Algebra Prelim

January 2003

Provide proofs for all statements, citing any theorems that may be needed.
If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{C}$ the field of complex numbers.

1. Let $T : U \longrightarrow V$ be a surjective linear transformation from the vector space U onto a vector space V of dimension $n \geq 1$.

   Argue that U has a positive dimensional subspace S such that the restriction of T to S is injective.

   Further argue that the dimension of S can be chosen to be as large as n and no larger.

2. If an abelian group G has an element g of order 10 and an element h of order 6, then prove that G must have an element of order 30.

3. Let p be a prime number. Let G be a finite group and let N be a normal subgroup of order p^k for some $k \geq 1$. Then argue the following:

   (a) N is a subgroup of every p-Sylow subgroup of G.

   (b) The number of p-Sylow subgroups of G divides $|G/N|$.

4. If k is a field, then explain why every ideal of the ring $\mathbb{Z} \times k[X] \times k$ is principal (i.e. generated by a single element).

5. Prove that the ring $\mathbb{Q}\,[X^2, X^5]$ is not isomorphic to $\mathbb{Q}\,[X]$.

6. Prove that in a cyclic group $< a >$ of finite order n, the order of the element a^i is given by the formula $\frac{n}{\gcd(i,n)}$.

7. Consider the symmetric group S_5. Answer the following questions:

   (a) Does the set $\{\sigma \in S_5 | \sigma^6 = id_{S_5}\}$ form a subgroup of S_5? Explain!

   (b) Is there an automorphism $\phi : S_5 \longrightarrow S_5$ of order 3 (i.e. $\phi \neq id_{S_5}$ but $\phi^3 = id_{S_5}$)?

   (c) Is there a homomorphism $\psi : S_5 \longrightarrow S_3$ which is nontrivial (i.e. image of ψ is different from $\{id_{S_3}\}$)?

   (d) If the answer in part c is yes, then determine the number of such possible distinct homomorphisms.

8. Let F_p be a finite field with p elements and let $f \in F_p[X]$ be an irreducible polynomial of degree n.

   Argue that if α is a root of f in some extension field L, then $\alpha, \alpha^p, \cdots, \alpha^{p^{n-1}}$ are all the **distinct** roots of f in L.

   Using this or otherwise, argue that $F_p(\alpha)$ is a Galois extension of F_p and determine its Galois group.

9. Suppose that α is a root of a monic polynomial $f(X) \in k[X]$ of degree $n > 1$ over a field k.

   (a) Show that $w = \alpha^2$ is a root of some monic polynomial $g(X) \in k[X]$ of degree n.

   (b) Generalize the above by replacing w by any polynomial in α.