Quadrilaterals

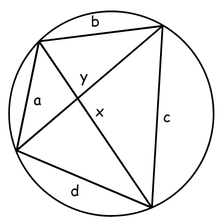
MA 341 – Topics in Geometry
Lecture 21

UK

Ptolemy's Theorem

Let a, b, c, and d be the lengths of consecutive sides of a cyclic quadrilateral and let x and y be the lengths of the diagonals. Then

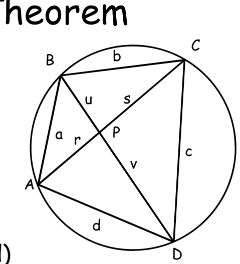
ac + bd = xy.



Ptolemy's Theorem

We have: ΔABP ~ ΔCDP &
ΔBCP ~ ΔDAP.
So

$$\frac{a}{c} = \frac{u}{s} = \frac{r}{v} \quad \text{and} \quad \frac{b}{d} = \frac{u}{r} = \frac{s}{v}$$



as = uc, br = ud, and uv = rs.
$sa^2 + rb^2 = uac+ubd = u(ac+bd)$
$xu^2+xrs = xu^2+xuv=xu(u+v)=uxy$
By Stewart's Theorem

$$u(ac+bd) = sa^2 + rb^2 = xu^2 + xrs = uxy$$

$$ac + bd = xy$$

The Converse of Ptolemy's Theorem

Let a, b, c, and d be the lengths of consecutive sides of a quadrilateral and let x and y be the lengths of the diagonals. If

$$ac + bd = xy,$$

then the quadrilateral is a cyclic quadrilateral.

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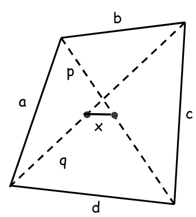
Euler's Theorem

Let a, b, c, and d be the lengths of consecutive sides of a quadrilateral, m and n lengths of diagonals, and x the distance between midpoints of diagonals. Then



$$a^2 + b^2 + c^2 + d^2 = m^2 + n^2 + 4x^2$$

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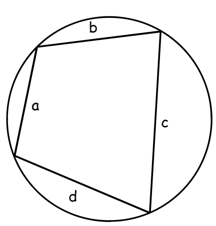
Brahmagupta's Theorem

There is an analog of Heron's Formula for special quadrilaterals.

Let a, b, c, and d be lengths of consecutive sides of cyclic quadrilateral, then



$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

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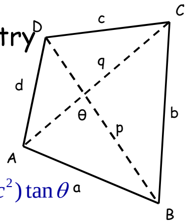
Area of a Quadrilateral

Using triangle trigonometry you can show that



$$A=\frac{1}{2}pq\sin\theta=\frac{1}{4}(b^2+d^2-a^2-c^2)\tan\theta$$

$$=\frac{1}{4}\sqrt{4p^2q^2-(b^2+d^2-a^2-c^2)^2}$$

$$=\sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos\tfrac{1}{2}(A+C)}$$

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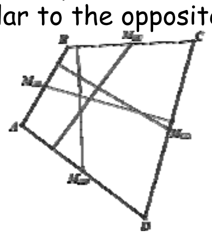
Maltitudes

For a quadrilateral the maltitude (midpoint altitude) is a perpendicular through the midpoint of one side perpendicular to the opposite side.



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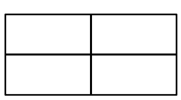
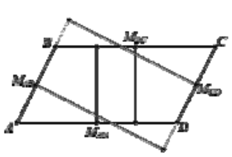
Maltitudes

Rectangle



Square – same

Parallelogram



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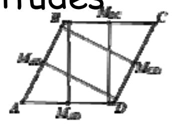
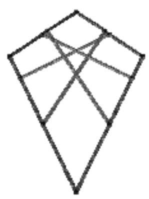
Maltitudes

Rhombus

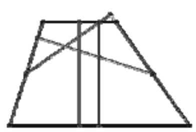


Kite

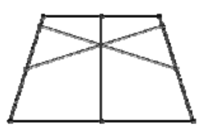


Maltitudes

Trapezoid

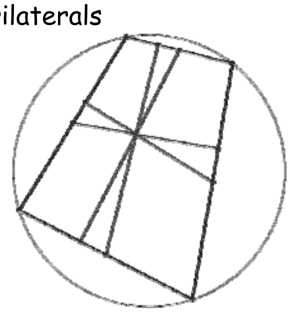


Isosceles trapezoid



Maltitudes

Cyclic quadrilaterals



Quadrilaterals and Circles

- For a cyclic quadrilateral, area is easy and there are nice relationships
- Maybe maltitudes of cyclic quadrilateral are concurrent
- Can we tell when a quadrilateral is cyclic?
- Can we tell when a quadrilateral has an inscribed circle?

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Theorem

For a cyclic quadrilateral the maltitudes intersect in a single point, called the anti-center.

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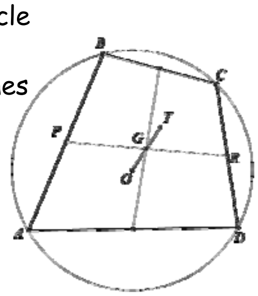
Proof

Let O = center of circle
G = centroid, intersection of midlines

Let T = point on ray OG so that OG = GT



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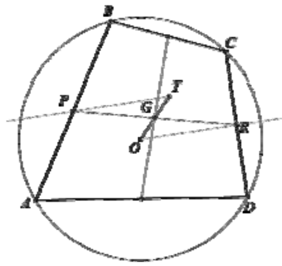
Proof

PG = RG
OG = GT
∠PGT = ∠RGO
ΔPGT = ΔRGO

∠PTG = ∠ROG

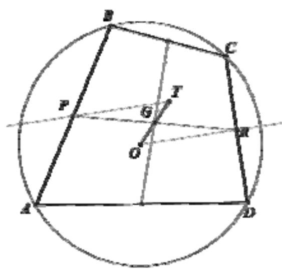
PT || OR



Proof

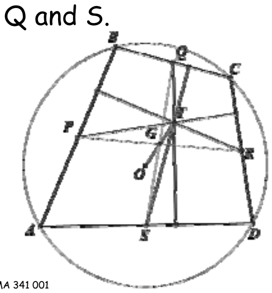
OR ⊥ CD
Thus, PT ⊥ CD
T lies on maltitude through P

Use ΔRGT = ΔPGO to show T lies on maltitude through R



Proof

Using other midline we show T lies on maltitudes through Q and S.



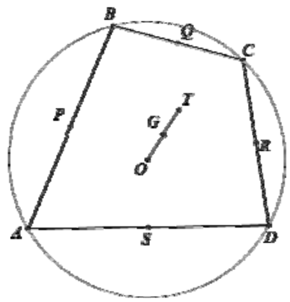
Anticenter

T = anticenter = intersection of maltitudes

G = midpoint

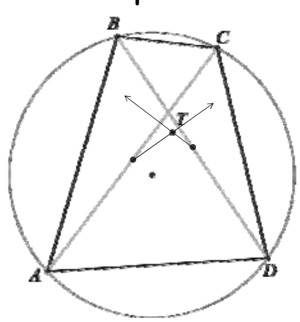
O = circumcenter

OG = GT



Other Anticenter Properties

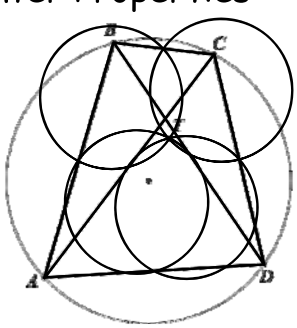
Perpendiculars from midpoint of one diagonal to other intersect at T.



Other Anticenter Properties

Construct 9-point circles of the four triangles ΔABD, ΔBCD, ΔABC, and ΔADC.

The 4 circles intersect at the anticenter.



Other Anticenter Properties

The centers of the 9-point circles are concyclic with center T.