4.5 A Formula for Gaussian Curvature

The Gaussian curvature can tell us a lot about a surface. We compute K using the unit normal U, so that it would seem reasonable to think that the way in which we embed the surface in three space would affect the value of K while leaving the geometry of M unchanged. This would mean that the Gaussian curvature would not be a geometric invariant and, therefore, would not be as helpful in studying surfaces. If we can find a formula for K which does not depend on U, we would then show that the value of K does not depend on how M is situated in space. We will give a formula for K when depends only on E, F, and G. These three quantities $\{E, F, G\}$ are called the ***metric*** of the surface. I will give the more general formula later, but we will derive this for the case that $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$. Note that this means that the u and v-parameter curves form perpendicular families of curves.

Theorem 4.2 (Gauss' Theorem Egregrium) *The Gaussian curvature depends only on the metric E, F, and G,*

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right). \tag{4.1}$$

where

$$E_v = \frac{\partial}{\partial v}E = \frac{\partial}{\partial v}(\mathbf{x}_u \cdot \mathbf{x}_u) \text{ and } G_u = \frac{\partial}{\partial u}G = \frac{\partial}{\partial u}(\mathbf{x}_v \cdot \mathbf{x}_v)$$

PROOF: This is nothing more than finding the coefficients of a vector with respect to a particular basis.

Since we assumed that our patch is regular, we know that $\{\mathbf{x}_u, \mathbf{x}_v, U\}$ forms a basis for $\mathbf{R}^3$. Now, we need to eliminate U from our formula for Gaussian curvature and $l = \mathbf{x}_{uu} \cdot U$, $m = \mathbf{x}_{uv} \cdot U$, $n = \mathbf{x}_{vv} \cdot U$. Expand $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$ in terms of this basis.

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + lU \\
\mathbf{x}_{uv} &= \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v + mU \\
\mathbf{x}_{vv} &= \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v + nU
\end{aligned} \tag{4.2}$$

Our job is to find the Γ's. While they are just the coefficients in the basis expansion, they are traditionally known as *Christoffel symbols*.

$$\begin{aligned}
\mathbf{x}_{uu} \cdot \mathbf{x}_u &= \Gamma^u_{uu}\mathbf{x}_u \cdot \mathbf{x}_u + 0 + 0 \\
&= \Gamma^u_{uu}E
\end{aligned}$$

likewise

$$\mathbf{x}_{uu} \cdot \mathbf{x}_v = \Gamma^v_{uu}G$$

so if we can compute $\mathbf{x}_{uu} \cdot \mathbf{x}_u$ we can find Γ^u_{uu}.

- $E = \mathbf{x}_u \cdot \mathbf{x}_u$ so by taking the derivative with respect to u, we get that

$$E_u = \mathbf{x}_{uu} \cdot \mathbf{x}_u + \mathbf{x}_u \cdot \mathbf{x}_{uu} = 2\mathbf{x}_u \cdot \mathbf{x}_{uu}.$$

  Therefore,

$$\mathbf{x}_{uu} \cdot \mathbf{x}_u = \frac{E_u}{2} \text{ and } \Gamma^u_{uu} = \frac{E_u}{2E}.$$

- $\mathbf{x}_u \cdot \mathbf{x}_v = 0$ so differentiating with respect to u gives

$$0 = \mathbf{x}_{uu} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{uv} \text{ or } \mathbf{x}_{uu} \cdot \mathbf{x}_v = -\mathbf{x}_u \cdot \mathbf{x}_{uv}$$

- $E = \mathbf{x}_u \cdot \mathbf{x}_u$ and differentiating with respect to v gives $E_v = 2\mathbf{x}_u \cdot \mathbf{x}_{uv}$. So,

$$\frac{E_v}{2} = \mathbf{x}_u \cdot \mathbf{x}_{uv} = -\mathbf{x}_{uu} \cdot \mathbf{x}_v.$$

  Thus,

$$\Gamma^v_{uu} = \frac{\mathbf{x}_{uu} \cdot \mathbf{x}_v}{G} = -\frac{E_v}{2G} \text{ and } \Gamma^u_{uv} = \frac{\mathbf{x}_{uv} \cdot \mathbf{x}_u}{E} = \frac{E_v}{2E}$$

- $G = \mathbf{x}_v \cdot \mathbf{x}_v$, so $G_u/2 = \mathbf{x}_{uv} \cdot \mathbf{x}_v$ so

$$\Gamma^v_{uv} = \frac{\mathbf{x}_{uv} \cdot \mathbf{x}_v}{G} = \frac{G_u}{2G}$$

- $0 = \mathbf{x}_u \cdot \mathbf{x}_v$ so differentiating with respect to v and following the same technique as above will give us

$$\Gamma^u_{vv} = \frac{\mathbf{x}_{vv} \cdot \mathbf{x}_u}{E} = -\frac{G_u}{2E}.$$

- Finally, $\mathbf{x}_v \cdot \mathbf{x}_v = G$ so $\mathbf{x}_{vv} \cdot \mathbf{x}_v = G_v/2$ and

$$\Gamma^v_{vv} = \frac{\mathbf{x}_{vv} \cdot \mathbf{x}_v}{G} = -\frac{G_v}{2G}.$$

We end up with the following formulas

$$\begin{aligned}
\mathbf{x}_{uu} &= \frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + lU \\
\mathbf{x}_{uv} &= \frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + mU \\
\mathbf{x}_{vv} &= -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + nU \\
U_u &= -\frac{l}{E}\mathbf{x}_u - \frac{m}{G}\mathbf{x}_v \\
U_v &= -\frac{m}{E}\mathbf{x}_u - \frac{n}{G}\mathbf{x}_v
\end{aligned}$$

In order to complete our proof, it is necessary to look at certain mixed third partial derivatives. We know that the mixed partials are equal, regardless of the order in which we take the partials. Thus, $\mathbf{x}_{uuv} = \mathbf{x}_{uvu}$, or $\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = 0$. This means that when we write $\mathbf{x}_{uuv} - \mathbf{x}_{uvu}$ in terms of $\{\mathbf{x}_u, \mathbf{x}_v, U\}$ all of the coefficients are zero. We will concentrate on the $\mathbf{x}_v$ term for our result. Other terms give other results.

$$\mathbf{x}_{uuv} = \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\mathbf{x}_{uv} - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v - \frac{E_v}{2G}\mathbf{x}_{vv} + l_v U + l U_v$$

Expand $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$ and U_v by their basis expansions.

$$\begin{aligned}
\mathbf{x}_{uuv} &= \left[\left(\frac{E_u}{2E}\right)_v + \frac{E_uE_v}{4E^2} + \frac{E_vG_u}{4EG} - \frac{lm}{E}\right]\mathbf{x}_u + \left[\frac{E_uG_u}{4EG} - \left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4G^2} - \frac{ln}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{mE_u}{2E} - \frac{nE_v}{2G} + l_v\right]U \\
\mathbf{x}_{uvu} &= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\mathbf{x}_{uu} - \left(\frac{G_u}{2G}\right)_u \mathbf{x}_v - \frac{G_u}{2G}\mathbf{x}_{uv} + m_u U + l U_u \\
&= \left[\left(\frac{E_v}{2E}\right)_u + \frac{E_uE_v}{4E^2} + \frac{E_vG_u}{4EG} - \frac{lm}{E}\right]\mathbf{x}_u + \left[-\frac{E_vE_v}{4EG} + \left(\frac{G_u}{2G}\right)_u + \frac{G_uG_u}{4G^2} - \frac{m^2}{G}\right]\mathbf{x}_v \\
&\quad + \left[\frac{lE_v}{2E} + \frac{mG_u}{2G} + m_u\right]U
\end{aligned}$$

Now, the $\mathbf{x}_v$ coefficient of $\mathbf{x}_{uuv} - \mathbf{x}_{uvu}$ must be zero, so

$$\begin{aligned}
0 &= \left[\frac{E_uG_u}{4EG} - \left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4G^2} - \frac{ln}{G}\right] - \left[-\frac{E_vE_v}{4EG} + \left(\frac{G_u}{2G}\right)_u + \frac{G_uG_u}{4G^2} - \frac{m^2}{G}\right] \\
\frac{ln - m^2}{G} &= \frac{E_uG_u}{4EG} - \left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4G^2} + \frac{E_vE_v}{4EG} - \left(\frac{G_u}{2G}\right)_u - \frac{G_uG_u}{4G^2} \\
\frac{ln - m^2}{EG} &= \frac{E_uG_u}{4E^2G} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4EG^2} + \frac{E_vE_v}{4E^2G} - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u - \frac{G_uG_u}{4EG^2} \\
K &= \frac{E_uG_u}{4E^2G} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4EG^2} + \frac{E_vE_v}{4E^2G} - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u - \frac{G_uG_u}{4EG^2}
\end{aligned}$$

We have to check that right hand side of the above equation is the same as the right hand side of Equation 4.1. That means we must compute the derivatives.

$$\begin{aligned}
&-\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right) = \\
&-\frac{1}{2\sqrt{EG}}\left(\frac{E_{vv}\sqrt{EG} - E_v(\frac{1}{2}(EG)^{-1/2}(E_vG - EG_v)}{EG} + \frac{G_{uu}\sqrt{EG} - G_u(\frac{1}{2}(EG)^{-1/2}(E_uG - EG_u)}{EG}\right) \\
&\qquad = -\frac{E_{vv}}{2EG} + \frac{E_v(E_vG - EG_v)}{4E^2G^2} - \frac{G_{uu}}{2EG} + \frac{G_u(E_uG - EG_u)}{4E^2G^2} \\
&\quad = -\frac{1}{(2EG)^2}\left[GE_uG_u - 2EGE_{vv} + EE_vG_v + GE_vE_v - 2EGG_{uu} + EG_uG_u\right] \\
&\quad = \frac{E_uG_u}{4E^2G} - \frac{2GE_vv - 2E_vG_v}{4EG^2} - \frac{E_vG_v}{4EG^2} + \frac{E_vE_v}{4E^2G} - \frac{2GG_{uu} - 2G_uG_u}{4EG^2} - \frac{G_uG_u}{4EG^2} = K
\end{aligned}$$

Thus, we have developed the Gaussian curvature as a quantity involving only E, G, and (implicitly) F. ∎

Lemma 4.4 *In general the Gaussian curvature is given by*

$$K = \frac{1}{((EG-F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2}+F_{uv}-\frac{G_{uu}}{2} & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right).$$

4.6 Some Effects of Curvature

Recall that a point on a surface is an *umbilic point* if the principal curvatures a p are equal.

Theorem 4.3 *A surface M consisting entirely of umbilic points is contained in either a plane or a sphere.*

A surface in $\mathbf{R}^3$ is *compact* if it is closed and bounded. Here bounded means that it is contained in a sphere of sufficiently large, but finite, radius. Closed means that every sequence of points on the surface converges to a point on the surface.

Theorem 4.4 *On every compact surface $M \subset \mathbf{R}^3$ there is some point p with $K(p) > 0$.*

Corollary 5 *There are no compact surfaces in $\mathbf{R}^3$ with $K \leq 0$. In particular, no minimal surface embedded in $\mathbf{R}^3$ is compact.*

Theorem 4.5 (Liebmann) *If M is a compact surface of constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$.*