Algebra Prelim

January 2004

Provide proofs for all statements, citing any theorems that may be needed.
If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{C}$ the field of complex numbers.

1. Let V be a vector space of finite dimension over a field K. Let $S_1, S_2 \subset V$ be subspaces of V such that $\dim S_1 = \dim S_2$. Argue that there is an isomorphism $\varphi : V \to V$ of vector spaces such that $\varphi(S_1) = S_2$.

2. Consider the abelian group $G = \mathbb{Z}/(6) \times \mathbb{Z}$.

   (a) Determine all the finite subgroups of G.

   (b) Give two distinct subgroups of G that have no elements of finite order except the identity.

3. Let $\varphi : G \to G$ an automorphism of the group G and let $\psi : G \to G$ be an inner automorphism of G (i.e. there is some $h \in G$ such that $\psi(g) = h^{-1}gh$ for all $g \in G$). Argue that $\varphi^{-1} \circ \psi \circ \varphi$ is also an inner automorphism of G.

4. Find a factorization by irreducible elements in $\mathbb{Q}[X]$ of the following polynomials:
   $X^4 + 5X + 1$
   $X^4 + 4$
   $X^3 - X^2 + X - 21$
   $X^{100} - X^{99} - 4X^{50} + 4X^{49} + 6X - 6$.

5. Let $f = X^8+X^5+X^4+X^3+1,\ g = X^3+1 \in K[X]$ be polynomials over the field $K := \mathbb{Z}/(2)$. Determine a representative of the multiplicative inverse of $(g \bmod f)$ in $K[X]/(f)$.

6. Let R be a commutative ring such that R has a field K has a subring. Furthermore assume that R is a 2-dimensional K-vector space. Show:

   (a) There is an $a \in R$ such that $R = K[a]$.

   (b) There is a ring isomorphism $R \cong K[X]/(p)$ where $p \in K[X]$ is a quadratic polynomial.

   (c) R is either a field or isomorphic to $K \times K$ or to $K[X]/(X^2)$.

   (d) $K \times K$ and $K[X]/(X^2)$ are not isomorphic and neither of them is a field.

7. Let F be a finite field. Consider the map $\varphi : F \to F$ such that $\varphi(\alpha) = \alpha^2$ for all $\alpha \in F$.

   (a) What must be true about F in order that φ is a field homomorphism?

   (b) If φ is a field homomorphism, argue that φ is in fact an isomorphism (so an automorphism of F).

   (c) Assume φ is a field homomorphism. Describe all automorphisms of F using φ.

8. (a) Determine $[\mathbb{Q}(\sqrt{5}, \sqrt{7}) : \mathbb{Q}]$.

   (b) Find all ring homomorphisms $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \to \mathbb{C}$ whose image is not trivial. You should completely justify your answer.