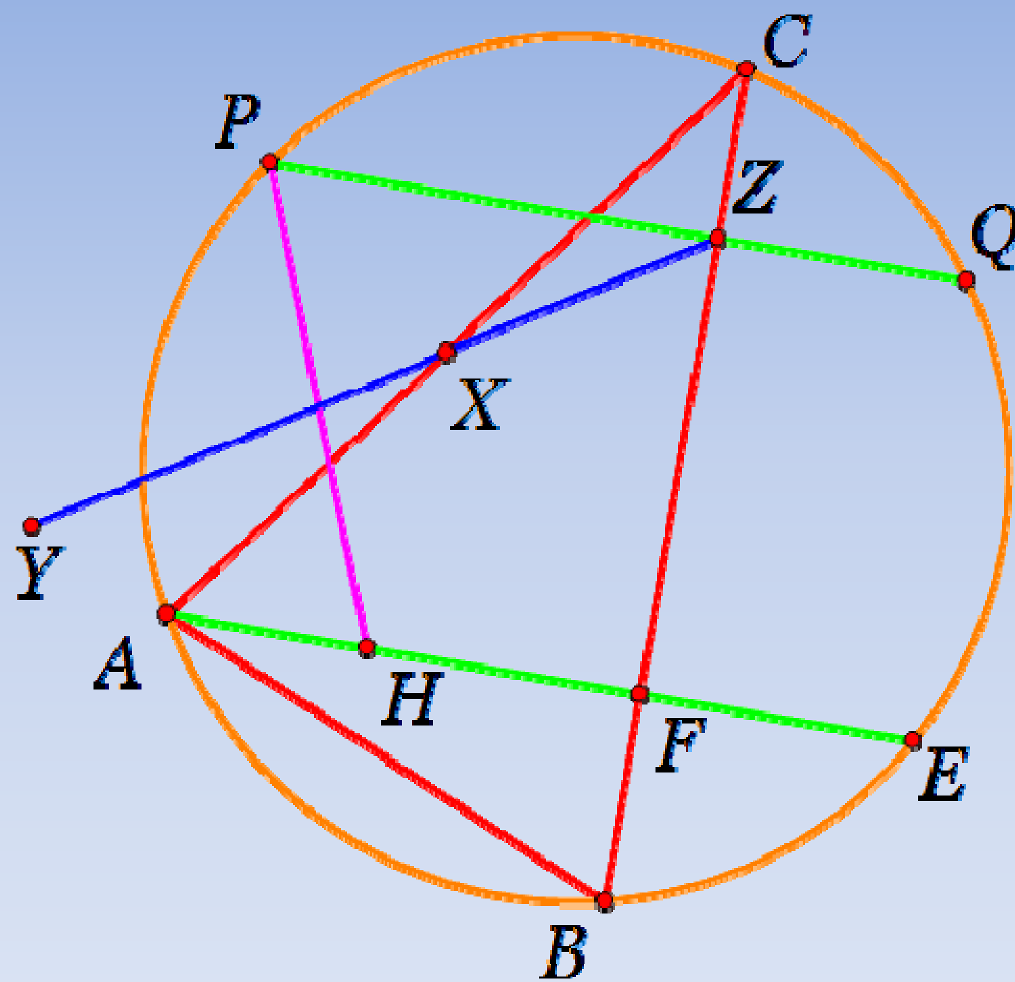
Construct PH.



# Proof

Construct PZ⊥BC.

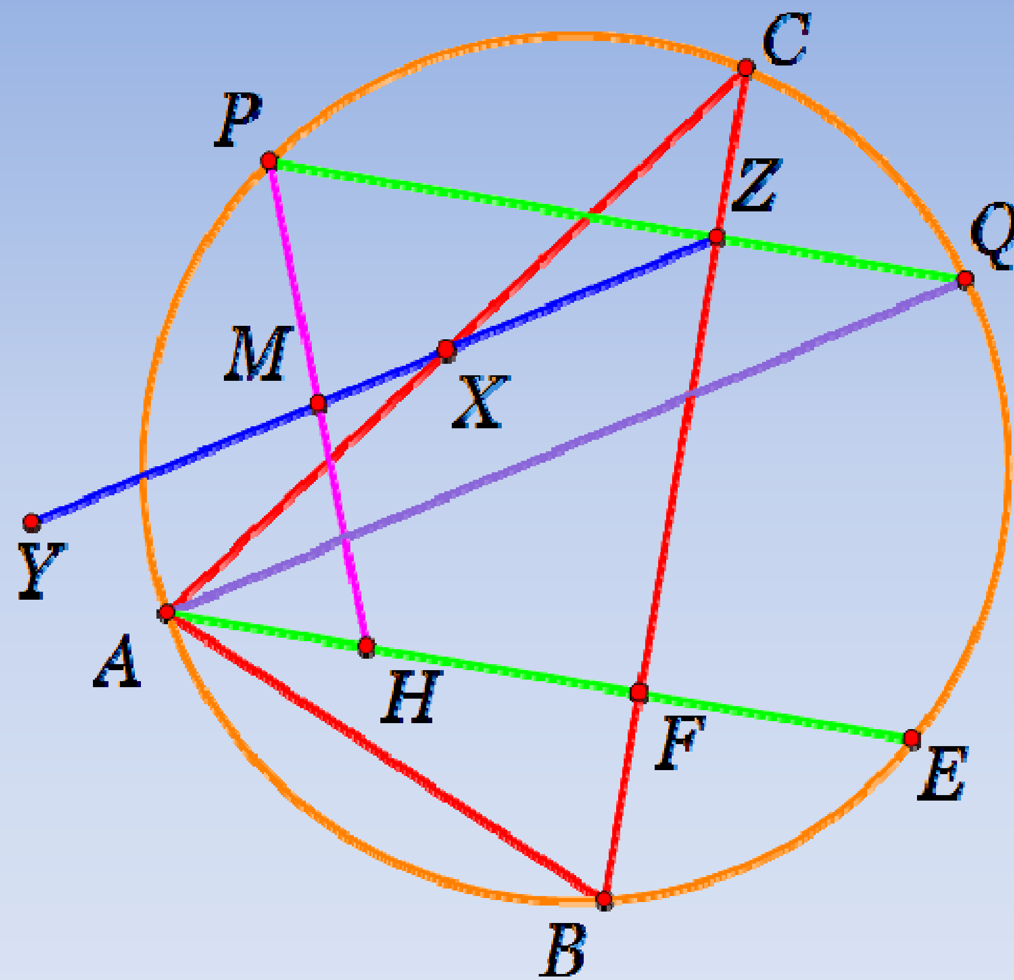
Extend to Q.



# Properties of Simson Line

Construct AQ.

YZ||AQ by Lemma 1



# Properties of Simson Line

Construct PE.

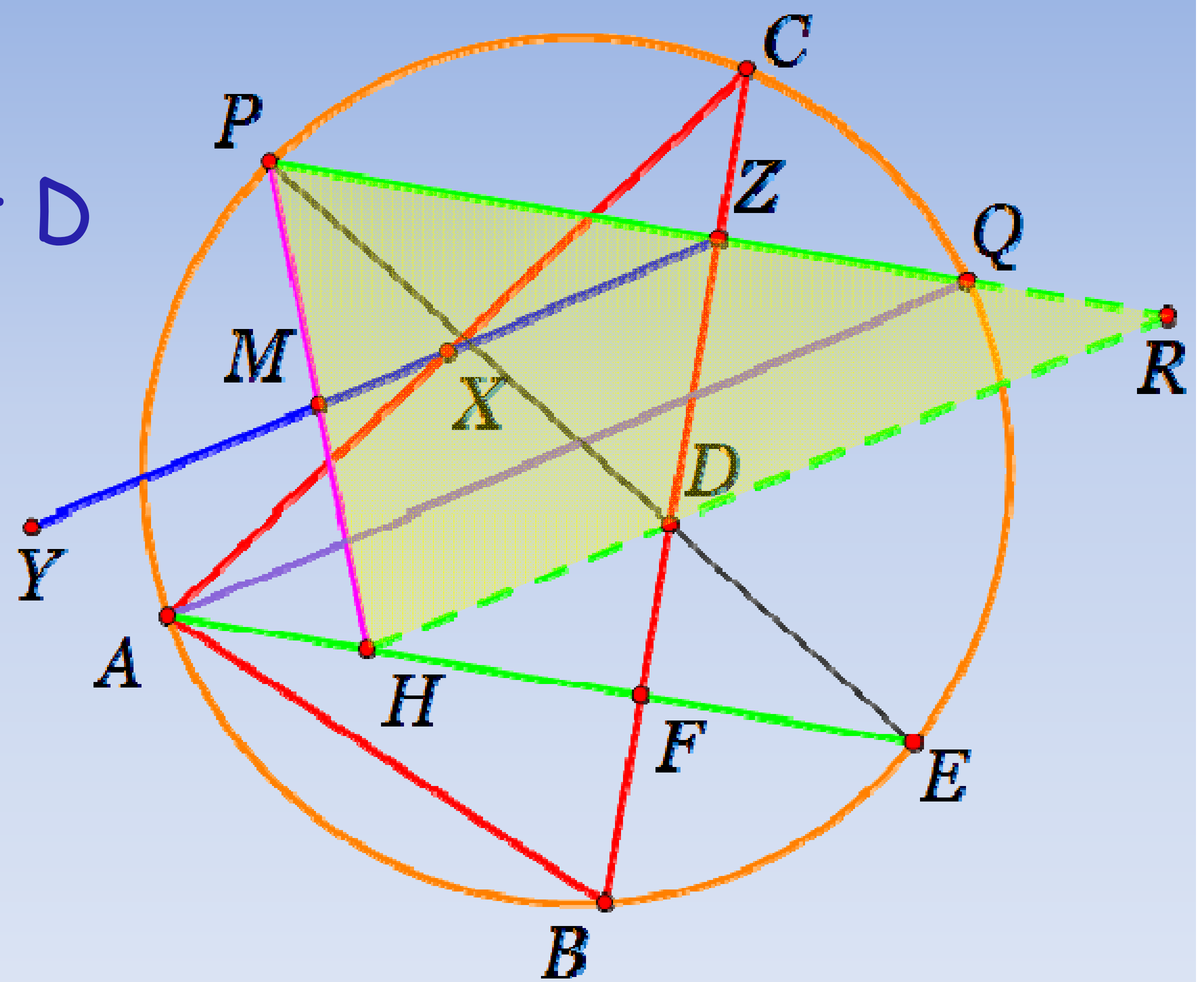
Intersects BC at D

Construct HD

Extend to meet

PQ at R

Consider ΔPHR.



# Properties of Simson Line

HF=FE (proven earlier)

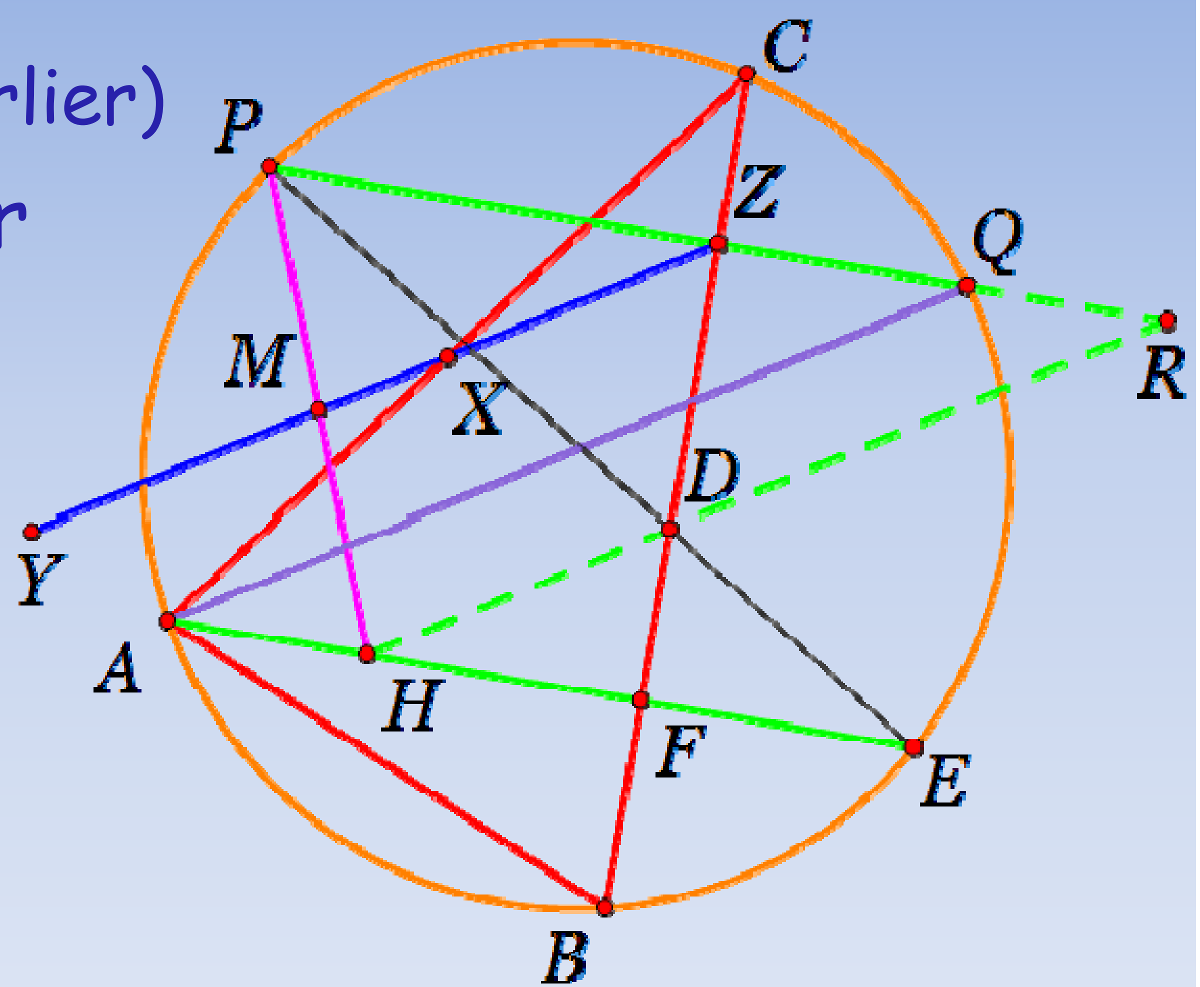
DF=perpendicular bisector of HE.

$\Rightarrow$DH=DE

$\angle$PQA= $\angle$PEA

= $\angle$RHE

= $\angle$PRH



# Properties of Simson Line

∠PQA = ∠PRH

⇒HR||AQ

⇒HR||YZ

Show: ΔPZD=ΔRZD

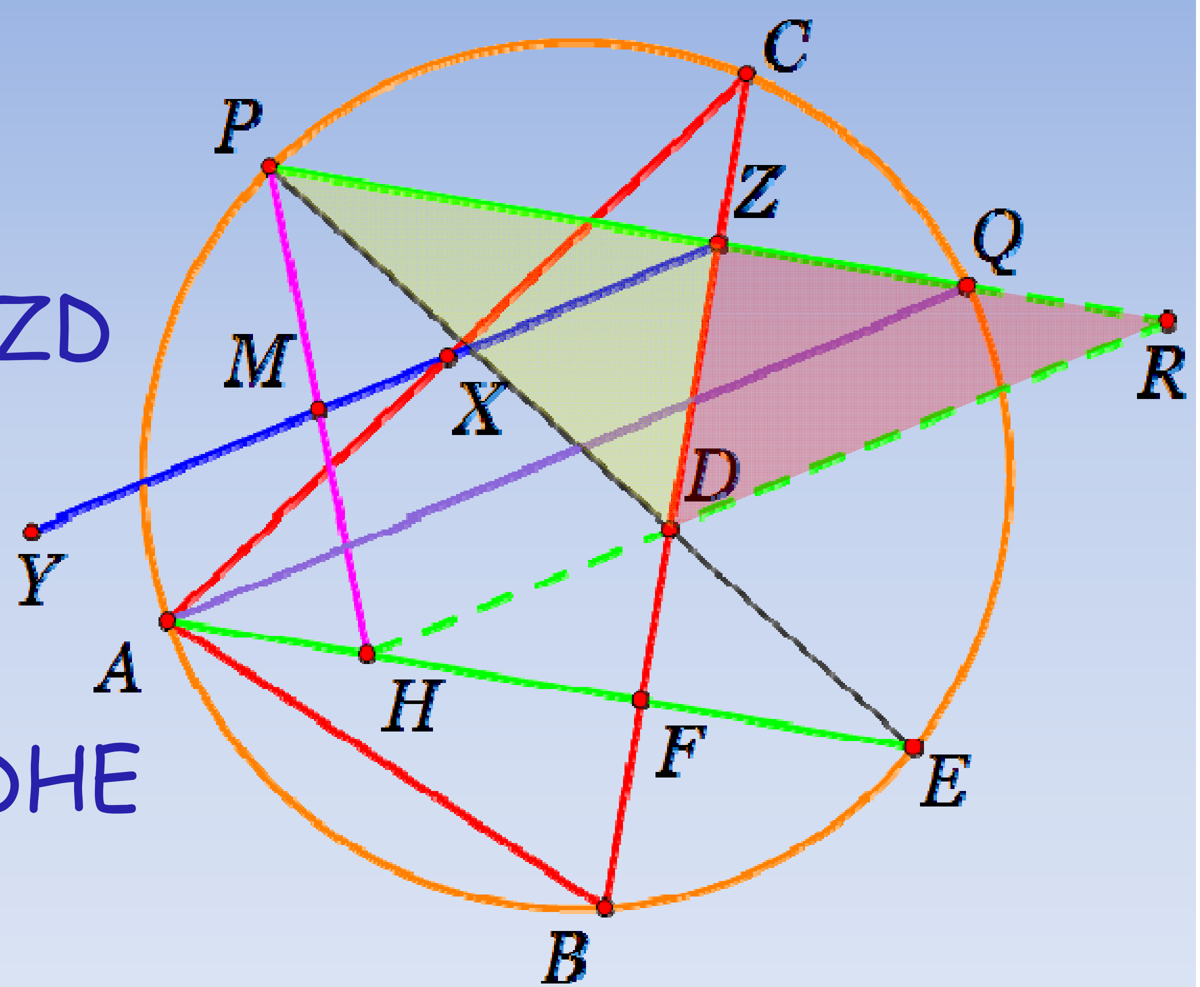
DZ=DZ

∠PZD=90=∠RZD

PR||AE ⇒

∠ZPD=∠DEH=∠DHE

=∠ZRD



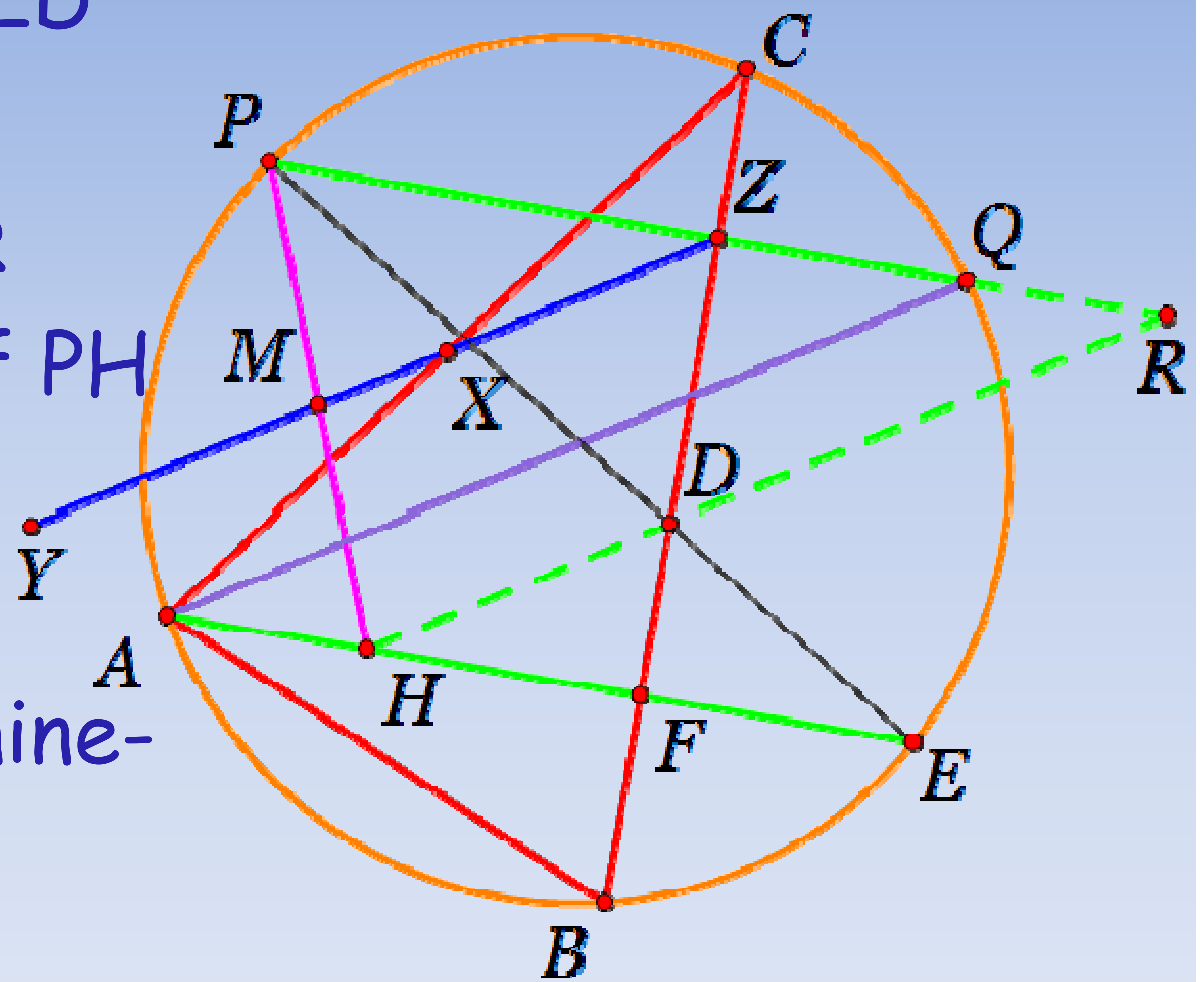
# Properties of Simson Line

Thus, ΔPZD=ΔRZD

⇒ PZ=ZR

⇒ Z=midpoint PR

⇒ M=midpoint of PH




Note: M lies on nine-point circle