Algebra Prelim, January 3, 2018

- Provide proofs for all statements, citing theorems that may be needed.
- If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.
- Do as many problems as you can and present your solutions as carefully as possible.

Good luck!

(1) a) Let A be an $n \times n$ matrix with entries in a field. Suppose that $A^5 - A$ is invertible. Argue that 1 is not an eigenvalue of A.

b) Let $V \subset \mathbb{F}_2^n$ be the subset of vectors with an even number of nonzero entries. Show that V is a vector space over $\mathbb{F}_2$ and determine its dimension.

(2) Let $\varphi\colon V \to W$ and $\psi\colon W \to V$ be linear maps of vector spaces. Suppose that $\psi \circ \varphi$ is the identity on V. Show that there is an isomorphism of vector spaces $W \cong V \oplus \ker \psi$.

(3) Let N be a normal subgroup of a finite group G, and let P be a Sylow p-subgroup of G. Show that $P \cap N$ is Sylow p-subgroup of N.

(4) Let G be a group of order $3825 = 17 \cdot 25 \cdot 9$. If N is a normal subgroup of order 17 in G, then prove that N is contained in the center of G.

(5) Let R be a UFD such that any ideal of R generated by two elements is principal. Prove that R is a PID.
(Hint: Recall that a UFD satisfies the ascending chain condition for principal ideals.)

(6) Determine all ring homomorphisms $\mathbb{Q}[x]/(x^{100}+2) \to \mathbb{Q}[x]/(x^{501}-2)$.

(7) Show that the ideal I of $\mathbb{Z}[x]$ generated by 29 and x^2+1 is not a maximal ideal.

(8) Determine the isomorphism type of the Galois group of the polynomial $f = (x^{12}-1)(x^2+5)$ over $\mathbb{Q}$, and describe the action of its elements on the splitting field of f.

(9) Consider the polynomial $f = x^5 - 5p^4x + p$, where p is a prime number.

a) Show that f has exactly three real roots.

b) Determine the Galois group of f over $\mathbb{Q}$.