Algebra Prelim

January , 2014

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. Let $a_i \in \mathbb{R}$ for $1 \le i \le n$ and set $f(x) = a_1 + a_2x + \cdots + a_nx^{n-1}$. Show that

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} = f(\zeta_1)f(\zeta_2)\cdots f(\zeta_n),$$

where $\{\zeta_1, \dots, \zeta_n\} \subset \mathbb{C}$ are the n-th roots of unity.

(**Hint:** show, for any i, that $\mathbf{v}_i = (1, \zeta_i, \zeta_i^2, \dots)^t$ is an eigenvector for the given matrix.)

The problem is incorrect as stated.

2. Let G be a group of order 2014. Determine, with proof, which of the following statements **must be true.**

(1) G is simple.

(2) G has a subgroup of **index** 2.

(3) G is abelian.

(4) G is cyclic.

3. Let G be a group such that $G/Z(G)$ is abelian, where $Z(G)$ denotes the center of G. Let H be a non-trivial normal subgroup of G. Show that $H \cap Z(G)$ is a non-trivial subgroup.

4. Let $I_c = (2Y^2 - X^3, Y - X - c)$ be an ideal in the polynomial ring $\mathbb{Q}[X, Y]$ where $c \in \mathbb{Z}$.
Answer the following:

(1) Determine a value of c for which I_c is a prime ideal. In this case, determine if I is a maximal ideal or not.

(2) Determine a value of c for which I_c is not a prime ideal.

5. Let I be the ideal in $\mathbb{Q}[x]$ generated by the product $f(x)g(x)$, where

$$f(x) = x^4 + 9x - 30 \qquad g(x) = x^2 + 2.$$

Show that $\mathbb{Q}[x]/I$ is isomorphic to a product of two fields.

6. Prove that the polynomial $x^4 + nx + 1$ is irreducible over $\mathbb{Q}$ for every integer $n \neq \pm 2$.

7. Consider the rings $R = \mathbb{Z}[\sqrt{-3}]$ and $S = \mathbb{Z}[i]$. Show that there is no ring homomorphism $\varphi : R \longrightarrow S$ such that $\varphi(1_R) = 1_S$.

8. Let F be the finite field $\mathbb{Z}_7$.

Answer the following:

(1) Let K be the splitting field of $X^3 + 3$ over the field $\mathbb{Z}_7$. Determine the degree $[K : F]$.

(2) Similarly, let L be the splitting field of $X^4 + 4$ over the field $\mathbb{Z}_7$. Determine the degree $[L : F]$.

(3) What is the degree of the compositum of L and K over F?

9. Let K denote the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $f(x) = x^5 + x^4 + 3x + 3$.

(a) What is $[K : \mathbb{Q}]$?

(b) Determine the Galois group $Gal(K/\mathbb{Q})$.