Neutral Geometry

How much of our "geometry" does not depend on a parallel axiom? How much is independent of this particular postulate? As you will see, a large portion of our knowledge does not depend on any parallel postulate. This study will then help us to see the true role of a parallel postulate in Geometry.

Alternate Interior Angles

DEFINITION: Let $\mathscr{L}$ be a set of lines in the plane. A line k is **transversal** of $\mathscr{L}$ if

(i) $k \notin \mathscr{L}$, and

(ii) $k \cap m \neq \varnothing$ for all $m \in \mathscr{L}$.

Let ℓ be transversal to *m* and *n* at points *A* and *B*, respectively. We say that each of the angles of intersection of ℓ and *m* and of ℓ and *n* has a *transversal side* in ℓ and a *non-transversal side* not contained in ℓ.

DEFINITION: An angle of intersection of *m* and *k* and one of *n* and *k* are **alternate interior angles** if their transversal sides are opposite directed and intersecting, and if their non-transversal sides lie on opposite sides of ℓ. Two of these angles are **corresponding angles** if their transversal sides have like directions and their non-transversal sides lie on the same side of ℓ.

DEFINITION: If *k* and ℓ are lines so that $k \cap \ell = \varnothing$, we shall call these lines **non-intersecting**.

We want to reserve the word *parallel* for later.

ALTERNATE INTERIOR ANGLE THEOREM: *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are non-intersecting.*

Proof: Let *m* and *n* be two lines cut by the transversal ℓ. Let the points of intersection be *B* and *B′*, respectively. Choose a point *A* on *m* on one side of ℓ, and choose $A' \in n$ on the same side of ℓ as *A*. Likewise, choose $C \in m$ on the opposite side of ℓ from *A*. Choose $C' \in n$ on the same side of ℓ as *C*. Hence, it is on the opposite side of ℓ from *A′*, by the *Plane Separation Axiom*.

We are given that $\angle A'B'B \cong \angle CBB'$. Assume that the lines *m* and *n* are not non-intersecting; *i.e.*, they have a nonempty intersection. Let us denote this point of intersection by *D*. *D* is on one side of ℓ, so by changing the labeling, if necessary, we may assume that *D* lies on the same side of ℓ as *C* and *C′*. By *Congruence Axiom 1* there is a unique point $E \in \overrightarrow{B'A'}$ so that $B'E \cong BD$. Since, $BB' \cong BB'$ (by Axiom C-2), we may apply the SAS Axiom to prove that

$$\Delta EBB' \cong \Delta DBB'.$$

From the definition of congruent triangles, it follows that $\angle DB'B \cong \angle EBB'$. Now, the supplement of $\angle DBB'$ is congruent to the supplement of $\angle EB'B$. The supplement of $\angle EB'B$ is $\angle DB'B$ and $\angle DB'B \cong \angle EBB'$. Therefore, $\angle EBB'$ is congruent to the supplement of $\angle DBB'$. Since the angles share a side, they are themselves supplementary. Thus, $E \in N$ and we have shown that $\{D, E\} \subset N$ or that $m \cap n$ is more that one point, a contradiction. Thus, *m* and *n* must be non-intersecting.

Corollary: *If m and n are distinct lines both perpendicular to the line ℓ, then m and n are non-intersecting.*

Proof: ℓ is the transversal to *m* and *n*. The alternate interior angles are right angles. By our previous proposition all right angles are congruent, so the *Alternate Interior Angle Theorem* applies. *m* and *n* are non-intersecting.

Corollary: *If P is a point not on* ℓ*, then the perpendicular dropped from P to* ℓ *is unique.*

The point at which this perpendicular intersects the line ℓ, is called the **foot** of the perpendicular.

Corollary: *If* ℓ *is any line and P is any point not on* ℓ*, there exists at least one line m through P which does not intersect* ℓ.

Proof: By Corollary 2 there is a unique line, *m*, through *P* perpendicular to ℓ. By previous proposition there is a unique line, *n*, through *P* perpendicular to *m*. By Corollary 1 ℓ and *n* are non-intersecting.

Note that while we have proved that there is a line through *P* which does not intersect ℓ, we have not (and cannot) proved that it is *unique*.

Weak Exterior Angle Theorem

Let ΔABC be any triangle in the plane. This triangle gives us not just three segments, but in fact three lines.

DEFINITION: An angle supplementary to an angle of a triangle is called an **exterior angle** of the triangle. The two angles of the triangle not adjacent to this exterior angle are called the **remote interior angles**.

EXTERIOR ANGLE THEOREM: An exterior angle of a triangle is greater than either remote interior angle.

Proof: We shall show that $\angle ACD > \angle A$. In a like manner, you can show that $\angle ACD > \angle B$. Then by using the same techniques, you can prove the same for the other two exterior angles.

By trichotomy:

$$\angle A < \angle ACD, \angle A \cong \angle ACD, \text{ or } \angle A > \angle ACD$$

If $\angle A = \angle BAC \cong \angle ACD$, then by the *Alternate Interior Angle Theorem*, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are non-intersecting. This is impossible, since they both contain B.

Assume, then, that $\angle A > \angle ACD$. By the definition of this ordering on angles, there exists a ray $\overrightarrow{AE}$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ so that

$$\angle CAE \cong \angle ACD.$$

By the *Crossbar Theorem*, $\overrightarrow{AE}$ intersects BC in a point *G*. Again by the *Alternate Interior Angle Theorem* $\overleftrightarrow{AE}$ and $\overleftrightarrow{CD}$ are non-intersecting. This is a contradiction. Thus, $\angle ACD > \angle A$.

PROPOSITION 1: [**SAA Congruence**] In triangles ΔABC and ΔDEF given that $AC \cong DF$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$, then $\Delta ABC \cong \Delta DEF$.

PROPOSITION 2: *Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.*

PROPOSITION 3: *Every segment has a unique midpoint.*

Proof: Let AB be any segment in the plane, and let C be any point not on $\overleftrightarrow{AB}$. Such a point exists by *Axiom I–3*. There exists a unique ray $\overrightarrow{BX}$ on the opposite side of $\overleftrightarrow{AB}$ from P such that $\angle PAB \cong \angle XBA$, by *Axiom C–4*. There is a unique point $Q \in \overrightarrow{BX}$ so that $AP \cong BQ$, by *Axiom C–1*. Q is on the opposite side of $\overleftrightarrow{AB}$ from P by Homework Problem 9, Chapter 3 and *Axiom B–4*. Since P and Q are on opposite sides of $\overleftrightarrow{AB}$, $PQ \cap \overleftrightarrow{AB} \neq \varnothing$. Let M denote this point of intersection. By *Axiom B–2*, either

$$A*M*B,\ M*A*B,\ A*B*M,\ M = A \text{ or } M = B$$

We want to show that $A*M*B$, so let us assume $\neg(A*M*B)$. Since $\angle PAB \cong \angle QBA$, by construction, we have from the *Alternate Interior Angle Theorem* that $\overleftrightarrow{AP}$ and $\overleftrightarrow{BQ}$ are non-intersecting. If $M = A$ then $A, P, M \in \overleftrightarrow{AP}$ and $\overleftrightarrow{AP} = \overleftrightarrow{AB}$ which intersects $\overleftrightarrow{BQ}$. We can dispose of the case $M = B$ similarly.

Thus, assume that $M*A*B$. This will mean that the line $\overleftrightarrow{PA}$ will intersect side MB of ΔMBQ at a point between M and B. Thus, by *Pasch's Theorem* it must intersect either MQ or BQ. It cannot intersect side BQ as $\overleftrightarrow{AP}$ and $\overleftrightarrow{BQ}$ are non-intersecting. If $\overleftrightarrow{AP}$ intersects MQ then it must contain MQ for P, Q, and M are collinear. Thus, $M = A$ which we have already shown is impossible. Thus, we have shown that $M*A*B$ is not possible.

In the same manner, we can show that $A*B*M$ is impossible. Thus, we have that $A*M*B$. This means that $\angle AMP \cong \angle BMQ$ since they are vertical angles. By *Angle-Angle-Side* we have that $\Delta AMP \cong \Delta BMQ$. Thus, $AM \cong MB$ and M is the midpoint of AB.

PROPOSITION 4:
- *(i) Every angle has a unique bisector.*
- *(ii) Every segment has a unique perpendicular bisector.*

PROPOSITION 5: *In a triangle ΔABC the greater angle lies opposite the greater side and the greater side lies opposite the greater angle; i.e., $AB > BC$ if and only if $\angle C > \angle A$.*

PROPOSITION 6: *Given ΔABC and $\Delta A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\angle B < \angle B'$ if and only if $AC < A'C'$.*

Theorems of Continuity

Elementary Continuity Principle

We will now take up the Axioms of Continuity. We will discuss some of the different uses of the Continuity Axioms in our work.

First, we shall need the famous *Triangle Inequality*. It is usually proved after we have given a measure to line segments, but that is not necessary.

PROPOSITION 7: [**Triangle Inequality**] *If A, B, and C are three noncollinear points, then $AC < AB + BC$, where the sum is segment addition.*

Measure of Angles and Segments

To avoid some of the difficulties that we faced in the previous proofs, and to facilitate matters at a later time, we will introduce a measure for angles and for segments.

The proof of the Theorem requires the axioms of continuity for the first time. The axioms of continuity are not needed if one merely wants to define the addition for congruence classes of segments and then prove the triangle inequality for these congruence classes. It is in order to prove several of our theorems that we need the measurement of angles and segments by real numbers, and for such measurement Archimedes's axiom is required. However, the fourth and eleventh parts of this theorem, the proofs for which require Dedekind's axiom, are never used in proofs in the text. It is possible to introduce coordinates without the continuity axioms, as in discussed in Appendix B of the text.

The notation $\angle A°$ will be used for the number of degrees in $\angle A$, and the length of segment AB will be denoted by $\overline{AB}$.

THEOREM 1

(i) *There is a unique way of assigning a degree measure to each angle such that the following properties hold*:
  - (a) $\angle A°$ *is a real number such that* $0° < \angle A° < 180°$.
  - (b) $\angle A° = 90°$ *if and only if* $\angle A$ *is a right angle.*
  - (c) $\angle A° = \angle B°$ *if and only if* $\angle A \cong \angle B$.
  - (d) *If* $\overrightarrow{AC}$ *is interior to* $\angle DAB$, *then* $\angle DAB° = \angle DAC° + \angle CAB°$.
  - (e) *For every real number x between 0 and 180, there exists an angle* $\angle A$ *such that* $\angle A° = x°$.
  - (f) *If* $\angle B$ *is supplementary to* $\angle A$, *then* $\angle A° + \angle B° = 180°$.
  - (g) $\angle A° > \angle B°$ *if and only if* $\angle A > \angle B$.

(ii) *Given a segment OI, called the* unit segment. *Then there is a unique way of assigning a length* $\overline{AB}$ *to each segment AB such that the following properties hold:*
  - (a) $\overline{AB}$ *is a positive real number and* $\overline{OI} = 1$.
  - (b) $\overline{AB} = \overline{CD}$ *if and only if* $AB \cong CD$.
  - (c) $A * B * C$ *if and only if* $\overline{AC} = \overline{AB} + \overline{BC}$.
  - (d) $\overline{AB} < \overline{CD}$ *if and only if* $AB < CD$.
  - (e) *For every positive real number x, there exists a segment AB such that* $\overline{AB} = x$.

DEFINITION: An angle $\angle A$ is **acute** if $\angle A° < 90°$, and is **obtuse** if $\angle A° > 90°$.

COROLLARY: *The sum of the degree measures of any* two *angles of a triangle is less than* $180°$.

This follows from the Exterior Angle Theorem and Theorem 1.

Proof: We want to show that $\angle A° + \angle B° < 180°$. From the Exterior Angle Theorem and Theorem 1.

$$\angle A° < \angle CBD°$$

$$\angle A° + \angle B° < \angle CBD° + \angle B° = 180°$$

since they are supplementary angles.

Corollary: [**Triangle Inequality**] *If A, B, and C are three noncollinear points, then* $\overline{AC} < \overline{AB} + \overline{BC}$.

Theorem 1 offers an easier proof of this than the one that we gave.

Saccheri-Legendre Theorem

This theorem gives us a setting for our later exploration into non-Euclidean geometry.

Saccheri-Legendre Theorem: *The sum of the degree measures of the three angles in any triangle is less than or equal to* $180°$

$$\angle A° + \angle B° + \angle C° \le 180°$$

Proof: Let us assume not; *i.e.*, assume that we have a triangle ΔABC in which $\angle A° + \angle B° + \angle C° > 180°$. So there is a positive real number, x, so that

$$\angle A° + \angle B° + \angle C° = 180° + x°$$

Let D be the midpoint of BC and let E be the unique point on $\overrightarrow{AD}$ so that $DE \cong AD$. Then by *SAS* $\Delta BAD \cong \Delta CED$. This makes

$$\angle B° = \angle DCE° \quad \angle E° = \angle BAD°.$$

Thus,

$$\begin{aligned}\angle A° + \angle B° + \angle C° &= \angle BAD° + \angle EAC° + \angle B° + \angle ACB° \\ &= \angle E° + \angle EAC° + \left(\angle DCE° + \angle ACD°\right) \\ &= \angle E° + \angle A° + \angle C°\end{aligned}$$

So, ΔABC and ΔACE have the same angle sum, even though they need not be congruent. Note that $\angle BAE° + \angle CAE° = \angle BAC°$, hence

$$\angle CEA° + \angle CAE° = \angle BAC°.$$

It is impossible for both of the angles $\angle CEA°$ and $\angle CAE°$ to have angle measure greater than $\frac{1}{2}\angle BAC°$, so at least one of the angles has angle measure greater than or equal to $\frac{1}{2}\angle BAC°$.

Therefore, there is a triangle ΔACE so that the angle sum is $180° + x$ but in which one angle has measure less than or equal to $\frac{1}{2}\angle A°$. Repeat this construction to get another triangle with angle sum $180° + x$ but in which one angle has measure less than or equal to $\frac{1}{4}\angle A°$. Now there is a positive integer n so that

$$\frac{1}{2^n}\angle A° < x,$$

by the Archimedean property of the real numbers. Thus, after a finite number of iterations of the above construction we obtain a triangle with angle sum $180° + x$ in which one angle has measure less than or equal to $\frac{1}{2^n}\angle A° < x$.

Then the other two angles must sum to a number greater than $180°$ contradicting Corollary 1 to Theorem 1. ■

Corollary: *In* ΔABC *the sum of the degree measures of two angles is less than or equal to the degree measure of their remote exterior angle.*

The Defect of a Triangle

Since the angle sum of any triangle in neutral geometry is not more than $180°$, we can compute the difference between the number 180 and the angle sum of a given triangle.

DEFINITION: The **defect** of a triangle ΔABC is the number

$$\text{defect}(\Delta ABC) = \delta(\Delta ABC) = 180° - \left(\angle A° + \angle B° + \angle C°\right)$$

In Euclidean geometry we are accustomed to having triangles whose defect is zero. Is this always the case? The Saccheri-Legendre Theorem indicates that it may not be so. However, what we wish to see is that the *defective* of triangles is preserved. That is, if we have one defective triangle, then all of the triangles are defective. By defective, we mean that the triangles have positive defect.

THEOREM: [**Additivity of Defect**] *Let* ΔABC *be any triangle and let D be a point between A and B. Then* $\text{defect}(\Delta ABC) = \text{defect}(\Delta ACD) + \text{defect}(\Delta BCD)$.

COROLLARY: $\text{defect}(\Delta ABC) = 0$ *if and only if* $\text{defect}(\Delta ACD) = \text{defect}(\Delta BCD) = 0$.

A **rectangle** is a quadrilateral all of whose angles are right angles. We cannot prove the existence or non-existence of rectangles in Neutral Geometry. Nonetheless, the following result is extremely useful.

THEOREM: *If there exists a triangle of defect 0, then a rectangle exists. If a rectangle exists, then every triangle has defect 0.*

Let me first outline the proof in five steps.

1) Construct a right triangle having defect 0.
2) From a right triangle of defect 0, construct a rectangle.
3) From one rectangle, construct arbitrarily large rectangles.
4) Prove that all right triangles have defect 0.
5) If every right triangle has defect 0, then every triangle has defect 0.

Having outlined the proof, each of the steps is relatively straightforward.

COROLLARY: *If there is a triangle with positive defect, then all triangles have positive defect.*