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Algebra Prelim

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1. Let K be an infinite field and $n \geq 1$. Argue that there is an ideal I in $K[X]$, the polynomial ring in one variable over K, such that $K[X]/I$ is isomorphic to the product of n fields.

2. Let H, K be normal subgroups of a group G. Assume that $H \bigcap K = \{1\}$. Also assume that both G/H and G/K are abelian.

   Prove that G is abelian.

3. Let K be a field and $R = K[X]$, the polynomial ring in one variable over K.

   Suppose $I \neq (1)$ is an ideal in R and there is an irreducible polynomial $f \in I$.

   (a) Argue that I is a prime ideal generated by f.

   (b) Give an example to prove that the result fails if we replace R by a polynomial ring in two variables. (Indeed, your example(s) should show that the ideal need not be prime or principal.)

4. Let F_p be the finite field with p elements. Let $F_p[X]$ be the polynomial ring in one variable over F_p.

   If $f(X)$ is a monic irreducible polynomial of degree $n > 1$ in $F_p[X]$ determine the order and the structure of the Galois group of $f(X)$ over F_p.

   You may use known theorems, after stating them precisely.

   Using this result or otherwise, determine the splitting field of the polynomial $g(X) = (X^2 - 2)(X^3 + X + 1)$ over the field F_5.

5. Let $G =< a >$ be a cyclic group of order n generated by a.

   Let d be a positive integer and let

   $$\phi : G \to G$$

   be defined by $\phi(x) = x^d$ for all $x \in G$.

   (a) Show that ϕ is a group homomorphism.

   (b) Explain why $ker(\phi)$ and $im(\phi)$ are cyclic groups.

   (c) If $n = 48$ and $d = 18$, then find $|ker(\phi)|$ and $|im(f)|$.
   Find an explicit generator of $ker(\phi)$ and a generator of $im(\phi)$, in this case.

   (d) Explain how to determine the generators for $ker(\phi)$ and $im(\phi)$ for general values of n, d.

Please turn over.

6. Let $M_2(K)$ be the ring of 2×2 matrices over a field K.

   Clarification. In this problem, use the following more general definition of a ring homomorphism:

   A map F from ring R to ring S is said to be a homomorphism if

   $$\forall\ x, y \in R \text{ we have } F(x+y) = F(x) + F(y) \text{ and } F(xy) = F(x)F(y).$$

   For $i, j = 1, 2$ define matrices E_{ij} thus:

   $$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Let $\psi : M_2(K) \to K$ be a ring homomorphism. Carry out the following steps to show that ψ must be the zero map.

   (a) Prove that for any ring homomorphism into K, the image of a nilpotent element is 0 and the image of an idempotent element is 1 or 0.

   Reminder: Definitions. An element x is nilpotent if $x^n = 0$ for some positive n. An element x is idempotent if $x^2 = x$.

   (b) Deduce that $\psi(E_{ij}) = 0$, if $i \neq j$ and $\psi(E_{ii}) = 1$ or 0 for $i = 1, 2$.

   (c) Establish a formula for ψ and conclude that ψ must be the zero map by considering the images of suitable products.

   (d) Comment on the possibility of extending the result to $M_n(K)$ for $n \geq 3$.