1.3 Curvature and Plane Curves

We want to be able to associate to a curve a function that measures how much the curve bends at each point.

Let $\alpha\colon (a,b) \to \mathbf{R}^2$ be a curve parameterized by arclength. Now, in the Euclidean plane any three non-collinear points lie on a unique circle, centered at the orthocenter of the triangle defined by the three points.

For $s \in (a,b)$ choose s_1, s_2, and s_3 near s so that $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$ are non-collinear. This is possible as long as α is not linear near $\alpha(s)$. Let $C = C(s_1, s_2, s_3)$ be the center of the circle through $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$. The radius of this circle is approximately $|\alpha(s) - C|$. A better function to consider is the square of the radius:

$$\rho(s) = (\alpha(s) - C) \cdot (\alpha(s) - C).$$

Since α is smooth, so is ρ. Now, $\alpha(s_1)$, $\alpha(s_2)$, and $\alpha(s_3)$ lie on the circle so $\rho(s_1) = \rho(s_2) = \rho(s_3)$. By Rolle's Theorem there are points $t_1 \in (s_1, s_2)$ and $t_2 \in (s_2, s_3)$ so that $\rho'(t_1) = \rho'(t_2) = 0$. Then, using Rolle's Theorem again on these points, there is a point $u \in (t_1, t_2)$ so that $\rho''(u) = 0$. Using Leibnitz' Rule we have $\rho'(s) = 2\alpha'(s) \cdot (\alpha(s) - C)$ and

$$\rho''(s) = 2[\alpha''(s) \cdot (\alpha(s) - C) + \alpha'(s) \cdot \alpha'(s)].$$

Since $\rho''(u) = 0$, we get

$$\alpha''(u) \cdot (\alpha(u) - C) = -\alpha'(s) \cdot \alpha(s) = -1.$$

Now, as s_1, s_2, and s_3 get closer to s, then the center of the circles will converge to a value $C_\alpha(s)$. Then t_1 and t_2 go to s, so $\rho'(s) = 0$ which forces $\alpha'(s) \cdot (\alpha(s) - C_\alpha(s)) = 0$. Furthermore, $\alpha''(s) \cdot (\alpha(s) - C_\alpha(s)) = -1$.

This says that the circle centered at $C_\alpha(s)$ with radius $\alpha(s) - C_\alpha(s)$ shares the point $\alpha(s)$ with the curve α. Furthermore, from the above the tangent to the circle at $\alpha(s)$ is a multiple of $\alpha'(s)$. Thus, this circle, called the **osculating circle**, is tangent to the curve at $\alpha(s)$. The point $C_\alpha(s)$ is called the **center of curvature** of α at s, and the curve given by the function $C_\alpha(s)$ is called the **curve of centers of curvature**.

Definition 1.5 The (unsigned) **plane curvature** of α at s is the reciprocal of the radius of the osculating circle:

$$\kappa_\pm(s) = \frac{1}{|\alpha(s) - C_\alpha s}|.$$

Theorem 1.4 $\kappa_\pm(s) = |\alpha''(s)|$.

Proof: Since $\alpha'(s) \cdot \alpha'(s) = 1$, differentiating gives $\alpha'(s) \cdot \alpha''(s) = 0$. This means that $\alpha''(s)$ is perpendicular to $\alpha'(s)$. Since we have seen that $\alpha(s) - C_\alpha(s)$ is also perpendicular to $\alpha'(s)$, there exists a $k \in \mathbf{R}$ so that

$$\alpha(s) - C_\alpha(s) = k\alpha''(s).$$

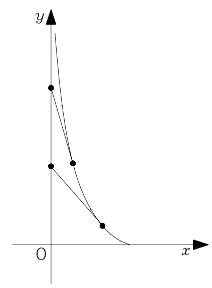

Figure 1.1: Tractrix curve

From above we have

$$\begin{aligned} -1 &= \alpha''(s) \cdot (\alpha(s) - C_\alpha(s)) \\ &= \alpha''(s) \cdot k\alpha''(s) \\ &= k|\alpha''(s)|^2. \end{aligned}$$

Thus,

$$|\alpha(s) - C_\alpha(s)| = |k|\, |\alpha''(s)| = \frac{1}{|\alpha''(s)|^2}|\alpha''(s)| = \frac{1}{|\alpha''(s)|}.$$

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We rarely can symbolically represent a curve as parameterized by arclength. Quite often, a different parameterization is more reasonable. To find the curvature, though, would require that we parameterize by arclength and then differentiate. There is an easier way.

Theorem 1.5 *The plane curvature of a regular plane curve* $\sigma(t) = (x(t), y(t))$ *is given by*

$$\kappa_\pm(t) = \left| \frac{x''y' - y''x'}{((x')^2 + (y')^2)^{3/2}} \right|.$$

1.3.1 Tractrix

Describe the curve followed by a weight being dragged on the end of a fixed straight length and the other end moves along a fixed straight line. The tractrix is the curve characterized by the condition that the length of the segment of the tangent line to the curve from the curve to the y-axis is constant. It has the following equation for a given constant a:

$$x = a\ln(\frac{a + \sqrt{a^2 - y^2}}{y}) - \sqrt{a^2 - y^2}.$$

and has graph shown in Figure 1.1.

Let the curve begin at $(a, 0)$ on the x-axis. Now, we can see that

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{\sqrt{a^2 - x^2}}{x}. \tag{1.1}$$

Square both sides of the equation and simplify

$$(x')^2 + (y')^2 = \left(\frac{a}{x}\right)^2 (x')^2. \tag{1.2}$$

Now, if we differentiate the first equation (1.1), we get

$$\frac{x'y'' - x''y'}{(x')^2} = \frac{-a^2x'}{x^2\sqrt{z^2 - x^2}} \tag{1.3}$$

$$x''y' - x'y'' = \frac{a^2(x')^3}{x^2\sqrt{a^2 - x^2}}. \tag{1.4}$$

Thus,

$$\kappa_{\pm}(x, y) = \left|\frac{-a^2x'}{x^2\sqrt{z^2 - x^2}}\frac{x^3}{a^3(x')^3}\right| = \left|\frac{x}{a\sqrt{a^2 - x^2}}\right|. \tag{1.5}$$

Of course, we can integrate Equation 1.1 to get

$$y(x) = \int \frac{\sqrt{a^2 - x^2}}{x}\,dx \tag{1.6}$$

A change of variables of the form $x = a\sin(t)$ gives:

$$\sigma(t) = (a\sin(t), a\ln(\tan(t/2)) + a\cos(t)),$$

which gives the plane curvature as $\kappa_{\pm}(t) = \left|\frac{\tan(t)}{a}\right|$.

Also, to parameterize the tractrix by arclength, we need $(x')^2+(y')^2 = 1$, thus $\left(\frac{a}{x}\right)^2 (x')^2 = 1$, which gives $x' = \pm\frac{1}{a}x$. Let's take $a = 1$ and consider just the case $x' = x$. Then, $x(s) = e^s$ from which it follows that

$$\begin{aligned} \frac{dy}{ds} &= \frac{\sqrt{1 - x^2}}{x}\frac{dx}{ds} = \sqrt{1 - e^{2s}} \\ y(x) &= \sqrt{1 - e^{2s}} - \operatorname{arccosh}(e^{-s}). \end{aligned}$$

This requires that $0 \le e^2s \le 1$. Take the curve traced out in the opposite direction by replacing s by $-s$. The parameterization is now:

$$\sigma(s) = (e^{-s}, \sqrt{1 - e^{-2s}} - \operatorname{arccosh}(e^s)), \quad s \ge 0.$$

For $a = 1$ we have the plane curvature:

$$\kappa_{\pm}(s) = |\sigma''(s)| = \frac{e^{-s}}{\sqrt{1 - e^{-2s}}}.$$

Let $\alpha\colon (a, b) \to \mathbf{R}^2$ be a curve. The **reverse curve** is $\hat{\alpha}\colon (a, b) \to \mathbf{R}^2$ is given by $\hat{\alpha}(t) = \alpha(b - t)$. We wish to distinguish between these two curves.

Definition 1.6 Let $\mathbf{e}_1, \mathbf{e}_2$ denote the standard basis vectors in $\mathbf{R}^2$. An ordered pair of vectors $[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$ is said to be in **standard orientation** if the matrix representing the transformation from $[\mathbf{u}, \mathbf{v}]$ to $[\mathbf{e}_1, \mathbf{e}_2]$ has a positive determinant.

If $\alpha(s)$ is a regular curve parameterized by arclength, then the unit tangent vector is $\mathbf{T}(s) = \alpha'(s)$. Let $\mathbf{N}(s)$ denote the unique unit vector perpendicular to $\mathbf{T}(s)$ with standard orientation $[\mathbf{T}(s), \mathbf{N}(s)]$. $\mathbf{N}(s)$ is the **unit normal vector** to α at s. Since $\mathbf{T}(s)$ is a unit vector, we see that $\mathbf{T}(s) \cdot \mathbf{T}'(s) = 0$. Thus, $\alpha''(s) = \mathbf{T}'(s)$ must be a multiple of $\mathbf{N}(s)$.

Definition 1.7 The **directed curvature** $\kappa(s)$ of a unit-speed curve α is given by the identity

$$\alpha''(s) = \kappa(s)\mathbf{N}(s).$$

Note that since $\mathbf{N}(s)$ is a unit vector, we see that $|\kappa(s)| = |\alpha(s)| = \kappa_{\pm}(s)$.

Theorem 1.6 (Fundamental Theorem for Plane Curves) *Given any continuous function $\kappa\colon (a,b) \to \mathbf{R}$, there is a curve $\sigma\colon (a,b) \to \mathbf{R}^2$, which is parameterized by arclength, such that $\kappa(s)$ is the directed curvature of σ at s for all $s \in (a,b)$. Furthermore, any other curve $\bar{\sigma}\colon (a,b) \to \mathbf{R}^2$ satisfying these conditions differs from σ by a rotation followed by a translation.*

The proof of this is a very neat, simple proof which uses differential equations.

PROOF: From the theorem, we have a function $f\colon (a,b) \to \mathbf{R}^2$ written as $f(s) = (f_1(s), f_2(s))$ satisfying the following system of differential equations:

$$\begin{gathered}(f_1'(s), f_2'(s)) = \kappa(s)(-f_2(s), f_1(s)),\\ \text{subject to } f(c) = \mathbf{u} \text{ and } |\mathbf{u}| = 1\end{gathered}$$

Note that if f is a solution to this differential equation, then it is a unit-speed curve because

$$\begin{aligned}\frac{d}{ds}(f_1^2(s) + f_2^2(s)) &= 2f_1(s)f_1'(s) + 2f_2(s)f_2'(s)\\ &= 2(f_1(s), f_2(s)) \cdot (f_1'(s), f_2'(s))\\ &= 2\kappa(s)(f_1(s), f_2(s)) \cdot (-f_2(s), f_1(s)) = 0\end{aligned}$$

Thus, $|f(s)|$ is a constant and since $|f(c)| = 1$, $|f(s)| = 1$ for all $s \in (a,b)$.

Lemma 1.1 *If $\mathfrak{g}(t)$ is a continuous $(n \times n)$-matrix-valued function on an interval, then there exist solutions, $F\colon (a,b) \to \mathbf{R}^n$, to the differential equation $F'(t) = \mathfrak{g}(t)F(t)$.*

Applying this lemma, we have a function $\mathfrak{g}(s)$ given by

$$\mathfrak{g}(s) = \begin{pmatrix} 0 & -\kappa(s) \\ \kappa(s) & 0 \end{pmatrix}$$

The equation $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ becomes $\mathbf{T}'(s) = \mathfrak{g}(s)\mathbf{T}(s)$. Thus, the above lemma gives us the function $\mathbf{T}(s)$ for the curve $\sigma(s)$ with the correct curvature. To find the curve $\sigma(s)$ we only need to integrate $\mathbf{T}(s)$. We can choose $\sigma(c)$ to be any point in $\mathbf{R}^2$ and we can choose $\mathbf{u}$ to be any unit vector in $\mathbf{R}^2$. Changing $\mathbf{u}$ at $\sigma(c)$ involves a rotation. That rotation passes through the differential equation so that another solution would appear as $\overline{\mathbf{T}}(s) = \rho_\theta \mathbf{T}(s)$, where ρ_θ is a rotation matrix. A translation resets the point $\sigma(c)$ to be any point in $\mathbf{R}^2$. Thus a second solution $\overline{\sigma}(s)$ must satisfy

$$\overline{\sigma}(s) = \rho_\theta \sigma(s) + \omega_0.$$

This proves the theorem. ∎