Algebra Prelim, January 4, 2017

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let A be an $n \times n$ matrix over an algebraically closed field K. Prove that $A^n = 0$ if and only if $\lambda I_n - A$ is invertible for all nonzero $\lambda \in K$. (Here I_n denotes the identity matrix.)

(2) Let V be a 4-dimensional vector space over a field K, and let $T : V \to V$ be a linear map with characteristic polynomial $\chi_T = x^4 - x^3$. Prove:

a) T is not surjective.

b) V has T-invariant subspaces of dimensions 1, 2, and 3.

(3) Let p be a prime number, and consider the group $G = C_{p^5} \times C_{p^6} \times C_{p^7} \times C_{p^8} \times C_{p^9}$, where C_n denotes a cyclic group of order n.

a) How many elements in C_{p^k} have order at most p^i if $i \leq k$?

b) How many elements in G have order p^7?

(4) a) Give the definition of a solvable group.

b) Let $p < q$ be prime numbers, and let G be a group of order pq^n, where n is any positive integer. Show that G is solvable.

(5) Consider the ring of Gaussian integers $R = \mathbb{Z}[i]$. Determine all ring homomorphisms $R \times R \to R$ that map the identity of $R \times R$ onto the identity of R.

(6) Let R be an integral domain such that the set of nonzero ideals of R contains a minimal element I (with respect to inclusion). Prove that R is a field. (Hint: For a nonzero $a \in I$ consider its square a^2.)

(7) Let $f \in K[x]$ be an irreducible polynomial of degree n over a field K. Let L/K be a field extension of degree m. If m and n are relatively prime, then show that f is irreducible in $L[x]$.

(8) Let $\mathbb{F}_q$ denote a finite field with $q = p^n$ elements, where p is a prime number.

a) Prove that the map $\varphi : \mathbb{F}_q \to \mathbb{F}_q,\ a \mapsto a^p - a$, is $\mathbb{F}_p$-linear.

b) Consider the polynomial $f = x^{p^{n-1}} + x^{p^{n-2}} + \cdots + x^p + x$ and the sets

$$S = \{a^p - a \mid a \in \mathbb{F}_q\},$$
$$T = \{b \in \mathbb{F}_q \mid f(b) = 0\}.$$

Show that $S = T$.

(9) Let p be a prime number and suppose the polynomial $f = x^p - a \in \mathbb{Q}[x]$ is irreducible. Let $\zeta \in \mathbb{C}$ be a primitive p-th root of unity, and consider the field $K = \mathbb{Q}(b, \zeta)$, where $b \in \mathbb{C}$ is any root of f.

a) Prove that the field extension $K/\mathbb{Q}$ is a Galois extension.

b) Determine the order of the Galois group G of K over $\mathbb{Q}$.

c) If P is a subgroup of G with order p, then show that P is a normal subgroup and that G/P is a cyclic group. Furthermore, describe the fixed field of K with respect to P explicitly.