Algebra Prelim

January 7, 2009

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let $n \in \mathbb{N}$ and F be a field. Suppose that $T : F \longrightarrow F^n$ is a linear transformation. Show the equivalence

$$T \text{ is injective} \iff T \text{ is not the zero map.}$$

(2) Consider the real vector space $V = \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \text{ continuously differentiable on } \mathbb{R}\}$ and the functions $p_0,\, p_1,\, p_2 \in V$ defined as

$$p_0(x) = 1, \;\; p_1(x) = x, \;\; \text{and } p_2(x) = x^2 \text{ for all } x \in \mathbb{R}.$$

Let W be the subspace of V generated by $p_0,\, p_1,\, p_2$ and let $D : W \longrightarrow W,\; f \longmapsto f'$ be the endomorphism of W given by differentiation.

(a) Argue that $\{p_0,\, p_1,\, p_2\}$ is a basis of W.

(b) Write the matrix representation of the endomorphism D with respect to the basis $\{p_0,\, p_1,\, p_2\}$.

(c) Compute the eigenvalues of D.

(d) For each eigenvalue λ you found in (c), compute the corresponding eigenspace, that is, the space $\{f \in W \mid D(f) = \lambda f\}$.

(3) Let $(G,\, \cdot\,)$ be a finite group with identity element e and let $H,\, K$ be cyclic normal subgroups of G such that $H \cap K = \{e\}$ and $|G| = |H| \cdot |K|$. Show

(a) H and K commute elementwise, that is, $hk = kh$ for all $h \in H$ and $k \in K$.

(b) If $|H|$ and $|K|$ are relatively prime, then G is cyclic.

(4) Show that there is no simple group of order 351.

(5) Let $a,\, b \in \mathbb{Z}$ be given integers. Find all solutions $x \in \mathbb{Z}$ to the simultaneous congruences

$$x \equiv a \bmod 8, \quad x \equiv b \bmod 3.$$

(6) Factor the following (possibly irreducible) polynomials into their irreducible factors in the given polynomial ring.

(a) $f := 2x^4 + 200x^3 + 2000x^2 + 20000x + 20 \in \mathbb{Z}[x]$.

(b) $g := x^3 + 2x^2 + x + 2 \in \mathbb{Z}_3[x]$.

(c) $h := 5x^4 + 4x^3 - 2x^2 - 3x + 21 \in \mathbb{Q}[x]$.

(7) Let $R = \{f : [0,1] \longrightarrow \mathbb{R} \mid f \text{ continuous}\}$ be the ring of all continuous functions from the interval $[0,1]$ to $\mathbb{R}$ and let $c \in [0,1]$ be any fixed number.

Show that the subset $M_c = \{f \in R \mid f(c) = 0\}$ is a maximal ideal in R.

[Hint: Consider the map $\psi : R \longrightarrow \mathbb{R},\ f \longmapsto f(c)$.]

(8) (a) Compute the minimal polynomial m_a of $a = \sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$.

(b) Show that $\mathbb{Q}(a)$ is the splitting field of m_a in $\mathbb{C}$.

[Hint: Show that $a^{-1} = \sqrt{2-\sqrt{2}}/\sqrt{2}$ and that $\sqrt{2} \in \mathbb{Q}(a)$.]

(c) Determine $\mathrm{Aut}(\mathbb{Q}(a) \,|\, \mathbb{Q})$.

(9) Let K be the splitting field of an irreducible and separable polynomial $f \in F[x]$ over the field F. Suppose that $\mathrm{Aut}(K \,|\, F)$ is abelian.

Show that $K = F(a)$ for each root $a \in K$ of f.

(10) Determine the automorphism type of the Galois group of $f = x^3 - 3x + 1 \in \mathbb{Q}[x]$.