Algebra Prelim

January 5, 2012

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let V be a finite-dimensional vector space and $T : V \longrightarrow V$ be a linear map.

a) Show that $\ker T^j \subseteq \ker T^{j+1}$ for all $j \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and conclude that there exists some $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$ such that $\ker T^j = \ker T^k$ for all $j \geq k$.

b) Show that $\ker T^k \cap \operatorname{im} T^k = \{0\}$ where k is as in (a).

(2) Let V be a two-dimensional vector space and $F : V \longrightarrow V$ be a linear map. Suppose F has exactly one eigenvalue, denoted by λ. Show that $F(v) - \lambda v$ is contained in $\operatorname{eig}(\lambda; F)$ for all $v \in V$, where $\operatorname{eig}(\lambda; F)$ is the eigenspace of F to the eigenvalue λ.

(3) Let G be a group with $5^2 \cdot 7$ elements.

a) Show that G is abelian.

b) Show that if G does not contain an element of order 25, then it contains an element of order 35.

(4) Let G be a group of order p^r for some prime p and $r \in \mathbb{N}$. Let X be the set of all subgroups of G and consider the group action $G \times X \longrightarrow X$, $(g, H) \longmapsto gHg^{-1}$. Furthermore, let $m = |X|$ (thus, m is the number of all subgroups of G), and let n be the number of all normal subgroups of G. Show that p divides $m - n$.

[Hint: A subgroup H is normal in G if and only if its orbit $\mathcal{O}_H$ satisfies $\mathcal{O}_H = \{H\}$.]

(5) Show that the kernel of the substitution homomorphism $\psi : \mathbb{Z}[x] \longrightarrow \mathbb{Q}, f \longmapsto f(\frac{1}{2})$ is a principal ideal.

(6) a) Let R be a domain. Prove that any prime element in R is irreducible.

b) Show that the element $4 + \sqrt{10}$ is irreducible in the ring $\mathbb{Z}[\sqrt{10}]$, but not prime.

(7) Let F be a finite field and let $f \in F[x]$ be a polynomial such that f and its derivative f' are relatively prime. Show that there exists an $n \in \mathbb{N}$ such that $f \mid (x^n - x)$.

(please turn over)

For the following problems recall that for a field extension $L \subseteq K$ the notation $\mathrm{Aut}(K \mid L)$ denotes the group of all automorphisms of K that leave the elements of L fixed.

(8) a) Show that if $\mathbb{Q} \subseteq K$ is a field extension of degree 2, then $|\mathrm{Aut}(K \mid \mathbb{Q})| = 2$.

b) Show that for any odd number $n \in \mathbb{N}$ there exists a field extension $\mathbb{Q} \subseteq K$ of degree n such that $|\mathrm{Aut}(K \mid \mathbb{Q})| = 1$.

(9) Let $\zeta \in \mathbb{C}$ be a 31^{st} primitive root of unity. Show that there exist exactly 8 distinct subfields L of $\mathbb{Q}(\zeta)$ (that is, 8 distinct subfields L satisfying $\mathbb{Q} \subseteq L \subseteq \mathbb{Q}(\zeta)$, including the two trivial ones).

(10) Consider the polynomial $f := (x^2 - 2)(x^3 - 3) \in \mathbb{Q}[x]$, and let K be a splitting field of f over $\mathbb{Q}$.

a) Determine the degree of the field extension $\mathbb{Q} \subseteq K$.

b) Determine the isomorphism type of $\mathrm{Aut}(K \mid \mathbb{Q})$.