Algebra Prelim

January 2001

Provide proofs for all statements, citing any theorems that may be needed. If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Z}$ denotes the ring of integers.

1. let R be a commutative ring with identity element $1 \neq 0$. Let I be an ideal of R.

   (a) Show that the following are equivalent:

      i. I is proper (i.e. different from R) and for all $x, y \in R$ we have $xy \in I \Rightarrow x \in I$ or $y \in I$.

      ii. R/I is an integral domain.

   (b) Give an example (with justification) of a nonzero prime ideal I in some ring R such that I is not a maximal ideal of R.

2. Argue that a group of order 561 has a normal Sylow subgroup.

3. Let G be a finite group.

   (a) State the class equation for G.

   (b) Assume that G has order p^n where p is a prime and n is a positive integer.

   Prove that the center $Z(G) \neq 1$. Further prove that G is solvable.

4. Let L be a finite extension of the Galois field $\mathbb{Z}_p$ (of prime characteristic p) and assume that $[L : \mathbb{Z}_p] = n$.

   (a) How many elements does L have?

   (b) Let $\sigma : L \longrightarrow L$ be the map defined by $\sigma(x) = x^p$. Argue that σ is an automorphism of L.

(c) Argue that $\mathbb{Z}_p$ is the fixed field of σ.

(d) Explain why the Galois group $G = Gal(L, \mathbb{Z}_p)$ is cyclic of order n.

5. Let V be a finite dimensional vector space and let $T : V \longrightarrow V$ be a linear transformation such that $T^2 = 0$. Argue that V has a basis such that the matrix of T with respect to that basis has only 1's or 0's as entries.

6. Let W be a vector space of dimension 8 with basis.

   If S, T are two subspaces of W with dimensions $4, 5$ respectively, what is the smallest possible dimension for $S \bigcap T$? Why?

   What is the largest possible dimension for $S \bigcap T$? Why?

   For each possible dimension between your two answers, exhibit concrete subspaces S, T with the chosen dimension of intersection.

7. Let K be an extension field of $\Re$ the reals, of odd degree ($[K : \Re]$).

   Argue that $K = \Re$.

8. (a) In $\mathbb{Q}\ [x, y]$ find a prime ideal requiring at least two generators.

   (b) In $\mathbb{Q}\ [x, y]$ find some ideal requiring at least three generators.

9. Let $\alpha = \sqrt{2} + \sqrt{5}$ and $K = \mathbb{Q}\ [\alpha]$. Answer the following:

   (a) Explain why K contains $\sqrt{2}$ and $\sqrt{5}$.

   (b) Argue that $K = \mathbb{Q}\ (\sqrt{2}, \sqrt{5})$.

   (c) Prove that K is the splitting field of $(X^2 - 2)(X^2 - 5)$.

   (d) Determine the Galois group of K over $\mathbb{Q}$.

   (e) Calculate $f(X)$ the minimum polynomial of α over $\mathbb{Q}$.

   (f) Prove that $f(X)$ is reducible modulo every prime. **(This may be hard.)**